# A STUDY OF A CLASS OF ASYMPTOTIC STABILITY OF DELAY DIFFERENTIAL EQUATIONS WITH A FORCING TERM

**IBRAHIM, MOHAMMED DAYA**
**(M.SC/ MA/04/0358)**

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# A STUDY OF A CLASS OF ASYMPTOTIC STABILITY OF DELAY DIFFERENTIAL EQUATIONS WITH A FORCING TERM

**BY**

**IBRAHIM, MOHAMMED DAYA**
**(MSc/MA/04/0358)**

**A Project Report Submitted to the Department of Mathematics, Modibbo Adama University of Technology, Yola,**
**in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics, School of Pure and Applied Sciences.**

**MAY, 2014**

## DECLARATION

I hereby declare that this project was written by me and it is a record of my own research work. It has not been presented before in any previous application for higher degree. All references cited have been dully acknowledged.

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| IBRAHIM, Mohammed Daya<br>(MSc/MA/04/358) | Date |

## DEDICATION

This project is dedicated to my late father, may his soul rest in perfect peace, Amen.

**APPROVAL PAGE**

This project entitled "A Study of a Class of Asymptotic Stability of Delay Differential Equation with a Forcing Term" meets the regulations governing the award of Master of Science (M. Sc) degree in Mathematics of the Modibbo Adama University of Technology, Yola, and is approved for its contribution to knowledge and literary presentation.

____________________________ ____________________

Prof. Michael Egwurube Date
(Supervisor)

____________________________ ____________________

Dr. Solomon O. Adee Date
(Internal Examiner)

____________________________ ____________________

Prof. Dauda G. Yakubu Date
(External Examiner)

____________________________ ____________________

Dr. Samuel Musa Date
(Head of Department)

Dr. Daniel T. Gungula (Dean, Postgraduate School) Date

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## ABSTRACT

This project studied a class of asymptotic stability of delay differential equations (DDE) with forcing term $x'(t) = -p(t)f(x(t-\tau)) + r(t), t \geq 0$ showing conditions which ensure the asymptotic stability of such DDE. Some examples are also given.

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## CHAPTER ONE

## INTRODUCTION

### 1.0 Introduction

The mathematical models or equations that describe phenomena are in most cases ordinary differential equations of the form

$$x'(t) = f(t, x) \tag{1.1}$$

with the initial data $x(t_0) = x_0$. Since the initial data, which often result from all types of measurements, may have errors, it is important to know the extent to which small disturbances in the initial data affect the desired behavior of the solutions of (1.1). If, by making a sufficiently small change in the initial data, a substantial deviation is observed in the corresponding solution then the solution obtained from the given initial data is unacceptable because it does not describe the required phenomenon even approximately. The area of Mathematics that deals with such problems relating to the behavior of the solutions of (1.1) is usually referred to as stability theory.

The problem of investigating the conditions that will not allow the solutions to remarkably deviate from the desired behavior is therefore vital. Karakostas and Sficas (1995) stated that, the problem of determining various stability properties of the solutions of differential equations with delay has been extensively studied. A large class of such equations is covered by the theory of Lyapunov functional developed by krasovskii, (Karakostas and Sficas, 1995; Zhang, 2005). Investigations motivated by a number of difficulties encountered in the study of stability by means of Lyapunov's direct method usually requires point wise conditions of an averaging nature.

### 1.1 Background of the Study

Delay differential equations (DDE) provide mathematical models for physical systems in which the rate of change of the systems depend not only on their present state but also on their past history. These types of differential equations appear in control theory, biological models, dynamics, economics and so on. The stability theory of (DDE) has been extensively developed during the past years.

The stability of nonlinear dynamical systems is a difficult issue to deal with. When we speak of stability in the context of a nonlinear dynamical systems, we usually mean stability in the sense of Lyapunov. Yoneyama and Sugie (1988), Lyapunov (19892); presented the fundamental concepts of the stability theory analysis of linear and nonlinear systems, both time-invariant and time variant.

In many applications, we know that the zero solution of a system of the type (1.1) is asymptotically stable, we have to often find out if every solution of the system, irrespective of its initial value, approaches zero as $t \to \infty$. In other words, we need to ensure that the region of asymptotic stability of the zero solution is the whole space $R^n$. If this is indeed so, then we say that the zero solution is asymptotically stable in the large or globally asymptotically stable or completely stable.

## 1.2 Statement of the Problem

Yorke (1970) developed a criterion for which the zero solution is asymptotically stable and give a region of attraction. His criterion called simple, readily applicable criterion for asymptotic stability.

The project is concerned with the paper of Liu (2003) on asymptotic stability of DDE where the most recent past does not affect the evolution of the system which uses a different approach hence our interest in this study.

## 1.3 Aim and Objectives of the Study

The aim isto consider the mathematics of his approach and supply all the missing parts where he claims it is obvious.

Further objectives are:

i. Highlight his established conditions for asymptotic stability of this class of DDE.

ii. Construct examples and test the applicability of the established conditions.

## 1.4 Significance of the Study

Since his approach is significantly different from already applicable criterion, it provides alternative method for the study of asymptotic stability of DDE.

## 1.5 Scope of the Study

The project looks only at the Liu (2003) paper in details even though quite a substantial other materials were consulted.

## 1.6 Definition of Terms

Let R be the real line and let $R^+$ be the set of all non negative real numbers. Let $\sigma$, r be two fixed real numbers such that $0 \le \sigma \le r$, if $x$ is a function from R to $R^+$ then for any t, $t \in R(R^+) \, x\,(\sigma, r)$ denotes the function.

$$x_t(\sigma, t)(s) = x(t+s), s \in [-r, -\sigma] \tag{1.2}$$

Definition 1. (Rama, 1981)

The solution of $x(t)$ of (1.1) is said to be stable if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon > 0)$ such that, for any solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ of (1.1), the inequality $|| \bar{x}_0 - x_0 || \le \delta_0$ implies $|| \bar{x}(t) - x(t) || < \varepsilon$ for all $t \ge t_0$.

Definition 2. (Rama, 1981)

The solution $x(t)$ of (1.1) is called asymptotically stable, if it is stable and if there exists a $\partial_0 > 0$ such that $|| \bar{x}_0 - x_0 || \le \delta_0$ implies $|| \bar{x}(t) - x(t) || \to 0$ as $t \to \infty$.

Definition 3. (Rama, 1981)

The solution $x(t)$ of (1.1) is known as unstable if it is not stable

Definition 4 (Rama, 1981)

A delay differential equation is said to be autonomous when the coefficient of the equation is a constant and non autonomous when the coefficient of the equation is a function itself.

Definition 5 (Rama, 1981)

The delay differential equation

$$x'(t) + \sum_{i=1}^{n} p(t) x(t - \tau_i(t)) = 0 \tag{1.3}$$

where the coefficients and the delay terms are functions is called a functional differential equation.

# CHAPTER TWO

# LITERATURE REVIEW

This chapter reviews some related literature on the uniform asymptotic stability of DDE.

## 2.1 Stability of Delay Differential Equation by Lyapunov Method

The formal definition of system stability is at the focus of differential and integral analysis, has engaged the attention of leading mathematicians and physicists including Torricelli, Laplace, Lagrange and others. However, it was only in 1892 that a clear criterion was established, with the publication of the work of the Russian mathematician, Lyapunov (Lyapunov, 1907). He defined a scalar function inspired by a classical energy function (Lyapunov's direct method), which has three important properties that are sufficient for establishing the domain of attraction of a stable equilibrium point:

(a) It must be a local positive definite function,

(b) it must have continuous partial derivatives, and

(c) its time derivative along any state trajectory must be negative semi definite (Slotine and Li, 1991). While Lyapunov theory provides powerful guarantees concerning a system's stability once an appropriate function is identified, it regrettably provides no guidance on how to select it.

Liu (1991), applied Lyapunov – Razumikhin technique to obtain the uniform asymptotic stability for linear integrodifferential equations in Hilbert Spaces.

Liu (1991), introduced heat equation for material with memory which Londen (1990) states

$$\begin{cases} q(t,x) = -\mathrm{E}u_x(t,x) - \int_{\#}^{t} b(t-s)u_x(s,x)dx, (\# = 0 \text{ or } -\infty\ ) \\ u_t(t,x) = -\partial q(t,x)/\partial x + f(t,x) \end{cases} \tag{2.1}$$

The equation gives the heat flux and the second is the balance equation. Equation (2.1) can be written as (assuming E=1)

$$u_t(t,x) = \frac{\partial^2}{\partial x^2}\left[u(t,x) + \int_{\#}^{t} b(t-s)u(s,x)dx\right] + f(t,x) \tag{2.2}$$

Thus the equations

$$x'(t) = A[x(t) + \int_0^t F(t-s)x(s)ds], t \geq t_0 \geq 0, x(s) = \phi(s), 0 \leq s \leq t_0 \tag{2.3}$$

and

$$x'(t) = A[x(t) + \int_\infty^t F(t-s)x(s)ds], t \geq t_0 \geq 0, x(s) = \phi(s), s \leq t_0 \tag{2.4}$$

can arise naturally in applications. Grimmer and Liu (1992), gave examples in visco elasticity. Here, operator A generates a strongly continuous semi group and F(t) is a bounded operator for t≥0 on a real Hilbert space (x. || . || ).

Lûkshmikanthamet *al* (1991) examined the stability of Delay Differential Equations. The approach used to study stability properties of differential equations with delay is that of Lyapunov functions on product spaces. Furthermore, they developed stability theory in terms of two measures in order to unify several known stability concepts such as partial stability, conditional stability and eventually stability. The presentation demonstrated the advantage of utilizing Lyapunov functions on product spaces. They considered Initial Value Problem

$$\acute{x}(t) = f(t, x_t), \quad x_{t_o} = \Phi_{t_o} \epsilon \xi \tag{2.5}$$

where $f \epsilon c[R_+ \chi \xi, R^n]$

The global exponential stabilizability for a class of differential inclusion systems with multiple time delays is considered in which some criteria for global exponential stabilizability of such systems via linear control are provided. The approach used is the frequency-domain approach in order to guarantee the global exponential stability for a class of differential inclusion system. Also, numerical example is provided to illustrate the use of the main results (Sun, 2001).

Sun (2001) considered a class of differential inclusion system

$$\dot{x}(t) = A_{t,o}x(t) + \sum_{i=1}^{p} A_{t,i}x(t-h_i), \quad t \geq 0$$

$$x(t) = \varphi(t), \qquad t \in [H, o]. \tag{2.6}$$

By using variational method, Fu and Feng (2001) established some criteria on the properties of solutions of the perturbed differential equation, such as stability and boundedness. In comparison, they found that the variationalLyapunov method is just an extension of the Lyapunov method. They considered two differential systems

$$y' = f(t,y), y(t_o) = x_o, \tag{2.7}$$

and$x' = F(t,x), x(t_o) = x_o.$ (2.8)

Zhang (2004), studied the stability of the scalar DDE of the form

$$x' = -a(t)x(t) + g(t, x_t) \tag{2.9}$$

by means of a fixed point theory and proved a general stability theorem for the equation (2.5). He compares his result with those obtained by Lyapunov's direct method, and give examples to illustrate how to apply the main theorem to specific equations. It is well-known that Lyapunov's direct method tends to require point-wise relation, but fixed point theory used averaged conditions for a comparison between two methods (Burton and Furumocli 2001).

Now consider a scalar DDE of the form

$$x'(t) = -a(t)x(t) + b(t)x(t - r(t)) \tag{2.10}$$

where b, r: $R^+ \to R$ are continuous functions with

$$r(t) \geq 0, t - r(t) \to \infty \text{ as } t \to \infty \tag{2.11}$$

and

$$a(t) \geq \alpha,\ J|b(t)| \leq a(t), t \geq 0 \tag{2.12}$$

for some constants $\alpha>0$, $J>1$. One can apply Lyapunov's direct method to solve the problem. Taking the derivative of the Lyapunov function $V(x) = x^2$ along a solution x = x(t) of (2.2), we obtain

$$v'(t) \leq \mu x^2(t), \mu > 0$$

whenever $x^2(s) < J\, x^2(t)$for all $s \in [t - r(t), t]$. Then it can be argued that the zero solution at (2.6) is asymptotically stable, Driver (1962).

Kiet and Phat (2000) studied asymptotic stability of nonlinear time-varying differential equations by Lyapunov direct method. Sufficient conditions for asymptotic stability are given in terms of non-differentiable Lyapunov-like functions. Also they gave an application to stabilizability of a class of nonlinear control system with feedback controls. They considered a nonlinear time-varying differential equation of the form

$$\dot{x}(t) = f(t, x(t)),\ \ t \geq 0 \tag{2.13}$$

Papachristodoulon and Prajna (2002) considered the relaxation of Lyapunov's direct method that allows for an algorithmic construction of Lyapunov functions to prove stability of equilibria in nonlinear systems where the search is restricted to systems with polynomial vector fields. Papachristodoulon and Prajna (2002) extended the above technique to include systems with equality, inequality and integral constraints which allows certain non-polynomial nonlinearities in the vector field to be handled exactly and the constructed Lyapunov functions to contain non-polynomial terms. It also allows robustness analysis to be performed. They considered nonlinear system

$$x' = f_x(x,u) \tag{2.14}$$

with the following constraints:

$$a_{i_1}(x,u) \leq 0 \text{ for } i = 1, \ldots, N_1 \tag{2.15}$$

$$b_{i_2}(x,u) = 0 \text{ for } i_2 = 1, \ldots, N_2 \tag{2.16}$$

$$\int_0^T c_{i_3}(x,u)dt \leq 0 \text{ for } i_3 = 1, \ldots, N_3 \text{ and } \forall T \geq 0 \tag{2.17}$$

Daino (2008) stated that stability of nonlinear dynamical system is a difficult issue to deal with. When we speak of stability in the context of a nonlinear dynamical system, we usually mean stability in the sense of Lyapunov. Demidovich (1967) presented the fundamental concepts of the stability theory known as the first method of Lyapunov. This method is widely used for the stability analysis of linear and nonlinear systems, both time-invariant and time-varying (Weissenberger, 1973). As such it is directly applicable to the stability analysis of neural networks. The study of neurodynamics may follow one of the two routes, (Daino, 2006, Hopfied, 1985, Sincak, 1996), depending on the application of interest:

1. Deterministic Neurodynamics: in which the neural network model has a deterministic behavior. In mathematical terms, it is described by a set of nonlinear delay differential equations that define the exact evolution of the model as a function of time (Busa, 2001, Daino, 1999, Daino, 2006).

2. Statistical Neurodynamics: in which the neural network model is perturbed by the presence of noise. In this case, we have to deal with stochastic nonlinear differential equations, expressing the solution in probabilistic terms. The combination of stochasticity and nonlinearity makes the subject more difficult to handle (Daino, 1986, Daino, 2004).

Daino (2008) restricted to deterministic neurodynamics.

### 2.1.1. Lyapunov Stability Principles

This study concerns stability analysis of autonomous systems of the general form:

$$x' = f(x) \qquad (2.18)$$

where $x = R^n$. A system is said to be autonomous if *f* does not depend explicitly on time. An equilibrium point of the system of (2.9), $x = x^*$, is one that satisfies: $f(x^*) = 0$ An equilibrium point is said to be stable in the sense of Lyapunov if for any *n*-dimensional ball of radius $\varepsilon > 0$ there exists an *n*-dimensional ball of radius $\delta(\varepsilon)$, such that for any trajectory $x(t_0, x_0)$, starting in $\delta$, then $x(t, x_0) < \varepsilon$ for any $t > 0$. Otherwise, the equilibrium point is unstable. These conditions are stated formally as:

$$x_0 < \delta(\varepsilon) \Rightarrow x(t) < \varepsilon \ \forall t \geq 0 \qquad (2.19)$$

This definition binds an equilibrium point to its domain of attraction without requiring it to be asymptotic stable. An equilibrium point is also said to be asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\chi_o < r$ implies that

$x(t, x_0) \to 0$ as $t \to \infty$

A function $v(x)$ is said to be a Lyapunov function if it satisfies two main properties:

(a) $v(x)$ must be locally positive definite, meaning that it is bounded from below by a constantly increasing function equal to zero at the origin;

(b) the derivative of $v(x)$ with respect to time:

$dv(x)\ v(x)\ f(x)\ dt\ x\ \partial = \partial$ ,

must be semi-negative definite in the domain of attraction. This type of function must exist around a stable equilibrium point (Shankar, 1999) encapsulated by an attraction domain in the ball *Br*, of radius *r*, where the above properties are satisfied. Furthermore, for systems of the type of Eq. (1), asymptotic stability is guaranteed if the function's derivative (Eq. 4) is locally negative definite (Slotine and Li, 1991).

## 2.2 **Asymptotic Behavior of Delay Differential Equation**

Bellman (1965), raised the question of the behavior of solutions of the functional differential equation

$$u'(t) + au(t - r(t)) = 0 \qquad (2.20)$$

when the delay function r(t) is nearly constant for large t, and also asked for conditions of the function r(t), under which all solutions tend to zero as $t \to +\infty$.

Cooke (1966, 1967), investigated the following state dependent delay equation in response to the questions set by Bellman (1965).

$$u'(t) + au(t - r(t)) = 0 \text{ ,a>0} \qquad (2.21)$$

as well as general linear equation. Under the conditions that either the delay is asymptotically zero, and/or its mean value for t ≥0 is zero, Cooke obtained various sharp results regarding the asymptotical behavior of solution of (2.20) and ( 2.21).

Cooke (1967) in response to a question of Bellman (1965), investigated the asymptotic behavior of the solutions of the functional differential equation

$$u'(t) + au(t - r(t)) = 0 \qquad (2.22)$$

Under the assumption that the lag function r(t) is continuous, nonnegative, tends to zero as $t \to +\infty$, and that $\int_t^{\infty} r(t)dt < \infty \; or \int_t^{\infty} [r(t)]^{2-\delta} dt < \infty$ for some positive δ.

Cooke ( 1967), similar investigation for delay differential equation of implicit type. In order to lay bare the essentials of the argument, Cooke (1967) restricted attention to the simple equation

$$u'(t) + au(t - r(u(t))) = 0, a > 0 \qquad (2.23)$$

in which r(u) is a given nonnegative, continuous function. The notable feature of equation (2.23) is that the amount of 'time lay' is itself a function of the unknown u(t), equations of this type have been encountered in electrodynamics by Driver ( 1963), and in population studies by Cooke (1965) and it appears that they may be of increasing importance in these and other fields.

Haddock and Kuang (1992) studied the following general nonlinear nonautonomous delay differential equation

$$x'(t) = -\int_{t-r}^{t} f(t, x(s)) d\mu(t, s), \qquad (2.24)$$

which includes

$$x'(t) = -\sum_{t=1}^{n} a_i(t) f(t, x(t - r_i(t))) \quad (2.25)$$

and

$$x'(t) = -\int_{t-r}^{t} f(t, x(s))k(t,s)ds \quad (2.26)$$

as special cases. Here Haddock and Kuang assume r(t), a$S_i$(t), f(t , x), k(t ,s) are continuous with respect to their arguments. μ(t ,s) is of bounded variation. In particular, r(t) may be unbounded, and f(t, x) is any function satisfying xf(t ,x) ≥0, f(t, 0) =0. Thus x( t) ≡0 is a solution of (2.24).

Haddock and Kuang (1992) established sufficient conditions for the solution of (2.24) to be bounded for locally or globally asymptotically stable of the zero solution. In this cases, an estimate on the supreme bound of solutions and the size of the region of attraction was obtained. These conditions are general, easy to verify, and improve several of the existing ones.

Rama (1981), pointed out that the asymptotic behavior of the solution of

$$x' = (A + B(t))x \quad (2.27)$$

where $x \in R^n$, A is an n x n constant matrix, and B(t) is an n x n continuous matrix on $0 \le t < \infty$. Rama (1981) said in so doing we shall confine ourselves to a smallness property on B(t), namely, B(t) is small in some sense as $t \to \infty$. More specifically, we assume that the matrix function B(t) satisfies either of the conditions

$$\| B(t) \| \to 0 \text{ as } t \to \infty \quad (2.28a)$$

$$\int^{\infty} \|B(s)\| \, ds < \infty \quad (2.28b)$$

It should be noted that the n-th order differential equation

$$u^{(n)} + \sum_{i=1}^{n} (a_i + p_i(t)) u^{(n-1)} = 0 \quad (2.29)$$

where $a_i$ are the constants and $p_i$(t)are continuous on [0,∞), is a special case of system (2.27). Equation (2.18) may be regarded as a perturbed system of the linear autonomous system

$$x' = Ax \,. \quad (2.30).$$

It is often necessary and useful to know if any property of the solutions of the unperturbed system (2.30) remains unchanged when the system is subjected to changes of the form (2.27). We now give several results on the asymptotic behaviour of the solutions of (2.27) assuming that the function B(t) satisfies either (2.28a) or (2.28b).

Zhang (1983), discussed the asymptotic behavior and structure of solutions of the linear delay differential equations of the form

$$x(t) = p(t(x(t) - \dot{x}(t-1))), t \in (0,8) \tag{2.31}$$

and obtained the following result.

If p(t)= const., or p(t)> 0 and $\int_t^{t+1} p(s)ds \geq 1$, then any solution x(t) of (2.31) can be expressed in the form

$$x(t) = c_o x_o(t) + \bar{x}(t) \tag{2.32}$$

where $c_0$ is a number, $x_0(t)$ is some fixed unbounded solution, whereas x(t) is a bounded solution.

Zhang (1983), used an equation of more general form than (2.29) but with periodic coefficients; i.e.

$$\dot{x}(t) = q(t)x(t) - p(t)x(t-1), t \in (0,\infty) \tag{2.33}$$

where q(t) and p(t) are real- valued, continuous functions on [0, ∞], with the same period 1/k (k is a positive integer).

Considering a general case

$$\dot{x}(t) = q(t)x(t) - p(t)x(t-r), t \in (0,\infty) \tag{2.34}$$

where r>0 is a real number, not necessarily equal to 1, Suppose that there is a real number ω>0 such that

$$q(t+w) = q(t), p(t+w) = p(t), \quad t \in (0,\infty)$$

and r= kω,

The problem of determining stability properties of the solutions of differential equation with delay has been extensively studied. A large class of such equation is covered by the theory of Lyapounov functions developed by Krasovskii (1963). Consider for instance, the equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t-h(t)) \tag{2.35}$$

where a> $0, 0 \le h(t) \le r$, *and* $b, h$ continuous Krasovskii(1963), proved that if

$|b(t)| \le a\theta$ for some, $\theta \in (0,1)$, then the zero solution is asymptotically stable. While Hale and Lunel (1993) proved uniform asymptotic stability of the equation (2.35) under the same condition, Karakostas and Sficas (1995) observed that as far as these conditions are concerned, the delay term is dominated by the instantaneous one; thus (2.35) can be viewed as a perturbation of the ordinary differential equation $x' = -ax$. If $a = 0$, then, clearly, previous arguments do not apply. In this case the equation (2.35) takes the form

$$\dot{x}(t) = b(t)x(t - h(t)) \tag{2.36}$$

where $0 \le h(t) \le r$ for all *t*. If $b(t) < 0$ for all t, then we speak of an equation of stable type which was studied by many authors see Lillo (1969). Barton and Hadlock (1976), Yoneyama (1987).For more specific case, Hale and Lunel (1993), considered a delay differential equation of the form

$$\dot{x}(t) = -p(t)x(t - r) \tag{2.37}$$

*p(t)* is a constant *p*, say, such that $0 < pr < \pi/2$, then uniformly asymptotically stable of (2.37).

Ladas *et al*. (1983) showed that if p(t)>0 and $\lim \int_{t-r}^{t} p(s) < \frac{\pi}{2}$, then (2.37) is uniformly asymptotically stable. Karakostans and Sficas (1995), said that the same is true if $p(t) > 0$ and $\lim \int_{t-r}^{t} p(s) < \frac{\pi}{2}$. But if in (2.36), $b(t)$ and/or $h(t)$ is not constant and $(\sup b(t))(\sup(h(t)) < 3/2$ then we have uniform asymptotic stability.

Karakostans and Sficas (1995), discussed the stability of a non autonomous equation of the form

$$\dot{x}(t) = f(t, x) \tag{2.38}$$

which has the following characteristic, there is a certain σ>0 with σ≤r such that the values of the solution is on the interval (σ-t,t] do not affect the evolution at the moment t. Thus is what we call amnesia and it plays an essential role in our approach.

Xilin and Feng (2005), consider the following systems

$$y' = f(t, y), y(t_0) = x_0 \tag{2.39}$$

and

$$x' = F(t,x), x(t_0) = x_0 \qquad (2.40)$$

where f, F∈C[$R_+$X $S_{(p)}$, $R^n$]. Here $S_{(p)}$={ x∈$R^n$ ,||x||<p}, where ||.|| is a norm in $R^n$.

Xilin and Feng (2005), assume that (H) the solution y (t)= y ( t, $t_0$ , $x_0$) of (2.41 ) exist for all t ≥ $t_0$, are unique, continuous with respect to the initial data and ,|| ( t, $t_0$ , $x_0$)|| is locally Lipschitzian in $x_0$.Since (H) implies that, || y( t, s , x)|| < p for t≥ s, x∈ $S_{(p)}$, for any v ∈C[$R_+$X $S_{(p)}$, $R^N{}_x$] and any fixed t ∈ ($t_0$,∞), variationalLyapunov function v(s, y (t, s, x)) was introduced and its derivative

$$D^+v(s, y(t,s,x)) = \lim_{h\to 0^+} \sup \frac{1}{h}[v(s+h, y(t,s+h, x+hF[(s,x))) - v(s, y(t,s,x))] \qquad (2.41)$$

for $t_0$≤ s≤t and x∈ $S_{(p)}$.

Xilin and Feng (2005) said, Lyapunov stability of the invariant set of a differential system does not rule out the possibility of asymptotic stability of the set.

Consider a scalar functional differential equation (FDE) of the general form,

$$x'(t) = f(t, xt), t \geq 0 \qquad (2.42)$$

where*f* : [0,∞) × *C* → IR is continuous, and *C* := *C*([−*h*, 0]; IR) is the phase space of continuous functions from [−*r*, 0] to IR, *r* >0, with the

sup norm_*ϕ*_ = *max*−*r*≤*θ*≤*0* |*ϕ*(*θ*)|. As usual, *xt*denotes the function in *C* defined by xt(θ) = x(t + θ),−r ≤ θ ≤ 0. Eq. (1.1) often appears as a model for the growth of a single population species, where *x*(*t*) denotes the population density at time *t*. For f(t, ϕ) = b(t)ϕ(0)[1−L(ϕ)] in (1.1), where b : [0,∞) → (0,∞) and L : C([−r, 0]; IR) → IR is a bounded linear operator, we obtain a general delayed logistic equation with autonomous linearity,

$$x'(t) = b(t)x(t)[1 - L(xt)], \qquad (2.43)$$

which was studied in Faria and Liz (2003).

Garroni and Langlais (1982), studied n age-structured dependent population diffusion with an external constraint using the variational approach. They considered the existence and uniqueness of solutions under a weak hypothesis and rediscovered all the biologically intuitive properties connecting the densities of the population to other parameters of the problem. Using this same technique with some suitable modifications, Tchuenche (2005) studied the existence and uniqueness of a weak solution of a population dynamics problem with an additional structure.

Liu (2003), studied the asymptotic behavior of solutions of the forced delay differential equation

$$x'(t) = -p(t) f(x(t-\tau)) + r(t), t \geq 0 \tag{2.44}$$

where $p \in C([0,+\infty),(0, +\infty))$, $r \in C([0, +\infty),R)$, $\tau>0$. $f:R \rightarrow R$ is increasing.

Liu (2003),assumed that

$$\lim_{x \to 0} \frac{f(x)}{x} = b \in (0, \infty) \tag{2.45}$$

and

$$|f(x)| \leq |x|, x \in R. \tag{2.46}$$

Obviously, the equation

$$x'(t) = -p(t)x(t-\tau) + r(t), t \geq 0 \tag{2.47}$$

Studies in Yan(1994) and Graef and Qian (2000), is a special case of equation (2.44). Although the more general case

$$x'(t) + \sum_{j=1}^{n} q_j(t) f(x(t-\tau_j)) = r(t), \tag{2.48}$$

was studied by Yan (1994).

Stoinski (1997), considered the linear initial value problem with delay

$$\frac{d(u(t) - \sum_{j=1}^{l} p_j u(t-r_j))}{dt} = \sum_{i=1}^{m} \varphi_i u(t-s_i), t \geq 0 \tag{2.49}$$

$u(t) = f(t), t \in [-r, 0],$

with $0 < r_1 < \ldots\ldots < r_l, 0 = s_1 < s_2 < \ldots < s_m, r = \max_{i,j}\{r_j s_i\}$

$p_j, \varphi_i \in M(R)$ for $1 \leq i \leq M$, *and* $f \in C([-r,0], R^n)$. As usual it was associate to (2.49), its so-called characteristic equation

$$\det(\lambda - \lambda \sum_{j=1}^{l} p_j e^{-\lambda r_j} - \sum_{i=1}^{m} \varphi_i e^{-\lambda s_i}) = 0 \tag{2.50}$$

It is known that the location of the roots of (2.41) determines the asymptotic behavior of (2.49).

**2.3 Application of Delay Differential Equation**

Cushing (1977) and Freedman (1980) stated that a single species growth equation takes the form

$$x'(t) = x(t)g(x(t)) , \tag{2.51}$$

According to Wright (1955), by introducing a discrete delay $\tau$ in g(x), where

g(x) = $\gamma$ (1 – x/k), the equation (2.51) reduces to

$$x'(t) = \gamma x(t)[1 - {x(t-\tau)}/{k})], \tag{2.52}$$

Motivated by this possible application Kuang and Smith (1991), study global stability in the following general first order real scalar delay differential equation.

$$x'(t) = -(1 + x(t))\sum_{i=1}^{n} \int_{t-r(t)}^{t} f_i(t, x(s))d\mu_i(t, s) \tag{2.53}$$

where $f_i$ (t , x) and r(t) are continuously differential with respect to their arguments; $\mu_i$(t, s) is continuous with respect to t, nondecreasing with respect to s and is defined for all (t,s) $\in R^2$.

Garroni and Langlais (1982), studied on an age- structured dependent population diffusion with an external constraint using the variation approach. They considered the existence and uniqueness of solutions under a weak hypothesis and rediscovered all the biologically intuitive properties connecting the densities of the population to other parameters of the problem. Using this same technique with some suitable modification, Tehuenche (2005), studies the existence and uniqueness of a weak solution of a population dynamics problem with an additional structure.

Various authors have used the abstract formalism in order to prove the well- posed ness, existence and uniqueness of solutions to various population models, see, Inaba (1990), Iannelli (1999), Da prato and Iannelli (1994) derived an abstract setting for a boundary control problem of an age-dependent model and proved the stabilizability result.

Tchuenche and Liadi (2006) recognizing the role that resolvent and semi group theories play in population dynamics and epidemiology, they considered the problem of analyzing the following delayed abstract functional differential equation

$$x' = Ax + F(x_t), x(0) = x_{0;} \qquad (2.54)$$

where A is a mapping from D(A)⊂ E, $x_t$is the section of t of the function x, such that $x_t(s) = x(t+s), s \in [-\gamma, 0], \gamma > 0$ E is the underlying Banach space, while $F : C([-\gamma, 0]; E) \to E$ is a boundary linear operator.

For any initial distribution starting near the origin, the solution of equation (2.54) for t≥0 converges asymptotically to zero.

Kuang (1991) studied a typical single-species growth equation which takes the Cushing (1977) and Freeman (1986) form,

$$x'(t) = x(t)g(x(t)) \qquad (2.55)$$

where g(0)>0, $g'(x)<0$, and there is a k>0 (the so- called carrying capacity) such that g(k) =0. He also say the per capita growth rate function g(x), frequently takes one of the following two forms

$$g(x) = \gamma(1 - x/k), or \ g(x) = \gamma(1 - \frac{ax}{(1+ck)}) \qquad (2.56)$$

where γ, a and c are positive constants. When $g(x) = (1 - \frac{x}{k})$, (2.55) becomes the well known logistic, equation. By introducing a discrete delay γ in g(x), (2.55) reduces to the so-called Wright's equation (1955)

$$x'(t) = yx(t)[1 - x(t-\tau)/k] \qquad (2.57)$$

By recalling the time (t=τ) and the variable x (x=k), (2.57) can be changed to

$$x'(t) = y\tau x(t)[1 - x(t-1)] \qquad (2.58)$$

letting $y(t) = x(t) - 1, \alpha = y\tau$ (2.58) reduces to

$$y'(t) = \alpha(1 + y(t))y(t-1) \qquad (2.59)$$

Kuang (1991), studied the following general nonlinear nonautonomous delay equation,

$$x'(t) = -(1 + x(t))\sum_{i=1}^{n} \int_{t-r(t)}^{t} f_i(t, x(s)) d\mu_i(t,s) \qquad (2.60)$$

where $f_i$ (t,0) =0, r()t>0, $\mu_i$ (t,s) are non decreasing, Kuang (1991),gave attention on the global asymptotical stability of the trivial solution in equation (2.60).

Faria (2006) studied the global attractivity of positive equilibria of delayed logistic differential equations which appear as models for the growth of a single species population, in ecology problems or in disease modelling.

There is an extensive literature dealing with scalar delayed logistic and Lotka-Volterra type equations. We refer the reader to the books of Gopalsamy (1992) and Kuang (1993), and references therein. The use of time-delays in differential equations arises naturally in mathematical models. The introduction of delays produces oscillations, which is a phenomenon in population biology observed from data. In general, large delays imply the loss of stability of equilibria, and even existence of unbounded solutions. To study the behaviour of solutions of delay differential equations, and in particular the stability of equilibria, one approach is to give conditions involving the size of the delays, such as the well-known 3/2-type conditions, so that the FDE is expected to behave similarly to an ordinary differential equation if the delays are small enough. This line of investigation goes back to the work of Wright (1955), with remarkable extensions given by Yorke (1970), Yoneyama (1992), So and Yu (1999), to mention only a few authors.

Zhang (2005), considered the scalar delay equation

$$x'(t) = -b(t)x(t-\tau(t)) \tag{2.61}$$

and its generalization

$$x'(t) = \sum_{j=1}^{N} b_j x(t-\tau_j \tau(t)) \tag{2.62}$$

where b, $b_j \in \subset(R^+, R)$ and $\tau, \tau_j \in \subset(R^+, R^+)$ with t- $\tau(t) \rightarrow \infty$ and

t- $\tau_j(t) \rightarrow \infty$ as $t \rightarrow \infty$. Considering that (2.2) become (2.1) for N= 1.

Delay differential equations of the type considered here arise in a variety of application including control systems, electrodynamics, mixing liquids, neutron transportation, population models. It is the stability and asymptotic behavior of solution of these model equations that is especially important to many investigators. Burton and Haddock (1976) and Yorke (1970) have given a historical background and discussions of applications to dynamical models.

Zhang (2005) investigation is motivated by a number of difficulties encounte in the study of stability by means of Lyapunov's direct method usually requires pointwise conditions, the stability result Zhang offer asks conditions of an averaging nature.

In the case of bounded delays, the stability and asymptotic behavior of solutions of (2.61), as well as more general cases, have been studied by many authors. It is well known that if there are positive numbers β and q such that

$$0 < b(t) \geq \beta, \tau(t) \leq q \text{ and } \beta q \leq \frac{3}{2} \tag{2.63}$$

then the zero solution of (2.61) is stable Yorke (1970).furthermore, the upper bound 3/2 is sharp in the sense that if βq ≤3/2,Yoneyama (1987), generalized (2.63) to

$$\int_t^{t+q} b(s)ds > 0, \tau(t) \leq q, \int_t^{t+q} b(s)ds < \frac{3}{2} \tag{2.64}$$

and showed that the zero solution of (2.61) is uniformly asymptotically stable under (2.64), Krisztin (1991), gave a generalization of Yorke's theorem with condition flexible for more delays.

Appleby (2004) stated that a number of researchers have considered the non–exponential asymptotic behaviour of stochastic equations with unbounded delay which are either linear, or have a linearization close to the equilibrium e.g., Appleby and Buckwar (2004), Appleby and Reynolds (2003), Appleby (2004) and Mao and Riedle (2004). However, apart from Appleby and Buckwar (2003), the principal concern of these works is the asymptotic behaviour of convolution Itˆo-Volterra equations.

Appleby (2004) considered many definitive results on the decay rates of linear equations with unbounded delay which may be found in Krisztin (1988), and in its list of citations, and this paper is perhaps the most germane to the analysis here. Also, an excellent survey article by Corduneanu and Lakshmikantham (1980) poses some questions addressed herein for stochastic equations. More recently, analysis of the asymptotic behaviour of functional differential equations with unbounded delay has been effected through the use of non-differential functional equations, e.g., Dibl´ık (1998) .

Appleby studied the almost sure rates of decay and growth of solutions of the stochastic delay differential equation

$$dx(t) = (-ax(t) + bx(t - \tau(t)))dt + \tau(x(t))dB(t) \tag{2.65}$$

when the delay is unbounded viz., τ(t)→ 1 as t→∞, and the state-dependent noise perturbation $\sigma(x)$ is in some sense small when $x \to 0$ (when we study asymptotic stability) or as $x \to \infty$ (when we study the growth to infinity of solutions). They restrict attention to equations where the unbounded delay does not tend to infinity very quickly, in the sense that $\tau(t)/t \to 0$ as $t\to\infty$. Some results on the asymptotic behaviour of a similar class of stochastic equations with proportional delay (when $\tau(t)/t\to 1-q$ as $t\to\infty$, for some $q\in(0, 1)$) have been earlier studied by Appleby and Buckwar (2003).

In particular, we supply the exact asymptotic rates of decay (in the case when $0 < b < a$) and growth (in the case when $0 < a < b$). In other work, Appleby and Buckwar (2004) have given the decay and growth rates of solutions of the deterministic analogue of (2.65), namely

$$x'(t) = -ax(t) + bx(t-\tau(t)) \qquad (2.66)$$

The method of proof in the case of decaying solutions is to first obtain an estimate on the decay rate in the first mean. Although such estimates are extremely conservative when the noise contribution is large, they are often quite sharp when the noisy component is small close to equilibrium. In this instance, we can exploit the sharpness of this estimate to obtain a bound on the size of the diffusion term. This bound enables us to establish the exact asymptotic behaviour of the solution, as we can factorise X into a nowhere differentiable process (which has a non–trivial limit at infinity) and a process with C1 paths which obeys a random delay–differential equation with the limiting form (2.66). From this, the asymptotic behaviour of X can be recovered by applying (on a pathwise basis) the results on deterministic equations such as (2.66) developed in (2004)

The results on the growth rate are proven using a similar philosophy. First, one obtains good a priori estimates on the asymptotic behaviour of solutions in the first mean, and in an almost sure sense. Then, one reduces the study of the nowhere differentiable solutions of the stochastic delay equation (2.65) to the study of the everywhere differentiable solutions of a random delay differential equation which, in a pathwise sense, has the limiting form (2.66). The asymptotic behaviour of such equations can then readily be established by recourse to deterministic theory. At this juncture, it is then possible to use some of the estimates earlier developed to sharpen asymptotic results.

The particular idea used in the study of growing solutions is to show that the differentiable process $t \to \int_0^t x(s)ds$ obeys a delay–differential equation which has limiting form (2.66). A crucial intermediate technical result proves that this process tends to infinity as $t\to\infty$, once the noisy contribution at zero is sufficiently small at zero to preclude stochastic stabilization to the zero equilibrium, and sufficiently small at infinity to prevent deflection from the growth rate of the underlying deterministic equation.

Javier (2007) stated that light water reactor (LWR) are inherently stable in a wide range of operating conditions. This fact has been extensively proved by means of analytical studies and confirmed by the large operative experience of current reactors. However, the continuous optimization of fuel management strategies has been leading to significant uranium enrichment increases. Javier (2007) goes back to fundamentals and relies on classic methods of non-linear stability and dynamics analysis. By means of the application of Lyapunov theorems to a simplified system of differential equations to a representative current 3 loop/12feet contemporary pressurized water reactors (PWR) (Generation II) with up-to date 17x17 lattice fuel design, the dynamics characteristics are obtained with no need of obtaining the solution.

Kiet and Phat (2000), consider a nonlinear time-varying differential equation of the form

$$\dot{x}(t) = f(t, x(t)), t \geq 0 \qquad (2.67)$$

Kiet and Phat, said that there are two major approaches to the Lyapunov stability analysis of system (2.67): the first linearization method and the second direct method. Stability of system (2.67) can be investigated via the first linearization method, but in general and the most powerful technique is the second direct method. For the method one usually assumes the existence of, so called Lyapunov function, a positive definite function with negative derivative along the trajectory of the system. In the last decade the Lyapunov direct method has been a fruitful technique in stability analysis of nonlinear differential equations and has gained increasing significance in the development qualitative theory of dynamical systems

(Hahn ,1963, Hong, 1997, Lasalle and Lefschetz, 1961 and Zubov, 1964).

# CHAPTER THREE

# METHODOLOGY

The conditions for stability of Delay Differential Equations (DDE) and the method used to obtain such conditions, will now be highlighted.

**3.1 One - Dimensional Delay Differential Equation**

Yoneyama (1986) utilized a simple Lyapunov function, $V(x) = x^2$, where $x \in R$ and assumed some positive solution $x(t)$ on $[t_1, t_2]$ of the one - dimensional delay differential equation

$$\dot{x}(t) = -a(t) f(x(t - r(t))) \qquad (3.1)$$

in proving the following theorems as his main results. The following assumptions are used: $a:[0,\infty) \to [0,\infty), F: R \to R, r:[0,\infty) \to [0,q], q \ge 0, xf(x) > 0$ for, $x \neq 0$ and $a(t), f(x)$ and $r(x)$ are continuous.

Theorem 3.1 (Rama, 1981)

Suppose that the zero solution of the equation $x'(t) = -ax(t) + b(t)x(t - h(t))$ is unique and that

$$\int_{t-r(t)}^{t} a(s)ds \to 0 \quad \text{as} \quad t \to \infty \qquad (3.2)$$

Then the zero solution of equation $x'(t) = b(t)x(t - h(t))$ is uniformly stable

Theorem 3.2 (Rama, 1981)

Suppose that the conditions in Theorem 3.1 are satisfied, and furthermore that $\int_0^\infty a(s)ds = \infty$, then the zero solution of $x'(t) = -ax(t) + b(t)x(t - h(t))$ is asymptotically stable.

Theorem 3.3 (Cooke, 1966)

Suppose that

$f(x) = x, a(t) \equiv \alpha > 0,$ and $r(t) \to 0$ as $t \to \infty$ and $\int_0^\infty r(t)dt < \infty$

Then every solution $x(t)$ of $x'(t) = b(t)x(t - h(t))$ satisfies

$$\lim_{t \to \infty} x(t)e^{\alpha t} = c \qquad (3.3)$$

for some constant c. Moreover, for each constant c, there is a solution of (3.2) which satisfies (3.3). Then it follows from Theorem 3.2 that the zero solution of (3.2) is uniformly asymptotically stable.

We note that proofs of theorems 3.1 and 3.2 utilize Lyapunov's method.

### 3.2 One-Dimensional Nonautonomous Neutral Differential Equations

Haipang and Guozhu (2001) dealt with the stability behavior of a different type of delay differential equation called one-dimensional nonautonomous neutral delay differential equation.

$$\frac{d}{dt}[x(t) - f(t - x(p(t)))] + g(t, x(q(t))) = 0, t \geq t_0 \qquad (3.4)$$

where $p, q : [t_0, \infty) \to R$ are continuous and strictly increasing, $p(t) < t, q(t) < t$ for all $t \geq t_0$, with the following assumptions

i. $p, q : [t_0, \infty) \to R$ are continuous and strictly increasing,

   $p(t) < t, q(t) < t$ for all $t \geq t_0$, and $\lim_{t\to\infty} p(t) = \infty, \lim_{t\to\infty} q(t) = \infty$.

ii. $f, g \in C([t_0, \infty) \times R, R), f(t,0) = 0$ for $t \geq t_0$.

iii. $xg(t,x) \geq 0$ for $x \in R$.

By a solution of (3.9), we mean a function $x(t)$ that is continuous and satisfies (3.4) on $[t, \infty)$ for some $t \geq t_0$.

Existing results in the literature are stated and utilized to consider the case where $t - p(t)$ and $t - q(t)$ are unbounded. Details of these are given in the article (Haipang and Guozhu, 2001).

We consider the conditions obtained by Wandi *et al.* (2001).

### 3.3 Application to Discrete Population Models

The application of stability of delay differential equations to Lokta-Volterra equation is demonstrated by taking the conditions obtained by Wandi *et al*. (2001). First some positive solution of a discrete population model of Volterra – type with delays is assumed. The model incorporates time delays and allow for fluctuating environment. The proofs of main result are approached in the following ways.

i. First some positive solutions $\{x*(k)\}$ of the discrete population model with time delays.

$$x_i(k+1) = x_i(k)\exp\{r_i(k) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}(k)x_j(k-l)\}, i = 1,2,...n \tag{3.5}$$

is assumed.

ii. A change of variable is introduced as $u_i(k) = x_i(k) - x_i^*(k), i = 1,...n$ which transforms model (3.3) into

$$u_i(k+1) = x_i(k)\{r_i(k) - \sum_{j=1}^{n}\sum_{i=0}^{m} a_{ij}(k)x_j(k-l)\} - x_i^*(k)\exp\{r_i(k) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(k)u_j(k-l)\}$$

$$= [(u_i(k) + x_i^*(k))\exp\{-\sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(k)u_j(k-l)\} - x_i^*(k)] \times \exp\{r_i(k) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(k)x_j^*(k-l)\},$$

which can be written as

$$u_i(k+1) = [(1 - a_{ii}^0(k)x_i^*(k))u_i(k) - \sum_{l=1}^{m} a_{ii}^l(k)x_i^*(k)u_i(k)(k-l) -$$
$$\sum_{j=1}^{n}{}_{j\neq i}\sum_{l=0}^{m} a_{ij}(k)x_i^*(k)u_j(k-l) + f_i(k.\bar{u}^k)] \times \exp\{r_i(k) - \tag{3.6}$$
$$\sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(k)x_j^*(k-l)\}, i = 1,...n$$

And in view of (3.3), system (3.4) can be transformed into

$$u_i(k+1) = x_i^*(k+1)[(1 - a_{ii}^0(k)x_i^*(k)u_i\, {}^{(k)}\!/_{x_i^*}) -$$
$$\sum_{l=1}^{m} a_{ii}^l(k)u_i(k-l) - \sum_{j=1}^{n}{}_{j\neq i}\sum_{l=0}^{m} a_{ij}^l(k)u_j(k-l) + f_i(k,\bar{u}^k)/x_i^*(k)], i = 1,..,n \tag{3.7}$$

iii. Then a functional V is defined and its difference along the solution of (3.7) is calculated. Using a hypothesis in the theorem together with V, the main result of the theorem is established i.e system (3.5) is strongly persistent and globally asymptotically stable. The application of this main result to two special population models is illustrated and it involves the use and calculations of functional to establishe permanence and global asymptotic stability; see Wandi *et al*. (2001) for details.

Finally, Saito *et al*. (2001) studied the permanence of the Lokta-Volterra discrete competition system with delays for which the obtained necessary and sufficient conditions for its permanence. The method used in proving the main result is in this order:

i. First a lemma is established.
ii. Then discrete functions $V_1(n)$ and $V_2(n)$ for $n \geq 0$ are constructed.
iii. And the ratio $V_i(n+1)/V_i(n), i = 1,2$ is calculated.
iv. Using the lemma and defining a region D and two curves $\gamma$ and $\eta$, the main result/theorem is established.

# CHAPTER FOUR
# RESULTS AND DISCUSSION

## 4.0 Introduction

A delay differential equations (DDE) form another class of ordinary differential equations. The stability of DDE has been extensively studied.

The asymptotic behavior of solutions of

$$x'(t) = -p(t) f(x(t-\tau)) + r(t), t \geq 0 \tag{4.1}$$

a delay partial differential equation with forcing term was studied by Liu (2003) and shown to exhibit asymptotic properties under the following conditions:

## 4.1 Assumptions used in the paper.

i. f is increasing

ii. $|f(x)| \leq |x| \; x \in R$

iii $\lim_{t \to +\infty} \frac{r(t)}{p(t)} = 0$

iv $\int_0^{+\infty} p(s)ds = +\infty$

v. $\mu = \lim_{t \to +\infty} \sup \int_{t-\tau}^{t} p(s)ds < \frac{3}{2}$

for sufficiently large t then every solution of (4.1) is asymptotically stable.

## 4.2 Further Assumptions used

i. $p \in C([0,+\infty),(0,\infty)), r \in C([0,\infty), R)$

i. $\lim_{x \to 0} \frac{f(x)}{x} = b \in (0,\infty)$

## 4.3 Note

A special case of this equation was studied by the following Yan (1994), Graef and Qian (2000):

$$x'(t) = -p(t)x(t-\tau) + r(t), t \geq 0 \tag{4.2}$$

### 4.4 Main Result of the Paper

The following theorem is the main result of Liu (2003),

Theorem1(Liu,2003)

Suppose that 4.1(ii), 4.2(ii) and 4.1(iv) with

$$\lim_{t\to\infty}\sup\int_{t-\tau}^{t} p(s)ds < \frac{3}{2}$$

together with 4.1(iii), then every solution of equation (4.1) tend to zero as $t \to \infty$

### 4.5 Remarks 4.1

(i) In Graef and Qian (2000), the special case of the paper under review, was proved that if 4.1(iv) and 4.1(iii) hold with $\lim_{t\to\infty}\sup\int_{t-\tau}^{t} p(s)ds < 1$ and $\int_{0}^{\infty} r(s)ds$ converges then every solution of equation (4.2) tends to zero.

(ii) Graef and Qian (2000) also proved that if 4.1(ii), 4.2(ii) hold and $\lim_{t\to\infty}\int_{t-\tau}^{t} p(s)ds < \frac{\Pi}{2}$ then every solution of equation (4.2) tends to zero.

(iii)To prove the main result above some other results were mentioned.

Lemma1.

Suppose 4.2 (ii) and 4.1 (iii) hold then there exist $\alpha > 0$ such that $\frac{f(x)}{x} > \frac{b}{2},\ for|x| \le \alpha$ and for any $\varepsilon \in (0,\alpha)$, there is T > 0 such that $\left|\frac{r(t)}{p(t)}\right| < \frac{b\varepsilon}{2}, t > T$ .

Proof

Consider equation (4.1)

$$x'(t) = -p(t)f(x(t-\tau)) + r(t)$$

$$= p(t)[-f(x(t-\tau)) + \frac{r(t)}{p(t)}]$$

But $\frac{f(x)}{x} > \frac{b\varepsilon}{2},\ for|x| \le \alpha$

Therefore, $x'(t) < p(t)[\frac{-b\varepsilon}{2} + \frac{r(t)}{p(t)}], |x| < \alpha$

By assumption 4.2 implies $\frac{r(t)}{p(t)} - \frac{b\varepsilon}{2} < 0$

Hence for any $0 < \varepsilon < \alpha$, there is T > 0 such that $\left|\frac{r(t)}{p(t)}\right| < \frac{b\varepsilon}{2}, t > T$ .

Lemma 2. (Liu, 2003). Suppose that

$\lim_{x\to 0} \frac{f(x)}{x} = b, b \in (0,\infty)$ and $\lim_{x\to 0} \frac{r(t)}{p(t)} = 0,$ hold, $x(t)$ is an oscillatory solution of

$x'(t) = -p(t)x(t-\tau) + r(t), t \geq 0$ and $A > 0, \delta > 1$ such that $x(t)$ satisfies that

$x'(t) \leq Ap(t) + r(t), t \geq T$ . (4.3)

$x'(t) \leq -p(t)x(t-\tau) + r(t)$ if $x(t-\tau) \leq 0$ and $t \geq T + \tau$ (4.4)

$\int p(s)ds \leq \delta,$ for all $t \geq T + \tau$ (4.5)

If $c > T + 2\tau, x'(c) \geq 0$ , then we have

$$x(c) \leq (\delta - \frac{1}{2})A + \varepsilon(b\delta + \frac{b\delta^2}{2} + 1) \qquad (4.6)$$

**4.5 Remark 4.2**

Liu (2003) proved the Lemma 2 above considering two cases in case 1 of his proof, there were two sub cases, all of which are proved in the paper.

Lemma 3.

Suppose $\lim_{x\to 0} \frac{f(x)}{x} = b, b \in (0,\infty)$ and $|f(x)| \leq |x|, x \in R$ hold. $x(t)$ is a solution of the equation

$x'(t) = -p(t)f(x(t-\tau)) + r(t), t \geq 0$ , B<0, such that $x'(t) \geq Bp(t) + r(t), r \geq T,$

$x'(t) \geq -p(t)x(t-\tau) + r(t), if x(t-\tau) \geq 0, and t \geq T + \tau$

If the equation $x'(t) = -p(t)x(t-\tau) + r(t), t \geq 0$ holds, $c > 0$ and $x'(c) \leq 0$, then we have that

$$x(c) \geq (\delta - \frac{1}{2})B - \varepsilon(1 + b\delta + \frac{b\delta^2}{2}).$$

Proof

If $x(t-\tau) \geq 0$ and $t \geq T + \tau$

$x'(t) = -p(t)x(t-\tau) + r(t),$ if $x(t-\tau) \geq 0$ and $t \geq T + \tau$ hold $x(c) > 0$ and $x'(c) \leq 0$.

If $0 < x(c-\tau) \leq \varepsilon$ for $t \in (c-\tau, c)$. We have $t - \tau \leq c - \tau$.

Integrating $x'(t) \geq Bp(t) + r(t), r \geq T$, *from* $t-\tau$ to $c-\tau$, we get

$$\int_{t-\tau}^{c-\tau} x'(t)dt \geq B\int_{t-\tau}^{c-\tau} p(s)ds + \int_{t-\tau}^{c-\tau} r(s)ds .$$

$$x(c-\tau) - x(t-\tau) \geq -B\int_{t-\tau}^{c-\tau} p(s)ds - b\frac{b\varepsilon}{2}\int_{t-\tau}^{c-\tau} p(s)ds$$

$$\geq -B\int_{t-\tau}^{c-\tau} p(s)ds - \frac{b\varepsilon\delta}{2}$$

if $x(t-\tau) \geq 0$ then the equation $x'(t) = -p(t)x(t-\tau) + r(t),$

we get ,

$$x'(t) \geq Bp(t) + r(t),$$

$$x'(t) \geq Bp(t)\int_{t-\tau}^{c-\tau} p(s)ds + p(t)\delta\frac{b\varepsilon}{2} + r(t), t \in [c-\tau, c] \qquad (4.7)$$

If $x(t-\tau) < 0,$ then equation (1) implies $x'(t) \geq r(t)$ and hence equation (4.07) is valid.

Suppose that

$$\int_{c-\tau}^{c} p(s)ds \leq 1$$

Integrating (4.7) from $c-\tau$ to $c$, applying $\left|\frac{r(t)}{p(t)}\right| < \frac{b\varepsilon}{2}, t > T$ . and $\int_{t-\tau}^{t} p(s)ds$ for all $t \geq T + \tau$

we get,

$$x(c) \geq x(c-\tau) + B\int_{c-\tau}^{c} p(t)\int_{t-\tau}^{c-\tau} p(s)dsdt + \frac{b\varepsilon\delta}{2}\int_{c-\tau}^{c} p(s)ds + \int_{c-\tau}^{c} r(t)dt$$

$$\geq \varepsilon(\frac{b\delta}{2} + \frac{b\delta^2}{2} + 1) + B\int_{c-\tau}^{c} p(t)(\delta - \int_{c-\tau}^{t} p(s)ds)dt$$

$$= \varepsilon(\frac{b\delta}{2} + \frac{b\delta^2}{2} + 1) + B\delta(\int_{c-\tau}^{c} p(t)dt - B(\int_{c-\tau}^{t} p(t)\int_{c-\tau}^{t} p(s)ds)dt$$

$$= \varepsilon(\frac{b\delta}{2} + \frac{b\delta^2}{2} + 1) + B\delta\int_{c-\tau}^{c} p(t)dt - \frac{1}{2}B(\int_{c-\tau}^{t} p(s)ds)^2$$

Since $\delta x - \frac{1}{2}x^2$ is decreasing for $0 \leq x \leq 1 \leq \delta$, then

$$x(c) \geq (\delta - \frac{1}{2})B - \varepsilon(1 + b\delta + \frac{b\delta^2}{2})$$

Lemma 4 (Liu, 2003)

Suppose that $x(t)$ is an eventually non-negative solution of the equation

$x'(t) = -p(t)f(x(t-\tau)) + r(t), t \geq 0$ and that the conditions state in lemma 2 and

$\int_0^{+\infty} p(s)ds = +\infty$ hold. Then $x(t)$ tend to zero as t tend to infinity.

Remark

Liu (2003), proved Lemma 4 and considered the poof in two cases, which is proved in the paper.

Lemma 5 (Liu, 2003).

Suppose that $x(t)$ is any eventually non-positive solution of

$x'(t) = -p(t)f(x(t-\tau)) + r(t), t \geq 0$

and that the conditions state in lemma 3 hold. Then $x(t)$ tend to zero.

Proof

Let $\limsup_{t\to+\infty} x(t) = v.$ If v=0, then the proof is complete. If v<0, we have two cases to consider.

If $x'(t)$ is eventually positive, then there is $T_1 < T + \tau$ such that $x(t)$ is increasing for $t \leq T_1$ . the assumption $\limsup_{t\to+\infty} x(t) = v$ implies $x(t-\tau) \leq v$ for all $t \leq T_1$. By (1), we have,

$$x'(t) \geq -p(t)f(v) + r(t) \;,\; t \leq T_1 \qquad (4.8)$$

Integrating (4.8) from $T_1$ to t, we get

$$x(t) - x(T_1) \geq -f(v)\int_{T_1}^{t} p(s)ds + \int_{T_1}^{t} r(s)ds$$

Since v>0, we get f(v)>0. Choosing $\varepsilon \in (0, f(v)), \lim_{x\to+\infty} \frac{r(t)}{p(t)} = 0$ implies there is $T_2 > T_1$ such that $|r(t) \leq \varepsilon p(t)|$ for $t \geq T_2$. Hence

$$x(t) - x(T_1) \leq (-f(v) + \varepsilon)\int_{T_2}^{t} p(s)ds - f(v)\int_{T_1}^{T_2} p(s)ds + \int_{T_1}^{T_2} r(s)ds \qquad (4.9)$$

Let $t \to \infty$, by (4.9), we get $v - x(T_1) \leq -\infty$ , a contradiction. Choosing $T_{1.} > T$ such that $x(t-\tau \geq 0)$ for all $t \geq T_1$,we get

$$x'(t) \leq r(t), t \geq T_1 \qquad (4.10)$$

Suppose that $t^* > T_1 + \tau$ is any left maximum point of $x(t)$, then we have

$x(t^* - \tau) > \varepsilon$, using $|r(t)| \le \frac{b\varepsilon}{2} p(t)$ and $\left|\frac{r(t)}{p(t)}\right| < \frac{b\varepsilon}{2}, t > T$, we have

$$0 \le x'(t^*) = -p(t^*) f(x(t^* - \tau)) + r(t^*)$$
$$< p(t^*)(-f(x(t^* - \tau)) + \frac{b\varepsilon}{2})$$
$$= \varepsilon p(t^*)(-\frac{b}{2} + \frac{b}{2}) = 0$$

A contradiction. Integrating (4.10) from $t^* - \tau$ to $t^*$, by $\left|\frac{r(t)}{p(t)}\right| < \frac{b\varepsilon}{2}, t > T$ and $\int_{t-\tau}^{t} p(s)ds$ for all $t \ge T + \tau$ we get

$$x(t^*) \le x(t^* - \tau) + \int_{t^*-\tau}^{t^*} r(t)dt \le \frac{b\delta}{2}\varepsilon + \varepsilon$$

which shows that it is bounded above and then $v < \infty$. Choosing $\{t_n\}$ such that

$T_2 + \tau < t_1 < t_2 < \ldots, \lim_{n\to\infty} t_n = \infty, x'(t_n) \ge 0, \lim_{n\to\infty} x(t_n) = v$,

we get $x(t_n - \tau) \le \varepsilon$. therefore, $f(x(t - \tau)) > 0$ implies $x'(t) \le r(t)$.

Integrating this inequality from $t_n - \tau$ to $t_n$, we get

$$x(t_n) = x(t_n - \tau) + \int_{t_n-\tau}^{t_n} r(t)dt \le \varepsilon(1 + \frac{b\delta}{2}).$$

Let $n \to \infty, \varepsilon \to 0,$ we have v=0.

EXAMPLES.

Remark: The condition $p(t) \to 0$ as $t \to \infty$ and $\int_0^\infty p(s)ds = 0$ are somewhat independent of each other, as the following example shows.

Example 1.

Consider $p(t) = 1/(1+t)$ on $0 \le t \le \infty$

This shows clearly that $p(t) \to 0$ as $t \to \infty$

On the other hand $\int_0^{\infty} p(s)ds = \infty$

i.e. $\int_0^{\infty} \frac{1}{1+t} dt = \infty$

Example 2.

Liu (2003) considered the equation $x'(t) = -\frac{1}{4}(\frac{6}{5} + \cos t)x(t-\tau) + \frac{1}{1+t}$

Solution

Since $p(t) = \frac{1}{4}(\frac{6}{5} + \cos t)$, we get

$$\lim_{t \to =\infty} \int_{t-\Pi}^{t} p(s)ds = \lim_{t \to \infty} \frac{1}{4}(\frac{6}{5}\Pi + 2\sin t) = \frac{6}{20}\Pi + \lim_{t \to \infty} \frac{2\sin t}{4} = \frac{6}{20}\Pi + \frac{2}{4} = \frac{2+6\Pi}{4} < \frac{3}{2}.$$

This show that $\int_0^{\infty} p(s)ds = \infty$ and $\lim_{t \to +\infty} \frac{r(t)}{p(t)} = 0$.

It also shows that the theorem stated by Yan(1994), every solution of the equation tends to zero as t tends to infinity.

# CHAPTER FIVE

## SUMMARY, CONCLUSION AND RECOMMENDATION

### 5.1 Summary

The project consists of five chapters. These include introduction, background of the study, objectives of the study, statement of the problem, the need of the study and lastly but not the least definition of terms used in the thesis.

Contribution of various authors on conditions for stability or nonstability of delay differential equations and equations analogous to DDE. These authors used the Lyapunov method to establish conditions for the stability of the equation of the type

$$x'(t)+\sum_{i=1}^{n}p(t)x(t-\tau_i(t))=0$$

where p is a function of t, i.e. the equation is nonautonomous.

Chapter three is concerned with the methods used by the various authors in establishing the conditions for the stability or nonstability of the delay differential equations or system of delay differential equations they considered.

Chapter four is solely on the results and discussion, where we reviewed the conditions for stability or otherwise given by Liu (2003) for a delay differential equation with a forcing term by utilizing some conditions.

Chapter five contains summary, conclusion and recommendation, also contribution to knowledge, suggestion for feature research were stated.

### 5.2 Conclusion

This paper by Liu (2003), though Mathematically very vigorous establishes another method of showing asymptotic stability of delayed differential equations particularly with forcing terms, instead of the usual method which involves the construction of Lyapunov functional. Also using appropriate examples we illustrated the conditions obtained by Liu (2003) for the one-dimensional delay differential equations with forcing term.

### 5.3 Contributions to Knowledge

In this research work, the following contributions are highlighted.

We studied the asymptotic behavior of the following force delay differential equation

$x'(t) = -p(t) f(x(t-\tau)) + r(t), t \geq 0$

It is shown that if f is increasing and $|f(x)| \leq |x|$ for all $x \in R$, $\lim_{t \to +\infty} \frac{r(t)}{p(t)} = 0$, $\int_0^{+\infty} p(s)ds = +\infty$ and $\lim_{t \to +\infty} \sup \int_{t-\tau}^{t} p(s)ds < \frac{3}{2}$ for sufficiently large t. Then every solution of the equation above tend to zero as t tends to infinity. Our result improves the recent results obtained by Greaf and Qian (2000).

### 5.5 Recommendation/Suggestion for Future Research

i. For future research, Liu method is an improved method over Graef and Qian (2000) and can be used for further studies.

ii. Finally, by applying the Liu method to real life situation such as dynamical models and host of others is an area for further investigation.

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