MATHEMATICAL MODEL OF THE TRANSMISSION DYNAMICS OF EBOLA VIRUS DISEASE IN THE PRESENCE OF THE NON-HUMAN VECTOR

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A THESIS SUBMITTED TO THE DEPERTMENT OF MATHEMATICS, SCHOOL OF PHYSICAL SCIENCES, IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS OF THE MODDIBO ADAMA UNIVERSITY OF TECHNOLOGY, YOLA

DECEMBER, 2019

DECLARATION

I hereby declare that this thesis was written by me and it is a record of my own research work. It has not been presented before in any previous application for a higher degree. All references cited have been duly acknowledged.

Mohammed Manga _____ _____

Signature Date

DEDICATION

This project is dedicated to my belove Late Father Muhammad Manga and my lovely children's Khadija, Abdurrashid, Hauwa'u, Safiyya, Yahuza, Zainab, Abdul-azeez, Fatima, Abdul-bass, Amina, Yusuf and Ibrahim.

APPROVAL

This thesis entitled 'Mathematical model of the transmission dynamics of Ebola virus diseases in the presence of the non-human vector' meets the regulations governing the award of Masters of the Modibbo Adama University of Technology, Yola and is approved for its contribution to knowledge and literary presentation.

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ABSTRACT

This thesis analyses the transmission dynamics of ebola virus disease using the modified SEIR model which is a system of ordinary differential equations. We modified the SEIR model due to Durojaye and Ajie, by incorporating vaccination, quarantine and isolation and study their effect on both the spread and control of the disease. We also study the effect of susceptible and infected bat. The numerical analysis is done using MATLAB ode 45, we also use linearized method of stability using LU decomposition by reducing the matrix into upper triangular matrix. The results of our simulations shows that EVD can be eliminated faster when vaccination is, quarantine and isolation measures are implemented together. We also carried out the local and global stability and we found out that Disease Free Equilibrium is locally asymptotically stable for some stated conditions and globally asymptotically stable.

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CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Humanity has been surrounded with infectious diseases for years. The mechanisms of transmission are known for most diseases; generally, diseases such as influenza, measles, rubella and chicken pox that are transmitted by virus confer immunity against reinfection, while diseases such as tuberculosis, meningitis and gonorrhea that are transmitted by bacteria confer no immunity against reinfection. Other diseases, such as malaria, are transmitted not directly from human to human but by vectors (usually insects), which are agents that are infected by humans and then transmit the disease to other humans. West Nile virus has mosquitoes as its vectors and birds as its hosts. For sexually transmitted diseases with heterosexual transmission, each sex acts as a vector and the disease is transmitted back and forth between the sexes (Althaus, 2014).

Technological advancement has brought about remarkable against these diseases. Anti-retroviral drug have been made available for people living with HIV. In 2010, 7.7 million were able to access anti-retroviral therapy, 17.1 million in 2015 and 20.9 million as of June 2017, which reveals great appreciable progress in combating this virus and invariably control due to technological advances, others are still ravaging lives, the reason being the diversity of the pathogens coupled with their ability to mutate and adapt to changing environments and the complexity of their transmission mechanisms (UNAID, 2017).

The disease transmissions are categorized into two major routes, direct and indirect transmission. Direct transmission involves the transmission from infected people to uninfected people through close contacts. Their medium includes body fluids such as blood, semen, breast milk, etc. or through shaking of hands with or touching an infected individual. In the other hand, indirect transmission involves transmission by non-human infectious agents such as mosquitoes, tsetse flies, contaminated food/ water, which serve as intermediate hosts for the disease and later transmit the disease to humans. The incidence rate of diseases describes the transmission of the disease (Madubueze, Kimbir, Onah & Aboiyar, 2016).

An infectious disease that spreads rapidly to a large number of people in a given population for a short period of time is known as an epidemic. And the infectious disease that

persists in the community or population is known as an endemic disease while, a pandemic is an epidemic of infectious disease that has spread through human populations across a large region (several continents, or even worldwide) (Madubueze, Kimbir, Onah & Aboiyar, 2016).

Scientists have used mathematical models, which involve the use of mathematical equations and formula to represent real life problems, solved and made remarkable prediction based on the solutions obtained from the problems. Epidemiologists (scientists that study infectious diseases) have played a vital role in investigating the transmission dynamics of some of these diseases and have been able come up with recommendations for direct intervention strategies which have helped to control the spread of some of these diseases (Ana, Anton, Ira, Conall, Edmunds & Matthias 2017).

Ebola virus disease (EVD) also known as Ebola *Hemorrhagic Fever* (EHF) is a severe disease of humans and other primates caused by an RNA virus of the family *filoviridae* and genus Ebola virus. The disease has a high fatality rate killing between 25%-90% of infected. Research conducted in 2006 suggested fruit bats to be the likely reservoir due to their ability to spread the virus without being infected by it. The largest outbreak to date took place in West Africa between March 2014 to June 2016, affecting primarily Guinea, Liberia, Sierra Leone, Senegal, and Nigeria. A total of 11296 cases recorded including suspected and probable cases.

The first case of EVD in Nigeria was confirmed in Lagos on 23rd July 2014 'and spread to involve 20 laboratory-confirmed EVD cases, 8 of the confirmed cases of EVD in Nigeria eventually died (case fatality rate of 42.1%) and 12 percent nursed back to good health. In Nigeria, the rapid control of the EVD was facilitated by the rapid detection of the index case, the comprehensive contact tracing measures and the isolation and treatment of the secondary cases. On October 20th, 2014 Nigeria was declared free of EVD (NCDC, 2017).

The outbreak of Ebola virus disease in some West African countries spread to other countries in other continents. This was triggered by the advent of modern means of transportation. Infected individuals were transported from one country to another and therefore fostered the spread of the virus. In July 2014, an infected individual was transported from Liberia to Nigeria and ended up infecting over ten individuals, who also infected several others. Many countries closed their borders as a result of the outbreak; several flights from Africa were canceled as a result. African students were denied admissions into Western world countries

while certain athletes were hindered from participating in competitions they were registered for. All these facts put together motivated me to study the dynamics of Ebola virus disease and to contribute to the body of existing knowledge on it. Ebola is a viral illness of which the initial symptoms can include a sudden fever, intense weakness, muscle pain and a sore throat. However, in Africa, Ebola may be spread as a result of handling bush meat(wild animals hunted for food)and contact with infected bats.

Modified susceptible-exposed-infectious-recovered (SEIR) deterministic nonlinear system of equations will be used to model the dynamics of Ebola virus disease. In addition to infectious individuals, which are known to be the major carriers of infectious diseases, this model incorporated the method of transmission of the disease by deceased (corpse) infectious individuals, since they also contribute to the transmission of the disease to the susceptible population (Rechah & Defim, 2018).

Durojaye and Ajie (2017) formulated an SEIR model that analyse the transmission dynamics of Ebola Virus to see the effect of vaccination on both the spread and control of the disease. Data used by Rechah and Tores (2018) were used to carry out numerical simulation using MATLAB ode 45. The simulations reveal that vaccination is a very efficient factor in reducing the number of infected individuals in a short period of time and increasing the number of recovered individuals. However, they did not consider isolation of infected individuals and reservoir host population dynamics.

1.2 Statement of the Problem

Ebola virus Disease is a severe disease of humans and other primates caused by an RNA virus of the family filoviridae and genus Ebola virus. The disease has a high fatality rate killing between 25%-90% of those infected. Several researches conducted suggested that the virus is highly hazardous hemorrhagic fever. To this time; Ebola Virus is receiving a good attention from health practitioners and Governments. Moreover, mathematical modelers plays a vital role in gaining understanding of the spread of the disease. Durojaye and Ajie (2017) developed a model of the spread and control of ebola virus disease, in which they considered the influence of vaccine and exposed classes but did not consider isolation of infected human and reservoir host population dynamics. In this work we intend to incorporate isolation of infected humans' compartments and also consider the reservoir host population dynamics.

1.3 Aim and Objectives of the Study

The aim of this study is to modify the model due to Durojaye and Ajie (2017) by incorporating vaccination, quarantine and isolation and the disease reservoir host population. The objectives of this research include;

i. establish the disease free equilibrium (DFE) state of the model
ii. obtain reproduction number of the disease
iii. analyse both local and global stability of the DFE state of the model
iv. Carryout numerical experiments using MATLAB R2015a.

1.4 Significance of the Study

Government, Non-Governmental Organizations, research institutions, public health workers and the general public will find this research useful in setting out policies and to sensitize people on the danger of the disease.

1.5 Scope of the Study

This research will be restricted to the effect of interaction and the transmission dynamic of EVD between bats and human. A system of non-linear ordinary differential equations will be formed and method of linearization will be used to study the stability of the Disease Free Equilibrium Points of the model followed by numerical simulation.

1.6 Operational Definition of Terms

1.6.1 ***Susceptible individuals*****:** Are individuals who can be infected but have not yet contacted with Ebola virus.

1.6.2 ***Exposed individuals:*** Are individuals who have contacted Ebola virus but not yet infectious.

1.6.3 ***Infectious individuals*****:** Are individuals that have contacted Ebola virus/ and can transmit the disease to a susceptible individual.

1.6.4 ***Recovered individuals*****:** Are individuals who have recovered from Ebola.

1.6.5 ***Quarantine individuals*****:** are exposed individuals that are isolated before becoming infectious.

***1.6.6 Deceased individual*:** Are those individual that died as a result of Ebola.

***1.6.7 Vaccinated individuals*:** Are those individual that are vaccinated

1.6.8 Isolated individuals: *Are those individuals that are isolated for the purpose of receiving treatment.*

***1.6.9 Susceptible bat*:** Are those bats that are likely to be infected

***1.6.10 Infected bat*:** Are those bats that are infected and infectious

CHAPTER TWO

LITERITURE REVIEW

In this section we review other authors' work with regard to *ebola* virus. Mathematical modeling has an important tool for understanding dynamics of many infectious diseases one of which is ebola virus.

Chinwendu, Anande and Terhemen (2018) studied a deterministic model of ebola virus disease (EVD) in cooperating contact tracing and quarantine as interactions. Their model analysed the existence and stability of the disease free equilibrium (DFE) and endemic equilibrium (EE) states. They obtain both local and global stabilities for their models. However they carried out numerical simulations to examine the impact of contact tracing and quarantine measures on the transmission dynamics of EVD. The result of their simulations shows that EVD could be eliminated faster when contact tracing and quarantine measures are implemented together.

Madubueze, Kimbir, Onah and Aboiyar (2016) formulated a modified Ebola SEIR model, incorporating contact tracing and quarantine as control measures, existence and uniqueness of the solution of the model were tested in a feasible region Ω, using Lipschitz condition and was proven that the a unique solution of the model exist in Ω.

Lise, Edouard, Chloe, Laura and Rodolphe (2018) in their work ebola vaccine development: systematic review of pre-clinical and clinical studies and meta-analysis of determinants of anti-body response variability after vaccination studied ebola vaccine and to assess factors associated with anti-body response variability in humans. The used PubMed and Scopus to search for electronic databases and established that anti-body titer was slightly higher in NHP confidence internal than in humans. Also vaccine platform and rival strain youth for anti-body detection were associated with anti-body response in humans, but adjusted heterogeneity remain at 95%.

Hussain, Ali, Chaudary, Jerral, Yasmeen and Chaudary (2017) studied the numerical solution of mathematical model of ebola virus. In their work they presented a model for the transmission of dynamics of ebola virus, for this purpose they reviewed the SIR model. They also used numerical techniques like Euler, RK -2 and RK -4 to analyse their model.

Rechah and Defim (2018) studied the dynamics of ebola virus with isolation. They presented a deterministic compartmental model for assessing the impact of isolation to contain the virus. Their model includes Demographic effect, the latent undetectable and latent detectable compartment with isolations of infectious individual. They carried out numerical simulation using ACADO solver which shows that the isolation of latent detectable and infectious individuals is the most effective in curtailing the virus.

Tae and Young (2015) developed a mathematical model to better understand the spread of ebola virus, the dynamic of the disease and preventive behaviors. Their model suggests that "safe burial practice should be provided with no traditional rituals to be followed. The model can also predict the total number of infected cases, number of deaths and duration of outbreaks among others. They computed the reproduction number R_0 using next generation matrix and found that the diseace will spread if $R_0>0$.

Danny and Robert (2016) presented an SEIR compartment model of ebola with a five compartment for infectious disease to model the dynamics of an ebola virus outbreaks. They analyzed the stability of the disease free equilibrium using Ruth Hurwitz stability criteria and also formulated an equation for the eradication threshold R_0. They also carry out sensitivity analysis of the disease free equilibrium which points out the directions of the transmission possibility and the contact rate with infectious individuals as target for interaction. They found that at the disease free equilibrium, R_0 is most influenced by the transmission probability, the amount of contact with the dead and the contact rate between the infected individuals and the susceptible individuals.

Deepa, Nallamalli, Singh and Teja (2015) work on mathematical model for transmission of ebola. In their work, they presented a model for better understanding and awareness of the disease. They used SEIR model. The model with respect to various parameters has been developed and was found that, as rate of immunity is increased the number of infected individuals decrease.

El-Rhoubari, Besbassi, Khalid, and Noura (2018) study the Mathematical Modeling of Ebola Virus Disease in Bat Population. In their work they develop a generalized epizootic model for the transmission dynamics of Ebola virus disease (EVD) in bat population by taking into account the environment contamination. The transmission process is modeled by two general incidence functions that include many incidence rates used in infectious diseases modeling. They first proved that the model is epidemiologically and mathematically well-posed by showing the existence, positivity, and roundedness of solutions by analyzing the characteristic equations and constructing suitable Lyapunov functional, the stability analysis of equilibrium is investigated.

Rechah and Defim (2018) worked on analysis, simulation and optimal control of a SEIR model for ebola virus with demographic effects. They presented a mathematical description of different Susceptible–Exposed–Infectious– Recovered (SEIR) models, by using mathematical modeling and analysis, the outbreak of Ebola virus in West Africa was described. Their aim was to study and discuss the properties of SEIR models with respect to Ebola virus, the information they provided, and when the models make sense. They added to the basic SEIR model demographic effects in order to analyze the disease free equilibrium with vital dynamics using next generation method and found the DFE to be stable if $R_0<1$ and vice versa. Numerical simulations confirm that the propagation of the virus can be controlled through vaccination, reducing the number of infected individuals.

Amenaghawon and Aboubakary (2015) worked on the mathematical modeling of the transmission dynamics of ebola virus. In their work they developed system of ordinary differential equation for the transmission and linearization method of stability analysis was used to solve the equations and found the disease free equilibrium to be always unstable. However they carried out numerical simulations with the model's parameters, which show that with ebola uncontrolled transmittable contact between the infected and susceptible individuals can lead to a serious outbreak with high mortality rate. Furthermore, their model considered the susceptible individuals, infected and infectious individuals, and quarantine individuals, deceased individuals due to the ebola virus and the recovered individuals from the virus. Here natural deaths, vaccinated and exposed individuals were not considered in their work.

Durojaye and Ajie (2017) formulated an SEIR model that analyse the transmission dynamics of Ebola Virus to see the effect of vaccination on both the spread and control of the

disease. Data used by Rechah and Tores (2018) were used to carry out numerical simulation using MATLAB ode 45. The simulations reveal that vaccination is a very efficient factor in reducing the number of infected individuals in a short period of time and increasing the number of recovered individuals.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

This section presents the existing and modified models and discusses the methods used in studying the modified model.

3.1 Assumptions of the Existing Model

i. the total population assumed constant during the short period of time

ii. the susceptible individuals are vaccinated at rate ν

iii. the susceptible become exposed to the disease before being infectious

Table 3.1Variables and parameters of the existing model

Variables	Description
$S(t)$	the number of Susceptible individuals at time t.
E(t)	the number Exposed individuals at time t.
$I(t)$	the number of Infected individuals at time t.
$R(t)$	the number of Recovered individuals at time t.
β	Is the progression rate from susceptible to exposed population
γ	The progression rate from exposed to infected population
μ	the progression rate from infected to recovered population
υ	the vaccination rate

3.2 Existing Model Diagram

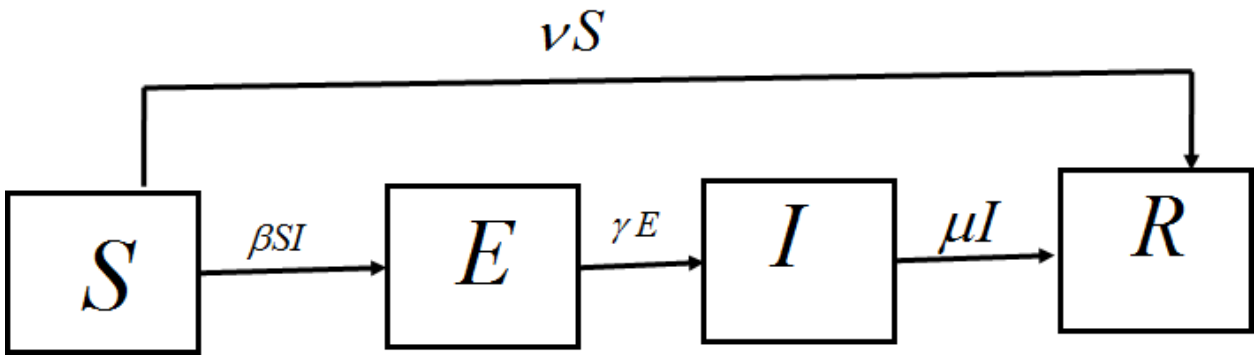

Fig. 3.2 Existing Model Diagram

3.3 Existing Model Equation

$$\frac{dS(t)}{dt} = -\beta S(t)I(t) - \nu S(t) \tag{3.1}$$

$$\frac{dE(t)}{dt} = \beta SI(t) - \gamma E(t) \tag{3.2}$$

$$\frac{dI(t)}{dt} = \gamma E(t) - \mu I(t) \tag{3.3}$$

$$\frac{dR(t)}{dt} = \mu I(t) + \nu S(t) \tag{3.4}$$

3.4 Assumptions of the Proposed Model

i. The susceptible individuals become exposed to the virus before becoming infected
ii. The susceptible individuals are vaccinated at a rate θ and they remain immune to the disease for the period of vaccine protection
iii. The vaccinated individuals return back to the susceptible compartment at a rate ψ when the vaccine wanes out.
iv. Isolated individuals do not transmit the virus
v. The disease does not confer immunity on its victims.

Table 3.2: Variables and parameters of the modified model

Variables	Description
$S(t)$	the number of Susceptible individuals at time t.
E(t)	the number of exposed individuals at time t.
D(t)	the number deceased individuals at time t.
$I(t)$	the number of Infected individuals at time t.
$R(t)$	the number of Recovered individuals at time t.
Q(t)	the number of quarantine individuals at time t
V(t)	the number vaccinated individuals at time t
$S_b(t)$	the number of susceptible bats at time t
$I_b(t)$	the number of susceptible bats at time t
ϕ	the birth rate of the population
μ	the natural mortality rate
α	the rate at which the susceptible be at the exposed population
τ	the progression rate form exposed to infected population
ω	the progression rate from infected to quarantine population
β	the progression rate from quarantined to the recovered populion
θ	the progression rate from susceptible to the vaccination population
ψ	the progression rate from vaccination to susceptible population
δ	the disease induced rate
η	the rate at which the susceptible bats become infected
χ	the rate at which the infected bats dies as a result of ebola
κ	the rate at which individuals from exposed move to the infected
ε	the rate at which individuals from quarantine move back to the susceptible

3.5 Diagram of the Proposed Model

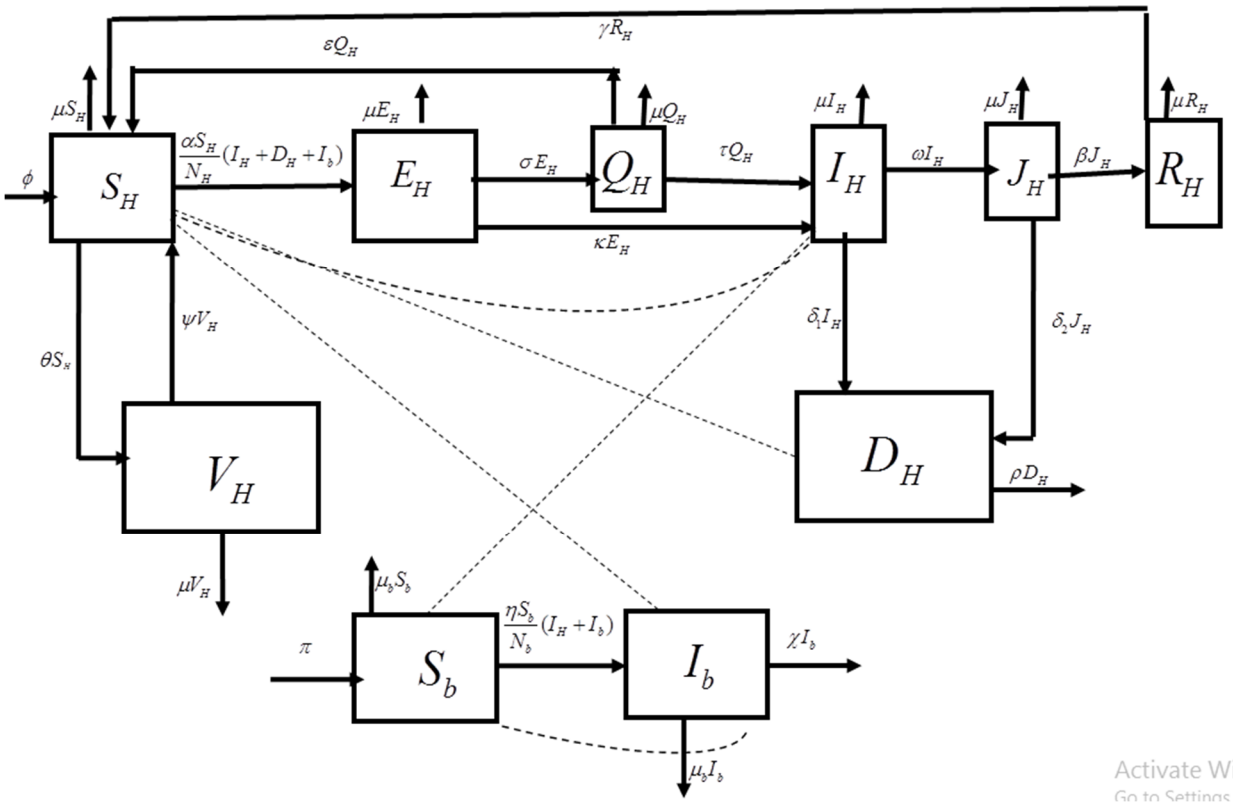

Figure 3.2 Diagram of the proposed model

3.6 Equations of the Proposed Model

$$\frac{dS_H}{dt} = \phi - \frac{\alpha S_H}{N_H}(I_H + D_H + I_b) + \psi V_H + \varepsilon Q_H - (\theta + \mu)S_H + \gamma R_H \tag{3.6}$$

$$\frac{dV_H}{dt} = \theta S_H - (\psi + \mu)V_H \tag{3.7}$$

$$\frac{dE_H}{dt} = \frac{\alpha S_H}{N_H}(I_H + D_H + I_b) - (\sigma + \mu - \kappa)E_H \tag{3.8}$$

$$\frac{dQ_H}{dt} = \sigma E_H - (\tau + \mu - \varepsilon)Q_H \tag{3.9}$$

$$\frac{dI_H}{dt} = \tau Q_H + \kappa E_H - (\delta_1 + \omega + \mu)I_H \tag{3.10}$$

$$\frac{dJ_H}{dt} = \omega I_H - (\beta + \mu + \delta_2) J_H \tag{3.11}$$

$$\frac{dR_H}{dt} = \beta J_H - (\gamma + \mu) R_H \tag{3.12}$$

$$\frac{dD_H}{dt} = \delta_1 I_H + \delta_2 J_H - \rho D_H \tag{3.13}$$

$$\frac{dS_b}{dt} = \pi - \frac{\eta S_b}{N_b}(I_H + I_b) - \mu_b S_b \tag{3.14}$$

$$\frac{dI_b}{dt} = \frac{\eta S_b}{N_b}(I_H + I_b) - \chi I_b - \mu_b I_b \tag{3.15}$$

$$N_H(t) = S_H(t) + E_H(t) + I_H(t) + J_H(t) + Q_H(t) + R_H(t) + V_H(t) + D_H(t) \tag{3.16}$$

$$N_b(t) = S_b(t) + I_b(t) \tag{3.17}$$

3.7 Method of Model Analysis

In this section, we present methods of model analysis.

3.7.1 Equilibrium states

The Equilibrium state for the model is obtained by setting the model equations to be zero i.e. $\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = \frac{dV}{dt} = \frac{dJ}{dt} = \frac{dD}{dt} = \frac{dS_b}{dt} = \frac{dI_b}{dt} = 0$

3.7.2 Basic reproduction number

Diekmann, and Heesterbeek (2000) defined the basic reproduction number, R_0 ,as the average number of secondary infections caused by an infectious individual during his/her entire life as an infectious person. In this model, we shall adopt the method of the Next Generation Matrix to compute our reproduction number. We call FV^{-1}, the next generation matrix for the model and set the reproduction number $R_0 = \rho(FV^{-1})$

where $F = \left[\frac{\partial F_i(x_0)}{\partial x_j} \right]$ and $V = \left[\frac{\partial V_i(x_0)}{\partial x_j} \right]$ for i ≥ 1 for the number of compartments and $1 \leq j \leq m$ for the infected compartments only. $\rho(FV^{-1})$ denotes the spectral radius of the matrix A.F and V are m×m matrices, where m is the number of infected classes (Diekmann & Heesterbeek, 2000).

3.7.3 Stability analysis

We obtained the steady state of the model and carryout local and global stability of the disease free equilibrium.

3.7.4 Numerical solutions

In this study, we used MATLAB R2015a inbuilt scheme of ode 45 as the scientific package for all numerical solutions.

3.7.5 Source of data

Theoretical data of certified literature is used.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.0 Introduction

In this chapter, we present the analytical study of our work; we established the existence and uniqueness of the solution, positivity of the solution, invariant region, and existence of disease free equilibrium (DFE) points and obtained the basic reproduction number of the disease. We also analyzed both local and global stability of the disease free equilibrium point of the model and carry out numerical simulation using MATLAB R2015a.

4.1 Analytic result

4.1.1 The Existence and the uniqueness of the model solution

The validity and authenticity of any mathematical model depends on whether the given system of equation has a solution and if the solution exists is unique. We used the Lipchitz condition to verify the existence and uniqueness of the system equation given by (3.7-3.16).

Theorem 4.1

Consider the system of equation below

$$\left.\begin{aligned} x_1^1 &= f_1(t, x_1, x_2, ...x_n), x_1(t_0) = x_{10} \\ x_2^1 &= f_1(t, x_1, x_2, ...x_n), x_2(t_0) = x_{20} \\ &. \\ &. \\ &. \\ x_n^1 &= f_n(t, x_1, x_2, ...x_n), x_n(t_0) = x_{n0} \end{aligned}\right\} \tag{4.1}$$

We may write equation (4.1) in compact form as

$$x^1 = f_1(t, x), x_1(t_0) = x_0 \tag{4.2}$$

Theorem: 4.2

Let D denote the region

$$|t - t_o| \le a, \|x - x_o\| \le b, \;\; x = (x_1, x_2, ..., x_n),\; x_o = (x_{10}, x_{20}, ..., x_{n0}) \tag{4.3}$$

And suppose that $F(t,x)$ satisfies the Lipchitz condition

$$\|f(t,x_1)-f(t,x_2\|\le k\|x_1-x_2\| \tag{4.4}$$

Whenever the pairs (t,x_1) and (t,x_2) belong to D^1, where K is a positive constant. Then there is a constant $\delta>0$ such that there exist a unique continues vector solution of $\underline{X}(t)$ of system (4.2) in the interval $|t-t_o|\le\delta.$ It is important to note that the condition (4.4) is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}, i, j=1,2,...$ be continues and bounded in D^1.

Theorem 4.3

Let D denote the region defined in $1\le\varepsilon\le R,$ such that $1\le\varepsilon\le R,$ and $0<R<\infty,$ hold then there exist a unique solution of equation (3.7) to (3.16) of the model equations which is bounded in the region D.

Proof

Let

$$F_1=\phi-\frac{\alpha S_H}{N_H}(I_H+D_H+I_b)+\psi V_H+\varepsilon Q_H-(\theta+\mu)S_H+\gamma R_H \tag{4.5}$$

$$F_2=\theta S_H-(\psi+\mu)V_H \tag{4.6}$$

$$F_3=\frac{\alpha S_H}{N_H}(I_H+D_H+I_b)-(\sigma+\mu+\kappa)E_H \tag{4.7}$$

$$F_4=\sigma E_H-(\tau+\varepsilon+\mu)Q_H \tag{4.8}$$

$$F_5=\kappa E_H+\tau Q_H-(\delta_1+\omega+\mu)I_H \tag{4.9}$$

$$F_6=\omega I_H-(\beta+\mu+\delta_2)J_H \tag{4.10}$$

$$F_7=\beta J_H-(\gamma+\mu)R_H \tag{4.11}$$

$$F_8=\delta_1 I_H+\delta_2 J_H-\rho D_H \tag{4.12}$$

$$F_9 = \pi - \frac{\eta S_b}{N_b}(I_H + I_b) - \mu_b S_b \tag{4.13}$$

$$F_{10} = \frac{\eta S_b}{N_b}(I_H + I_b) - \chi I_b - \mu_b I_b \tag{4.14}$$

It is sufficient to show that $\frac{\partial f_i}{\partial x_j}, i, j = 1,2,3,...,10$ are continuous.

Let $S_H = x_1, V_H = x_2, E_H = x_3, G_H = x_4, I_H = x_5, J_H = x_6, R_H = x_7, D_H = x_8, S_b = x_9 and I_b = x_{10}$

Also let $\frac{\alpha}{N_H}(I_H + D_H + I_b) = \varpi, \frac{\eta}{N_b}(I_H + I_b) = \xi$

Consider the partial derivatives below

From equation (4.5)

$$F_1 = \phi - \varpi S_H + \psi V_H + \varepsilon Q_H - (\theta + \mu) S_H + \gamma R_H$$

$$\left|\frac{\partial F_1}{\partial x_1}\right| = \left|-(\varpi + \theta + \mu)\right| < \infty$$

$$\left|\frac{\partial F_1}{\partial x_3}\right| = \left|\frac{\partial F_1}{\partial x_4}\right| = \left|\frac{\partial F_1}{\partial x_5}\right| = \left|\frac{\partial F_1}{\partial x_6}\right| = \left|\frac{\partial F_1}{\partial x_8}\right| = \left|\frac{\partial F_1}{\partial x_9}\right| = \left|\frac{\partial F_1}{\partial x_{10}}\right| = 0 < \infty$$

$$\left|\frac{\partial F_1}{\partial x_2}\right| = \left|\psi\right| < \infty, \left|\frac{\partial F_1}{\partial x_7}\right| = \left|\gamma\right| < \infty$$

From equation (4.6)

$$F_2 = \theta S_H - (\psi + \mu) V_H$$

$$\left|\frac{\partial F_2}{\partial x_1}\right| = \left|\theta\right| < \infty$$

$$\left|\frac{\partial F_2}{\partial x_2}\right| = \left|-(\psi+\mu)\right| < \infty$$

$$\left|\frac{\partial F_2}{\partial x_3}\right| = \left|\frac{\partial F_2}{\partial x_4}\right| = \left|\frac{\partial F_2}{\partial x_5}\right| = \left|\frac{\partial F_2}{\partial x_6}\right| = \left|\frac{\partial F_2}{\partial x_7}\right| = \left|\frac{\partial F_2}{\partial x_8}\right| = \left|\frac{\partial F_2}{\partial x_9}\right| = \left|\frac{\partial F_2}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.7)

$$F_3 = \varpi S_H - (\sigma+\mu+\kappa)E_H$$

$$\left|\frac{\partial F_3}{\partial x_1}\right| = \left|\varpi\right| < \infty$$

$$\left|\frac{\partial F_3}{\partial x_3}\right| = \left|-(\sigma+\mu+\kappa)\right| < \infty$$

$$\left|\frac{\partial F_3}{\partial x_2}\right| = \left|\frac{\partial F_3}{\partial x_4}\right| = \left|\frac{\partial F_3}{\partial x_5}\right| = \left|\frac{\partial F_3}{\partial x_6}\right| = \left|\frac{\partial F_3}{\partial x_7}\right| = \left|\frac{\partial F_3}{\partial x_8}\right| = \left|\frac{\partial F_3}{\partial x_9}\right| = \left|\frac{\partial F_3}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.8)

$$F_4 = \sigma E_H - (\tau+\varepsilon+\mu)Q_H$$

$$\left|\frac{\partial F_4}{\partial x_3}\right| = \left|\sigma\right| < \infty$$

$$\left|\frac{\partial F_4}{\partial x_4}\right| = \left|-(\tau+\mu+\varepsilon)\right| < \infty$$

$$\left|\frac{\partial F_4}{\partial x_1}\right| = \left|\frac{\partial F_4}{\partial x_2}\right| = \left|\frac{\partial F_4}{\partial x_5}\right| = \left|\frac{\partial F_4}{\partial x_6}\right| = \left|\frac{\partial F_4}{\partial x_7}\right| = \left|\frac{\partial F_4}{\partial x_8}\right| = \left|\frac{\partial F_4}{\partial x_9}\right| = \left|\frac{\partial F_4}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.9)

$$F_5 = \kappa E_H + \tau Q_H - (\delta_1+\omega+\mu)I_H$$

$$\left|\frac{\partial F_5}{\partial x_3}\right| = |\kappa| < \infty$$

$$\left|\frac{\partial F_5}{\partial x_4}\right| = |\tau| < \infty$$

$$\left|\frac{\partial F_5}{\partial x_5}\right| = |-(\delta_1 + \omega + \mu)| < \infty$$

$$\left|\frac{\partial F_5}{\partial x_1}\right| = \left|\frac{\partial F_5}{\partial x_2}\right| = \left|\frac{\partial F_5}{\partial x_4}\right| = \left|\frac{\partial F_5}{\partial x_7}\right| = \left|\frac{\partial F_5}{\partial x_8}\right| = \left|\frac{\partial F_5}{\partial x_9}\right| = \left|\frac{\partial F_5}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.10)

$$F_6 = \omega I_H - (\beta + \mu + \delta_2) J_H$$

$$\left|\frac{\partial F_6}{\partial x_5}\right| = |\omega| < \infty \quad , \left|\frac{\partial F_6}{\partial x_6}\right| = |-(\beta + \mu + \delta_2)| < \infty \quad ,$$

$$\left|\frac{\partial F_6}{\partial x_1}\right| = \left|\frac{\partial F_6}{\partial x_2}\right| = \left|\frac{\partial F_6}{\partial x_3}\right| = \left|\frac{\partial F_6}{\partial x_4}\right| = \left|\frac{\partial F_6}{\partial x_7}\right| = \left|\frac{\partial F_6}{\partial x_8}\right| = \left|\frac{\partial F_6}{\partial x_9}\right| = \left|\frac{\partial F_6}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.11)

$$F_7 = \beta J_H - (\gamma + \mu) R_H$$

$$\left|\frac{\partial F_7}{\partial x_6}\right| = |\beta| < \infty$$

$$\left|\frac{\partial F_7}{\partial x_7}\right| = |-(\gamma + \mu)| < \infty$$

$$\left|\frac{\partial F_7}{\partial x_1}\right| = \left|\frac{\partial F_7}{\partial x_2}\right| = \left|\frac{\partial F_7}{\partial x_3}\right| = \left|\frac{\partial F_7}{\partial x_4}\right| = \left|\frac{\partial F_7}{\partial x_5}\right| = \left|\frac{\partial F_7}{\partial x_8}\right| = \left|\frac{\partial F_7}{\partial x_9}\right| = \left|\frac{\partial F_7}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.12)

$$F_8 = \delta_1 I_H + \delta_2 J_H - \rho D_H$$

$$\left|\frac{\partial F_8}{\partial x_5}\right| = \left|\delta_1\right| < \infty$$

$$\left|\frac{\partial F_8}{\partial x_6}\right| = \left|\delta_2\right| < \infty$$

$$\left|\frac{\partial F_8}{\partial x_8}\right| = \left|-\rho\right| < \infty$$

From equation (4.13)

$$F_9 = \pi - \xi S_b - \mu_b S_b$$

$$\left|\frac{\partial F_9}{\partial x_9}\right| = \left|-(\xi + \mu_b)\right| < \infty$$

$$\left|\frac{\partial F_9}{\partial x_1}\right| = \left|\frac{\partial F_9}{\partial x_2}\right| = \left|\frac{\partial F_9}{\partial x_3}\right| = \left|\frac{\partial F_9}{\partial x_4}\right| = \left|\frac{\partial F_9}{\partial x_5}\right| = \left|\frac{\partial F_9}{\partial x_6}\right| = \left|\frac{\partial F_9}{\partial x_7}\right| = \left|\frac{\partial F_9}{\partial x_8}\right| = \left|\frac{\partial F_9}{\partial x_{10}}\right| = 0 < \infty$$

From equation (4.14)

$$F_{10} = \xi S_b - \chi I_b - \mu_b I_b$$

$$\left|\frac{\partial F_{10}}{\partial x_1}\right| = \left|\frac{\partial F_{10}}{\partial x_2}\right| = \left|\frac{\partial F_{10}}{\partial x_3}\right| = \left|\frac{\partial F_{10}}{\partial x_4}\right| = \left|\frac{\partial F_{10}}{\partial x_5}\right| = \left|\frac{\partial F_{10}}{\partial x_6}\right| = \left|\frac{\partial F_{10}}{\partial x_7}\right| = \left|\frac{\partial F_{10}}{\partial x_8}\right| = 0 < \infty$$

$$\left|\frac{\partial F_{10}}{\partial x_9}\right| = \left|\xi\right| < \infty$$

$$\left|\frac{\partial F_{10}}{\partial x_8}\right| = \left|-(\chi + \mu_b)\right| < \infty$$

Clearly all these partial derivatives are continuous and bounded. Hence by Theorem 2, there exists a unique solution of equation (3.7)-(3.16) in the region D.

4.1.2 Invariant region

Lemma 4.1.3

Let $(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b)$ be the solution of the model equations (3.7) to (3.16) with the initial conditions and biological feasible region given by the set $\Omega=\Omega_H \cup \Omega_b$ where

$$\Omega_H = \left\{ (S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H) \in R_+^8 : N_H \leq \frac{\phi}{\mu} \right\} \tag{4.15}$$

$$\Omega_b = \left\{ (S_b, I_b,) \in R_+^2 : N_b \leq \frac{\pi}{\mu_b} \right\} \tag{4.16}$$

Proof

Adding the equation (3.7) – (3.14)

$$N_H(t) = S_H(t) + V_H(t) + E_H(t) + Q_H(t) + I_H(t) + J_H(t) + R_H(t) + D_H(t)$$

$$\frac{dN_H}{dt} = \frac{dS_H}{dt} + \frac{dV_H}{dt} + \frac{dE_H}{dt} + \frac{dQ_H}{dt} + \frac{dI_H}{dt} + \frac{dJ_H}{dt} + \frac{dR_H}{dt} + \frac{dD_H}{dt} \tag{4.17}$$

$$\Rightarrow \frac{dN_H}{dt} = \phi - \frac{\alpha(I_H + D_H + I_b)S_H}{N_H} + \psi V_H + \varepsilon Q_H + \gamma R_H - (\theta + \mu)S_H + \theta S_H - (\psi + \mu)V_H$$

$$+ \frac{\alpha(I_H + D_H + I_b)S_H}{N_H} - (\sigma + \kappa + \mu)E_H + \sigma E_H - (\tau + \varepsilon + \mu)Q_H + \tau Q_H - (\delta_1 + \omega + \mu)I_H$$

$$+ \kappa E_H + \omega - (\beta + \delta_2 + \mu)J_H + \beta J_H - (\gamma + \mu)R_H + \delta_1 I_H + \delta_2 J_H - \rho D_H$$

$$\frac{dN_H}{dt} = \phi - \mu(S_H + V_H + E_H + Q_H + I_H + J_H + R_H) - \rho D_H$$

$$\frac{dN_H}{dt} = \phi - \mu N_{H-} - \rho D_H$$

$$\frac{dN_H}{dt} \leq \phi - \mu N_H$$

$$\frac{dN_H}{dt} + \mu N_H \leq \phi \tag{4.18}$$

Let integrate (4.18) using integrating factor of the variable

$$N_H . IF \leq \phi \int IF$$

$$IF = \ell^{\int pdt} = \ell^{\int \mu dt} = \ell^{\mu t}$$

$$N_H \ell^{\mu t} \le \phi \int \ell^{\mu t}$$

$$N_H \ell^{\mu t} \le \frac{\phi \ell^{\mu t}}{\mu} + C \tag{4.19}$$

Dividing both sides of (4.19) by $\ell^{\mu t}$

$$N_H \le \left(\frac{\phi \ell^{\mu t}}{\mu} + C \right) \ell^{-\mu t}$$

$$N_H(t) \le \frac{\phi}{\mu} + C\ell^{-\mu t} \tag{4.20}$$

At t=0

$$N_H(0) \le \frac{\phi}{\mu} + C$$

$$N_H(0) - \frac{\phi}{\mu} \le C \tag{4.21}$$

Substituting C into (4.20)

$$N_H(t) \le \frac{\phi}{\mu} + \left(N_H(0) - \frac{\phi}{\mu} \right) \ell^{-\mu t}$$

At $t \to \infty$, $N_H(t) \to \frac{\phi}{\mu}$

$$N_b(t) = S_b(t) + I_b(t)$$

Adding eaquation 3.15 and 3.16, we have,

$$\frac{dN_b}{dt} = \frac{S_b}{dt} + \frac{I_b}{dt} \tag{4.22}$$

$$\frac{dN_b}{dt} = \pi - \frac{\eta(I_H + I_b)S_b}{N_b} - \mu_b S_b + \frac{\eta(I_H + I_b)S_b}{N_b} - (\chi + \mu)I_b$$

$$\frac{dN_b}{dt} = \pi - \mu_b S_b - (\chi + \mu)I_b$$

$$\frac{dN_b}{dt} = \pi - \mu_b S_b - \chi I_b - \mu I_b$$

$$\frac{dN_b}{dt} = \pi - \mu_b (S_b - I_b) - \chi I_b$$

$$\frac{dN_b}{dt} \leq \pi - \mu_b (S_b - I_b)$$

$$\frac{dN_b}{dt} \leq \pi - \mu_b N_b$$

$$\frac{dN_b}{dt} + \mu N_b \leq \pi \tag{4.23}$$

Let integrate (4.23) using integrating factor of the variable

$$N_b.IF \leq \pi \int IF$$

$$IF = \ell^{\int pdt} = \ell^{\int \mu_b dt} = \ell^{\mu_b t}$$

$$N_b \ell^{\mu_b t} \leq \pi \int \ell^{\mu_b t}$$

$$N_b \ell^{\mu_b t} \leq \frac{\pi \ell^{\mu_b t}}{\mu_b} + C \tag{4.24}$$

Dividing both sides of (4.24) by $\ell^{\mu_b t}$

$$N_b \leq \left(\frac{\pi \ell^{\mu_b t}}{\mu_b} + C \right) \ell^{-\mu_b t}$$

$$N_b(t) \leq \frac{\pi}{\mu_b} + C\ell^{-\mu_b t} \tag{4.25}$$

At t=0

$$N_b(0) \le \frac{\pi}{\mu_b} + C$$

$$N_b(0) - \frac{\pi}{\mu_b} \le C \tag{4.26}$$

Substituting C into (4.25)

$$N_b(t) \le \frac{\pi}{\mu_b} + \left(N_b(0) - \frac{\pi}{\mu_b} \right) \ell^{-\mu_b t} \tag{4.27}$$

At $t \to \infty,\ N_b(t) \to \frac{\pi}{\mu_b}$

Therefore, the basic model is well posed epidemiologically and mathematically. Hence it is sufficient to study the dynamics of the basic model.

4.1.3 Positivity of solution of the model

Since the model given by (3.7 to 3.16) monitors human and bat population, it is significant to show that all the state variables in the model is non negative for all time.

Theorem 3: For non negative initial conditions of the model equations given by (3.7 to 3.16), the solution $(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b)$ of the model equations (3.7 to 3.16) are all non negative for all time $t \ge 0$

Proof:

From equation (3.6) we have

$$\frac{dS_H}{dt} = \phi - \lambda_H S_H \psi V_H + \varepsilon Q_H - K_1 S_H + \gamma R_H$$

$$Where\ \lambda_H = \frac{\alpha}{N_H}(I_H + D_H + I_b) \quad \text{and } K_1 = (\theta + \mu)$$

$$\frac{dS_H}{dt} \geq -(\lambda_H + K_1)S_H \tag{4.28}$$

Integrating (4.28) using separation of variables

$$\int \frac{dS_H}{S_H} \geq -\int (\lambda_H + K_1)dt$$

$$\ln S_H \geq -(\lambda_H + K_1)t + C \tag{4.29}$$

$$S_H(t) > A_1 \ell^{-(\lambda_H + K_1)t} \tag{4.30}$$

At $t = 0$

$$S_H(0) = A_1$$

Substitute $S_H(0) = A_1$ in (4.30)

$$S_H(t) \geq S_H(0)\ell^{-(\lambda_H + K_1)t}$$

Since $S_H(0) > 0$

$$S_H(t) \geq 0$$

From equation (3.7)

$$\frac{dV_H}{dt} = \theta S_H - K_2 V_H$$

Where $\mathrm{K}_2 = (\psi + \mu)$

$$\frac{dV_H}{dt} \geq -K_2 V_H \tag{4.31}$$

By integrating using separation of the variables

$$\int \frac{dV_H}{V_H} \geq -\int K_2 dt$$

$$\ln V_H \geq -K_2 t + C$$

$$V_H = \ell^{-K_2 t + C}$$

$$V_H(t) = A_2 \ell^{-K_2 t} \tag{4.32}$$

At $t = 0$

$$V_H(0) = A_2$$

Substitute $V_H(0) = A_2$ in (4.32)

$$V_H(t) = V_H(0) \ell^{-K_2 t}$$

Since $V_H(0) > 0$

$$V_H(t) \geq 0$$

From equation (3.8)

$$\frac{dE_H}{dt} = \lambda_H S_H - K_3 E_H$$

Where $\mathrm{K}_3 = (\sigma + \mu - \kappa)$

$$\frac{dE_H}{dt} \geq -K_3 E_H \tag{4.33}$$

Integrating (4.33) using separation of variables

$$\int \frac{dE_H}{E_H} \geq -\int K_3 dt$$

$$\ln E_H \geq -K_3 t + C$$

$$E_H \geq \ell^{-K_3 t + C}$$

$$E_H(t) \geq A_3 \ell^{-K_3 t} \tag{4.34}$$

Applying the initial condition

At $t = 0$

$E_H(0) = A_3$

Substitute $E_H(0) = A_2$ in (4.34)

$E_H(t) > E_H(0)\ell^{-K_3 t}$

Since $E_H(0) > 0$

$E_H(t) \geq 0$ for $t > 0$

From equation (3.9)

$$\frac{dQ_H}{dt} = \sigma E_H - K_4 Q_H$$

$where\ K_4 = (\tau + \mu - \varepsilon)$

$$\therefore \frac{dQ_H}{dt} \geq -K_4 Q_H \tag{4.35}$$

Integrating (4.35)

$$\int \frac{1}{Q_H} dQ_H \geq -\int K_4 dt$$

$$\ln Q_H \geq -K_4 t + C$$

$$Q_H = \ell^{-K4t+C}$$

$$Q_H = A_4 \ell^{-K_4 t} \tag{4.36}$$

Applying the initial condition

At $t = 0$

$Q_H(0) = A_4$

Substitute $Q_H(0) = A_4$ in (4.36)

$Q_H(t) > Q_H(0)\ell^{-K_5 t}$

Since $Q_H(0) > 0$

$Q_H(t) \geq 0$ for $t > 0$

From equation (3.10)

$$\frac{dI_H}{dt} = \tau Q_H - K_5 I_H$$

where $\mathrm{K}_5 = (\omega + \mu + \delta_1)$

$$\therefore \frac{dI_H}{dt} \geq -K_5 I_H \qquad (4.37)$$

Integrating (4.36)

$$\int \frac{1}{I_H} dI_H \geq -\int K_5 dt$$

$$\ln I_H \geq -K_S t + C$$

$$I_H \geq \ell^{-K5t+C}$$

$$I_H \geq A_5 \ell^{-K_5 t} \qquad (4.38)$$

At $t = 0$

$I_H(0) = A_5$

Substitute $I_H(0) = A_5$ in (4.38)

$I_H(t) > I_H(0)\ell^{-K_5 t}$

Since $I_H(0) > 0$

$I_H(t) \geq 0$ for $t > 0$

From equation (3.11)

$$\frac{dJ_H}{dt} = \omega I_H - K_6 J_H$$

where $\mathrm{K}_6 = (\beta + \mu + \delta_2)$

$$\frac{dJ_H}{dt} \geq -K_6 J_H \qquad (4.39)$$

Integrating (4.39)

$$\int \frac{1}{J_H} dJ_H \geq -\int K_6 dt$$

$$\ln J_H \geq -K_6 t + C$$

$$J_H \geq \ell^{-K_6 t + C}$$

$$J_H(t) \geq A_6 \ell^{-K_6 t} \qquad (4.40)$$

At $t = 0$

$$J_H(0) = A_6$$

Substitute $I_H(0) = A_6$ in (4.40)

$$J_H(t) > J_H(0) \ell^{-K_6 t}$$

Since $J_H(0) > 0$

$J_H(t) \geq 0$ for $t > 0$

From equation (3.12)

$$\frac{dR_H}{dt} = \beta J_H - K_7 R_H$$

where $\mathrm{K}_7 = (\gamma + \mu)$

$$\therefore \frac{dR_H}{dt} \geq -K_7 R_H \qquad (4.41)$$

Integrating (4.41) using seperation of the variables

$$\int \frac{1}{R_H} R_H \geq -\int K_7 dt$$

$$\ln R_H \geq -K_7 t + C$$

$$R_H \geq A_7 \ell^{-K_7 t + C}$$

$$R_H = A_7 \ell^{-K_7 t} \qquad (4.42)$$

At $t = 0$

$$R_H(0) = A_7$$

Substitute $R_H(0) = A_7$ in (4.42)

$$R_H(t) > R_H(0) \ell^{-K_7 t}$$

Since $R_H(0) > 0$

$R_H(t) \geq 0$ for $t > 0$

From equation (3.13)

$$\frac{dD_H}{dt} = \delta_1 I_H + \delta_2 I_H - \rho D_H$$

$$\frac{dV_H}{dt} \geq -\rho D_H \qquad (4.43)$$

Integrating (4.43)

$$\int \frac{1}{D_H} dD_H \geq -\int \rho dt$$

$$\ln D_H \geq -\rho t + C$$

$$D_H = \ell^{-\rho t + C}$$

$$D_H = A_8 \ell^{-\rho t} \tag{4.44}$$

At $t = 0$

$D_H(0) = A_8$

Substitute $D_H(0) = A_{10}$ in (4.44)

$D_H(t) > D_H(0)\ell^{-\rho t}$

Since $D_H(0) > 0$

$D_H(t) \geq 0$ for $t > 0$

From equation (3.14)

$$\frac{dS_b}{dt} = \pi N - \lambda_b S_b - \mu_b S_b$$

$$\frac{dS_b}{dt} \geq -\lambda_b S_b - \mu_b S_b$$

where $\lambda_b = \frac{\eta}{N_b}(\mathrm{I}_H + \mathrm{I}_b)$

$$\frac{dS_b}{dt} \geq -(\lambda_b - \mu_b)S_b \tag{4.45}$$

Integrating (4.45)

$$\int \frac{1}{S_b} dS_b \geq -\int (\lambda_b - \mu_b)dt$$

$$\ln S_b \geq -(\lambda_b - \mu_b)t + C$$

$$S_b \geq \ell^{-(\lambda_b - \mu_b)+C}$$

$$S_b \geq A_9 \ell^{-(\lambda_b - \mu_b)t} \tag{4.46}$$

At $t = 0$

$S_B(0) = A_9$

Substitute $S_b(0) = A_{12}$ in (4.46)

$S_b(t) > S_b(0)\ell^{-(\lambda_b-\mu_b)t}$

Since $S_b(0) > 0$

$S_b(t) \geq 0$ for $t > 0$

From equation (3.15)

$\frac{dI_b}{dt}\lambda_b S_b - K_8 I_b$

where $K_8=(\chi+\mu_b)$

$\frac{dI_b}{dt} \geq -K_8 I_b$ (4.47)

Integrating (4.47)

$\int \frac{1}{I_b} dI_b \geq -\int K_8 dt$

$\ln I_b \geq -K_8 t + C$

$I_b \geq A_{10}\ell^{-K_8 t}$ (4.48)

At $t = 0$

$I_b(0) = A_{10}$

Substitute $I_b(0) = A_{13}$ in (4.48)

$I_b(t) > I_b(0)\ell^{-k_8 t}$

Since $I_b(0) > 0$

$I_b(t) \geq 0$ for $t > 0$

4.1.4 Disease free equilibrium state

Consider the system of equation below

In order to obtain the equilibrium point we set the right hand sides of the equations (3.7) to (3.16) to zero that is

$$\phi - \frac{\alpha S_H^*}{N_H}(I_H^* + D_H^* + I_H^*) + \psi V_H^* + \varepsilon Q_H^* - (\theta + \mu)S_H^* + \gamma R_H^* = 0 \tag{4.49}$$

$$\theta S_H^* - (\psi + \mu)V_H^* = 0 \tag{4.50}$$

$$\frac{\alpha S^*_H}{N_H}(I_H^* + D_H^* + I_b^*) - (\sigma + \kappa + \mu)E_H^* = 0 \tag{4.51}$$

$$\sigma E_H^* - (\tau + \varepsilon + \mu)Q_H^* = 0 \tag{4.52}$$

$$\tau Q_H^* + \kappa E_H^* - (\delta_1 + \omega + \mu)I_H^* = 0 \tag{4.53}$$

$$\omega I_H^* - (\beta + \mu + \delta_2)J_H^* = 0 \tag{4.54}$$

$$\beta J_H^* - (\gamma + \mu)R_b^* = 0 \tag{4.55}$$

$$\delta_1 I_H^* + \delta_2 J_H^* - \rho D_H^* = 0 \tag{4.56}$$

$$\pi - \frac{\eta S_b^*}{N_b}(I_H^* + I_b^*) - \mu_b S_b^* = 0 \tag{4.57}$$

$$\frac{\eta S_b^*}{N_b}(I_H^* + I_b^*) - (\chi - \mu_b)I_b^* = 0 \tag{4.58}$$

$$\frac{dS_H}{dt} = \frac{dV_H}{dt} = \frac{dE_H}{dt} = \frac{dQ_H}{dt} = \frac{dI_H}{dt} = \frac{dJ_H}{dt} = \frac{dR_H}{dt} = \frac{dD_H}{dt} = \frac{dS_b}{dt} = \frac{dI_b}{dt} = 0 \tag{4.59}$$

From (4.49) and (4.50)

$$\phi + \psi V_H^* - (\theta + \mu)S_H^* = 0 \tag{4.60}$$

$$\theta S_H^* - (\psi + \mu)V_H^* = 0 \tag{4.61}$$

Then from (4.61), $V^*_H = \dfrac{\theta S^*_H}{(\psi+\mu)}$ (4.62)

By putting (4.62) in to (4.60) then we have

$$\phi + \psi\left(\frac{\theta S^*_H}{\psi+\mu}\right) - (\theta+\mu)S^*_H = 0$$
$$\Rightarrow \phi(\psi+\mu) + \psi\theta S^*_H - (\psi+\mu)(\theta+\mu)S^*_H = 0$$
$$\Rightarrow (\psi+\mu)(\theta+\mu)S^*_H - \psi\theta S^*_H = \phi(\psi+\mu)$$
$$\Rightarrow S^*_H((\psi+\mu)(\theta+\mu) - \psi\theta) = \phi(\psi+\mu)$$
$$\Rightarrow S^*_H = \frac{\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu) - \psi\theta} \quad (4.63)$$

By putting S_H in to (4.62) then we have

$$V^*_H = \frac{\theta\phi(\psi+\mu)}{(\psi+\mu)(\psi+\mu)(\theta+\mu) - \psi\theta}$$
$$\Rightarrow V^*_H = \frac{\theta\phi}{(\psi+\mu)(\theta+\mu) - \psi\theta} \quad (4.64)$$

Then from (4.13)

$$\pi - \mu_b S_b = 0 \quad (4.65)$$

$$S_b = \frac{\pi}{\mu_b} \quad (4.66)$$

Therefore, the disease -free equilibrium state is

$$E_O = \left(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b\right)$$
$$= \left(\frac{\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu) - \psi\theta}, \frac{\theta\phi}{(\psi+\mu)(\theta+\mu) - \psi\theta}, 0, 0, 0, 0, 0, 0, \frac{\pi}{\mu_b}, 0\right) \quad (4.67)$$

4.1.5 Basic reproduction number R_0

In order to obtain the basic reproduction number R_0, we use the approach of Diekmann and Heesterbeek (2000).

Firstly, we arrange the system to get group of infection s classes only that is $\left(E_H, Q_H, I_H D_h, J_H, , I_b\right)$. Let $f_i(x)$ be the rate of appearance of new infections (transmission) in compartment i, $V_i^+(x)$ be the transmission after new infections (transission rate by all other means) and $V_i^-(x)$ be the rate of transfer of individual out of compartment i.

Consider the equations for the infected population given by (3.9),(3.10),(3.11),(3.12),(3.13),(3.16) these are

$$\left.\begin{aligned}
\frac{dE_H}{dt} &= \frac{\alpha S_H}{N_H}(I_{.H} + D_{.H} + I_b) - (\sigma + \phi + \mu)E_H \\
\frac{dQ_H}{dt} &= \sigma E_H - (\tau + \mu + \varepsilon)Q_H \\
\frac{dI_H}{dt} &= \tau Q_H + \phi E_H - (\delta_1 + \omega + \mu)I_H \\
\frac{dJ_H}{dt} &= \omega I_H - (\beta + \mu + \delta_2)J_H \\
\frac{dD_H}{dt} &= \delta_1 I_H + \delta_2 J_H - \rho D_H \\
\frac{dI_b}{dt} &= \frac{\eta S_b}{N_b}(I_H + I_b) - (\chi + \mu_b)I_b
\end{aligned}\right\} \qquad (4.68)$$

From equation (4.68) we have

$$F_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \frac{\alpha S_H}{N_H}(I_H + D_H + I_b) \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\eta S_b}{N_b}(I_H + I_b) \end{bmatrix} \qquad (4.69)$$

From equation (4.13) we have to consider those terms that doesn't have new infection and multiply it by (-1)

That is we have

From equation (4.68) we have

$$V_i = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} (\sigma+\phi+\mu)E_H \\ (\tau+\mu+\varepsilon)Q_H - \sigma E_H \\ -\tau Q_H + \phi E_H + (\delta_1+\omega+\mu)I_H \\ (\beta+\mu+\delta_2)J_H + \omega I_H \\ \rho D_H - \delta_1 I_H - \delta_2 J_H \\ -(\chi+\mu_b)I_b \end{bmatrix} \tag{4.70}$$

Now,

$$\frac{\partial F_i}{\partial x_i} = \begin{bmatrix} \frac{\partial F_1}{\partial E_H} & \frac{\partial F_1}{\partial Q_H} & \frac{\partial F_1}{\partial I_H} & \frac{\partial F_1}{\partial J_H} & \frac{\partial F_1}{\partial D_H} & \frac{\partial F_1}{\partial I_b} \\ \frac{\partial F_2}{\partial E_H} & \frac{\partial F_2}{\partial Q_H} & \frac{\partial F_2}{\partial I_H} & \frac{\partial F_2}{\partial J_H} & \frac{\partial F_2}{\partial D_H} & \frac{\partial F_2}{\partial I_b} \\ \frac{\partial F_3}{\partial E_H} & \frac{\partial F_3}{\partial Q_H} & \frac{\partial F_3}{\partial I_H} & \frac{\partial F_3}{\partial J_H} & \frac{\partial F_3}{\partial D_H} & \frac{\partial F_3}{\partial I_b} \\ \frac{\partial F_4}{\partial E_H} & \frac{\partial F_4}{\partial Q_H} & \frac{\partial F_4}{\partial I_H} & \frac{\partial F_4}{\partial J_H} & \frac{\partial F_4}{\partial D_H} & \frac{\partial F_4}{\partial I_b} \\ \frac{\partial F_5}{\partial E_H} & \frac{\partial F_5}{\partial Q_H} & \frac{\partial F_5}{\partial I_H} & \frac{\partial F_5}{\partial J_H} & \frac{\partial F_5}{\partial D_H} & \frac{\partial F_5}{\partial I_b} \\ \frac{\partial F_6}{\partial E_H} & \frac{\partial F_6}{\partial Q_H} & \frac{\partial F_6}{\partial I_H} & \frac{\partial F_6}{\partial J_H} & \frac{\partial F_6}{\partial D_H} & \frac{\partial F_6}{\partial I_b} \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & \frac{\alpha S_H}{N_H} & 0 & \frac{\alpha S_H}{N_H} & \frac{\alpha S_H}{N_H} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\eta S_b}{N_b} & 0 & 0 & \frac{\eta S_b}{N_b} \end{bmatrix}$$

Therefore valuating F at disease free equilibrium, then we have

$$F = \frac{\partial F_{i(E_O)}}{\partial F_i} \begin{bmatrix} 0 & 0 & \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta} & 0 & \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta} & \dfrac{\alpha\phi}{(\psi+\mu)(\theta+\mu)-\psi\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\eta\pi}{\mu_b} & 0 & 0 & \dfrac{\eta\pi}{\mu_b} \end{bmatrix}$$

Let $B_1 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}, B_2 = \dfrac{\eta\pi}{\mu_b}, B_3 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}, B_4 = \dfrac{\alpha\phi}{(\psi+\mu)(\theta+\mu)-\psi\theta}, B_5 = \dfrac{\eta\pi}{\mu_b}$

$$F = \begin{bmatrix} 0 & 0 & B_1 & 0 & B_3 & B_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & B_5 \end{bmatrix}$$

Now

$$V = \frac{\partial V_i}{\partial x_i} = \begin{bmatrix} \dfrac{\partial V_1}{\partial E_H} & \dfrac{\partial V_1}{\partial Q_H} & \dfrac{\partial V_1}{\partial I_H} & \dfrac{\partial V_1}{\partial J_H} & \dfrac{\partial V_1}{\partial D_H} & \dfrac{\partial V_1}{\partial I_b} \\ \dfrac{\partial V_2}{\partial E_H} & \dfrac{\partial V_2}{\partial Q_H} & \dfrac{\partial V_2}{\partial I_H} & \dfrac{\partial V_2}{\partial J_H} & \dfrac{\partial V_2}{\partial D_H} & \dfrac{\partial V_2}{\partial I_b} \\ \dfrac{\partial V_3}{\partial E_H} & \dfrac{\partial V_3}{\partial Q_H} & \dfrac{\partial V_3}{\partial I_H} & \dfrac{\partial V_3}{\partial J_H} & \dfrac{\partial V_3}{\partial D_H} & \dfrac{\partial V_3}{\partial I_b} \\ \dfrac{\partial V_4}{\partial E_H} & \dfrac{\partial V_4}{\partial Q_H} & \dfrac{\partial V_4}{\partial I_H} & \dfrac{\partial V_4}{\partial J_H} & \dfrac{\partial V_4}{\partial D_H} & \dfrac{\partial V_4}{\partial I_b} \\ \dfrac{\partial V_5}{\partial E_H} & \dfrac{\partial V_5}{\partial Q_H} & \dfrac{\partial V_5}{\partial I_H} & \dfrac{\partial V_5}{\partial J_H} & \dfrac{\partial V_5}{\partial D_H} & \dfrac{\partial V_5}{\partial I_b} \\ \dfrac{\partial V_5}{\partial E_H} & \dfrac{\partial V_6}{\partial Q_H} & \dfrac{\partial V_6}{\partial I_H} & \dfrac{\partial V_6}{\partial J_H} & \dfrac{\partial V_6}{\partial D_H} & \dfrac{\partial V_6}{\partial I_b} \end{bmatrix} \tag{4.71}$$

Therefore by evaluating F at disease free equilibrium, then we have

$$V = \begin{bmatrix} (\sigma+\phi+\mu) & 0 & 0 & 0 & 0 & 0 \\ -\sigma & (\tau+\varepsilon+\mu) & 0 & 0 & 0 & 0 \\ -\phi & -\tau & (\delta_1+\omega+\mu) & 0 & 0 & 0 \\ 0 & 0 & -\omega & (\beta+\mu+\delta_2) & 0 & 0 \\ 0 & 0 & -\delta_1 & -\delta_2 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & (\chi+\mu_b) \end{bmatrix}$$

$$Let A_1 = (\sigma+\mu), A_2 = (\tau+\mu), A_3 = (\delta_1+\omega+\mu), A_4 = (\delta_1+\omega+\mu), A_5 = (\chi+\mu_b)$$

$$\Rightarrow V = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma & A_2 & 0 & 0 & 0 & 0 \\ -\phi & -\tau & A_3 & 0 & 0 & 0 \\ 0 & 0 & -\omega & A_4 & 0 & 0 \\ 0 & 0 & -\delta_1 & -\delta_2 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & A_5 \end{bmatrix}$$

Using the scientific work place 5.5 to compute V^{-1} gives

$$V^{-1}=\begin{vmatrix} \frac{1}{A_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sigma}{A_1A_2} & \frac{1}{A_2} & 0 & 0 & 0 & 0 \\ \frac{\sigma\tau+\phi A_2}{A_1A_2A_3} & \frac{\tau}{A_2A_3} & \frac{1}{A_3} & 0 & 0 & 0 \\ \frac{\sigma\tau\omega+\phi\omega A_2}{A_1A_2A_3A_4} & \frac{\tau\omega}{A_2A_3A_4} & \frac{\omega}{A_3A_4} & \frac{1}{A_4} & 0 & 0 \\ \frac{\sigma\tau(\omega\delta_2+\tau\sigma\delta_1A_4+\phi\omega\delta_2A_2+\phi\delta_1A_2A_4)}{A_1A_2A_3A_4\rho} & \frac{\tau(\omega\delta_2+A_4\delta_1)}{A_2A_3A_4\rho} & \frac{\omega\delta_2+A_4\delta_1}{A_3A_4\rho} & \frac{\delta_2}{A_4\rho} & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{A_5} \end{vmatrix}$$

LET $C_1=\frac{1}{A_1}$, $C_2=\frac{\sigma}{A_1A_2}$, $C_3=\frac{\sigma\tau+\phi A_2}{A_1A_2A_3}$, $C_4\frac{\sigma\tau\omega+\phi\omega A_2}{A_1A_2A_3A_4}=$,

$C_5=\frac{\sigma\tau(\omega\delta_2+\tau\sigma\delta_1A_4+\phi\omega\delta_2A_2+\phi\delta_1A_2A_4)}{A_1A_2A_3A_4\rho}$, $C_{6=}\frac{1}{A_2}$, $C_7=\frac{\tau}{A_2A_3}$

$C_8=\frac{\tau\omega}{A_2A_3A_4}$, $C_9=\frac{\tau(\omega\delta_2+A_4\delta_1)}{A_2A_3A_4\rho}$, $C_{10}=\frac{1}{A_3}$, $C_{11}=\frac{\omega}{A_3A_4}$ $C_{12}=\frac{\omega\delta_2+A_4\delta_1}{A_3A_4\rho}$, $C_{13}=\frac{1}{A_4}$ $C_{14}=\frac{\delta_2}{A_4\rho}$ $C_{15}=\frac{1}{\rho}$

$C_{16}=\frac{1}{A_5}$

$$V^{-1}=\begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ C_2 & C_6 & 0 & 0 & 0 & 0 \\ C_3 & C_7 & C_{10} & 0 & 0 & 0 \\ C_4 & C_8 & C_{11} & C_{13} & 0 & 0 \\ C_5 & C_9 & C_{12} & C_{14} & C_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{16} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} 0 & 0 & B_1 & 0 & B_3 & B_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & B_5 \end{bmatrix} \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ C_2 & C_6 & 0 & 0 & 0 & 0 \\ C_3 & C_7 & C_{10} & 0 & 0 & 0 \\ C_4 & C_8 & C_{11} & C_{13} & 0 & 0 \\ C_5 & C_9 & C_{12} & C_{14} & C_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{16} \end{bmatrix}$$

Therefore, $FV^{-1} = \begin{bmatrix} d_1 & d_3 & d_5 & d_7 & d_8 & d_9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d_2 & d_4 & d_6 & 0 & 0 & d_{10} \end{bmatrix}$

where $d_1 = B_1C_3 + B_3C_5$, $d_2 = B_2C_3$, $d_3 = B_1C_7 + B_3C_9$, $d_4 = B_2C_7$, $d_5 = B_1C_{10} + B_3C_{12}$, $d_6 = B_2C_{10}$, $d_7 = B_3C_{14}$ $d_8 = B_3C_{15}$, $d_9 = B_4C_{16}$ and $d_{10} = B_5C_{16}$

We can now find $\left|FV^{-1} - \lambda I\right| = 0$

$$= \begin{vmatrix} d_1 - \lambda & d_3 & d_5 & d_7 & d_8 & d_9 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & (-\lambda) & 0 \\ d_2 & d_4 & d_6 & 0 & 0 & d_{10} - \lambda \end{vmatrix} = 0$$

$$(-\lambda) \begin{vmatrix} d_1 - \lambda & d_3 & d_5 & d_7 & d_9 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & (-\lambda) & 0 \\ d_2 & d_4 & d_6 & 0 & d_{10} - \lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) \begin{vmatrix} d_1 - \lambda & d_3 & d_5 & d_9 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & (-\lambda) & 0 \\ d_2 & d_4 & d_6 & d_{10} - \lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda)(-\lambda) \begin{vmatrix} d_1 - \lambda & d_3 & d_9 \\ 0 & -\lambda & 0 \\ d_2 & d_4 & d_{10} \end{vmatrix} = 0$$

Clearly,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

therefore, the dominant eigenvalue of FV^{-1} is eigenvalue of

$$\begin{vmatrix} d_1 - \lambda & d_9 \\ d_2 & d_{10} - \lambda \end{vmatrix} = 0$$

$$(d_1 - \lambda)(d_{10} - \lambda) - d_2 d_9 = 0$$
$$d_1 d_{10} - d_1\lambda - d_{10}\lambda + \lambda^2 - d_2 d_9 = 0$$
$$\lambda^2 - (d_1 + d_{10})\lambda + d_1 d_{10} - d_2 d_9 = 0$$
$$\lambda^2 - (d_1 + d_{10})\lambda = -(d_1 d_{10} - d_2 d_9)$$

$$\lambda^2 - (d_1 + d_{10})\lambda + \left[\frac{-(d_1 + d_{10})}{2}\right]^2 = -(d_1 d_{10} - d_2 d_9) + \left[\frac{-(d_1 + d_{10})}{2}\right]^2$$

$$\Rightarrow \left[\lambda - \frac{(d_1 + d_{10})}{2}\right]^2 = d_2 d_9 - d_1 d_{10} + (-\frac{d_1 + d_{10}}{4})^2 = \frac{d_2 d_9 - d_1 d_{10}}{1} + \frac{d_1 + 2d_1 d_{10} + d_{10}^2}{4}$$

$$= \frac{4d_2 d_9 - 4d_1 d_{10} + d_1^2 2d_1 d_{10} + d_{10}^2}{4}$$

$$\Rightarrow \left[\lambda - \frac{(d_1 + d_{10})}{2}\right]^2 = \frac{d_1^2 + 4d_2 d_9 - 2d_1 d_{10} + d_{10}^2}{4}$$

$$\therefore \lambda - (\frac{d_1 - d_{10}}{2}) = \frac{\sqrt{d_1^2 + 4d_2 d_9 - 2d_1 d_{10} + d_{10}^2}}{2}$$

$$\Rightarrow \lambda = \frac{d_1 - d_{10}}{2} \pm \frac{\sqrt{d_1^2 + 4d_2 d_9 - 2d_1 d_{10} + d_{10}^2}}{2}$$

This is also implies

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$$

$$\lambda_5 = \frac{d_1 + d_{10} - \sqrt{d_1^{\,2} + 4d_2 d_9 - 2d_1 d_{10} + d_{10}^{\,2}}}{2}$$

$$\lambda_6 = \frac{d_1 + d_{10} + \sqrt{d_1^{\,2} + 4d_2 d_9 - 2d_1 d_{10} + d_{10}^{\,2}}}{2}$$

$$R_0 = \rho\left(FV^{-1}\right) = \left|\lambda_5\right|, \left|\lambda_6\right|$$

$$Let R_{01} = \lambda_{5,} R_{02} = \lambda_6$$

therefore $R_0 = Max\left\{R_{01} R_{02}\right\}$

4.1.6 Local stability of the disease fee equilibrium

The local stability of the disease free equilibrium point of model equations given by (3.6 to 3.16) can be established by showing that all the eigenvalues of the Jacobian matrix of the linearized system evaluated at E_0 are negative is locally asymptotically stable (LAS).

Theorem 4

The DFE of the modified model given by (3.6) to (3.16) is locally asymptotically stable if $\psi\theta < A_1 A_5$ and $\sigma\tau A_7 < A_6 A_{10} A_{11}$

Proof:

We shall use the method of linearization to proof this theorem,

Let F_i, *for* i = 1,2,...,10 denote the equations (3.6) to (3.16)
Thus,

$$F_1 = \phi + \varepsilon Q_H - \frac{\alpha S_H}{N_H}\left(I_H + D_H + I_b\right) + \psi V_H - \left(\theta + \mu\right) S_H + \gamma R_H$$

$$F_2 = \theta S_H - \left(\psi + \mu\right) V_H$$

$$F_3 = \kappa E_H + \frac{\alpha S_H}{N_H}\left(I_H + D_H + I_b\right) - \left(\sigma + \mu\right) E_H$$

$$F_4 = \sigma E_H - \left(\tau + \mu + \varepsilon\right) Q_H$$

$$F_{5} = \tau Q_H - \left(\delta_1 + \omega + \mu\right) I_H - \kappa I_H$$

$$F_6 = \omega \mathrm{I}_H - \left(\beta + \mu + \delta_2\right) I_H$$

$$F_7 = \beta J_H - \left(\gamma + \mu\right) R_H$$

$$F_8 = \delta_1 I_H + \delta_2 J_H - \rho D_H$$

$$F_9 = \pi - \left(\frac{\eta}{N_b} \left(I_H + I_b \right) + \mu_b \right) S_b$$

$$F_{10} = \frac{\eta S_b}{N_b} \left(I_H + I_b \right) - \left(X + \mu_b \right) I_b$$

Thus, the Jacobean matrix J for the system $(F_1 \ \ to \ F_{10})$

$$J = \begin{bmatrix}
\frac{\partial F_1}{\partial S_H} & \frac{\partial F_1}{\partial V_H} & \frac{\partial F_1}{\partial E_H} & \frac{\partial F_1}{\partial Q_H} & \frac{\partial F_1}{\partial I_H} & \frac{\partial F_1}{\partial J_H} & \frac{\partial F_1}{\partial R_H} & \frac{\partial F_1}{\partial D_H} & \frac{\partial F_1}{\partial S_b} & \frac{\partial F_1}{\partial I_b} \\
\frac{\partial F_2}{\partial S_H} & \frac{\partial F_2}{\partial V_H} & \frac{\partial F_2}{\partial E_H} & \frac{\partial F_2}{\partial Q_H} & \frac{\partial F_2}{\partial I_H} & \frac{\partial F_2}{\partial J_H} & \frac{\partial F_2}{\partial R_H} & \frac{\partial F_2}{\partial D_H} & \frac{\partial F_2}{\partial S_b} & \frac{\partial F_2}{\partial I_b} \\
\frac{\partial F_3}{\partial S_H} & \frac{\partial F_3}{\partial V_H} & \frac{\partial F_3}{\partial E_H} & \frac{\partial F_3}{\partial Q_H} & \frac{\partial F_3}{\partial Q_H} & \frac{\partial F_3}{\partial J_H} & \frac{\partial F_3}{\partial R_H} & \frac{\partial F_3}{\partial D_H} & \frac{\partial F_3}{\partial S_b} & \frac{\partial F_3}{\partial I_b} \\
\frac{\partial F_4}{\partial S_H} & \frac{\partial F_4}{\partial V_H} & \frac{\partial F_4}{\partial E_H} & \frac{\partial F_4}{\partial Q_H} & \frac{\partial F_4}{\partial Q_H} & \frac{\partial F_4}{\partial J_H} & \frac{\partial F_4}{\partial R_H} & \frac{\partial F_4}{\partial D_H} & \frac{\partial F_4}{\partial S_b} & \frac{\partial F_4}{\partial I_b} \\
\frac{\partial F_5}{\partial S_H} & \frac{\partial F_5}{\partial V_H} & \frac{\partial F_5}{\partial E_H} & \frac{\partial F_5}{\partial Q_H} & \frac{\partial F_5}{\partial Q_H} & \frac{\partial F_5}{\partial J_H} & \frac{\partial F_5}{\partial R_H} & \frac{\partial F_5}{\partial D_H} & \frac{\partial F_5}{\partial S_b} & \frac{\partial F_5}{\partial I_b} \\
\frac{\partial F_6}{\partial S_H} & \frac{\partial F_6}{\partial V_H} & \frac{\partial F_6}{\partial E_H} & \frac{\partial F_6}{\partial Q_H} & \frac{\partial F_6}{\partial Q_H} & \frac{\partial F_6}{\partial J_H} & \frac{\partial F_6}{\partial R_H} & \frac{\partial F_6}{\partial D_H} & \frac{\partial F_6}{\partial S_b} & \frac{\partial F_6}{\partial I_b} \\
\frac{\partial F_7}{\partial S_H} & \frac{\partial F_7}{\partial V_H} & \frac{\partial F_7}{\partial E_H} & \frac{\partial F_7}{\partial Q_H} & \frac{\partial F_7}{\partial Q_H} & \frac{\partial F_7}{\partial J_H} & \frac{\partial F_7}{\partial R_H} & \frac{\partial F_7}{\partial D_H} & \frac{\partial F_7}{\partial S_b} & \frac{\partial F_7}{\partial I_b} \\
\frac{\partial F_8}{\partial S_H} & \frac{\partial F_8}{\partial V_H} & \frac{\partial F_8}{\partial E_H} & \frac{\partial F_8}{\partial Q_H} & \frac{\partial F_8}{\partial Q_H} & \frac{\partial F_8}{\partial J_H} & \frac{\partial F_8}{\partial R_H} & \frac{\partial F_8}{\partial D_H} & \frac{\partial F_8}{\partial S_b} & \frac{\partial F_8}{\partial I_b} \\
\frac{\partial F_9}{\partial S_H} & \frac{\partial F_9}{\partial V_H} & \frac{\partial F_9}{\partial E_H} & \frac{\partial F_9}{\partial Q_H} & \frac{\partial F_9}{\partial Q_H} & \frac{\partial F_9}{\partial J_H} & \frac{\partial F_9}{\partial R_H} & \frac{\partial F_9}{\partial D_H} & \frac{\partial F_9}{\partial S_b} & \frac{\partial F_9}{\partial I_b} \\
\frac{\partial F_{10}}{\partial S_H} & \frac{\partial F_{10}}{\partial V_H} & \frac{\partial F_{10}}{\partial E_H} & \frac{\partial F_{10}}{\partial Q_H} & \frac{\partial F_{10}}{\partial Q_H} & \frac{\partial F_{10}}{\partial J_H} & \frac{\partial F_{10}}{\partial R_H} & \frac{\partial F_{10}}{\partial D_H} & \frac{\partial F_{10}}{\partial S_b} & \frac{\partial F_{10}}{\partial I_b}
\end{bmatrix}$$

The Jacobian evaluated at the DFE id given by

$$J(E_0)=\begin{bmatrix} -A_1 & \psi & 0 & 0 & -A_2 & 0 & \gamma & -A_3 & 0 & -A_4 \\ \theta & -A_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_6 & 0 & A_7 & 0 & 0 & A_8 & 0 & A_9 \\ 0 & 0 & \sigma & -A_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & -A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & -A_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & -A_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 & \delta_2 & 0 & -\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{14} & 0 & 0 & 0 & -A_{15} & A_{16} \\ 0 & 0 & 0 & 0 & A_{17} & 0 & 0 & 0 & 0 & -A_{18} \end{bmatrix}$$

where $A_1 = (\theta+\mu)$, $A_2 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$, $A_3 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$,

$A_4 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$, $A_5 = (\psi+\mu)$, $A_6 = (\sigma+\mu)$

$A_7 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$, $A_8 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$, $A_9 = \dfrac{\phi\alpha(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}$,

$A_{10} = (\tau+\mu)$, $A_{11} = (\delta_1+\omega+\mu)$, $A_{12} = (\beta+\mu+\delta_2)$

$A_{13} = (\gamma+\mu)$, $A_{14} = \dfrac{n\pi}{N_b}$, $A_{15} = -\mu_b$, $A_{16} = \dfrac{n\pi}{N_b}$, $A_{17} = \dfrac{n\pi}{N_b}$, $A_{18} = (\chi+N_b)$

Therefore using maple 18 we reduce the matrix to upper triangular matrix using Lu decomposition the transformed matrix evaluated at E_0 is

$$J(E_0)=\begin{bmatrix} -A_1 & \psi & 0 & 0 & -A_2 & 0 & \gamma & -A_3 & 0 & -A_4 \\ 0 & U_1 & 0 & 0 & -U_2 & 0 & U_3 & -U_4 & 0 & -U_5 \\ 0 & 0 & -A_6 & 0 & A_7 & 0 & 0 & A_8 & 0 & A_9 \\ 0 & 0 & 0 & -A_{10} & U_6 & 0 & 0 & U_7 & 0 & U_8 \\ 0 & 0 & 0 & 0 & U_9 & 0 & 0 & U_{10} & 0 & U_{11} \\ 0 & 0 & 0 & 0 & 0 & -A_{12} & 0 & U_{12} & 0 & U_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_{13} & U_{14} & 0 & U_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U_{16} & 0 & U_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A_{15} & -U_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U_{19} \end{bmatrix} \quad (4.72)$$

$$Where\ U_1 = \frac{\psi\theta - A_1A_5}{A_1},\ U_2 = \frac{\theta A_2}{A_1},\ U_3 = \frac{\theta\gamma}{A_1},\ U_4 = \frac{\theta A_3}{A_1},\ U_5 = \frac{\theta A_4}{A_1},\ U_6 = \frac{\sigma A_7}{A_6},\ U_7 = \frac{\sigma A_8}{A_6},$$

$$U_8 = \frac{\sigma A_9}{A_6},\ U_9 = \frac{\sigma\tau A_7 - A_6A_{10}A_{11}}{A_{10}A_6},\ U_{10} = \frac{\sigma\tau A_8}{A_{10}A_6},\ U_{11} = \frac{\sigma\tau A_9}{A_{10}A_6},\ U_{12} = \frac{\sigma\tau\omega A_8}{\sigma\tau A_7 - A_{10}A_6A_{11}},$$

$$U_{13} = \frac{\sigma\tau\omega A_9}{\sigma\tau A_7 - A_{10}A_6A_{11}},\ U_{14} = \frac{\beta\sigma\tau\omega A_8}{A_{12}(\sigma\tau A_7 - A_{10}A_6A_{11})},\ U_{15} = \frac{\beta\sigma\tau\omega A_9}{A_{12}(\sigma\tau A_7 - A_{10}A_6A_{11})},$$

$$U_{16} = \frac{\sigma\tau\omega A_8 + \rho\sigma\tau A_7A_{12} - \rho A_{10}A_6A_{11}A_{12} + \sigma\tau\delta_1A_8A_{12}}{A_{12}(\sigma\tau A_7 - A_{10}A_6A_{11})},\ U_{17} = \frac{\sigma\tau A_9\left(\omega\delta_2 + A_{12}\delta_1\right)}{A_{12}(\sigma\tau A_7 - A_{10}A_6A_{11})},$$

$$U_{18} = \frac{\sigma\tau\omega\delta_2A_8A_{16} + \rho\sigma\tau A_7A_{12}A_{16} + \rho\sigma\tau A_9A_{12}A_{14} - \rho A_6A_{10}A_{11}A_{12}A_{16} + \sigma\tau\delta_1A_8A_{12}A_{16}}{\sigma\tau\omega A_8\delta_2 + \rho\sigma\tau A_7A_{12} - \rho A_6A_{10}A_{11}A_{12} + \sigma\tau\delta_1A_8A_{12}}$$

$$U_{19} = \frac{\sigma\tau\omega\delta_2A_8A_{18} + \rho\sigma\tau A_7A_{12}A_{18} + \rho\sigma\tau A_9A_{12}A_{17} - \rho A_6A_{10}A_{11}A_{12}A_{18} + \sigma\tau\delta_1A_8A_{12}A_{18}}{\sigma\tau\omega A_8\delta_2 + \rho\sigma\tau A_7A_{12} - \rho A_6A_{10}A_{11}A_{12} + \sigma\tau\delta_1A_8A_{12}}$$

The eigen values of (4.72) are presented as follows

$\lambda_1 = -A_1$
$\lambda_2 = -A_6$
$\lambda_3 = -A_{10},$
$\lambda_4 = -A_{12}$
$\lambda_5 = -A_{13}$
$\lambda_6 = -U_{16}$
$\lambda_7 = -A_{15}$
$\lambda_8 = -U_{19}$
$\lambda_9 = U_1$
$\lambda_{10} = U_9$

It is clear that $\lambda_i < 0,$ for $i = 1, 2, ..., 8 =$

Since $A_i > 0,$ for $i = 1, 2, ..., 16$ and $U_i > 0,$ for $i = 1, 2, ..., 19$

For $\lambda_9 < 0,$ we must have $\psi\theta < A_1A_5$ (4.73)

Similarly,

λ_{10} satisfies negativity requirement for stability if $\sigma\tau A_7 < A_6 A_{10} A_{11}$ (4.74) in U_9

Thus, the DFE is locally asymptotically stable if the inequalities (4.73) and (4.74) are satisfied

4.1.7 Global stability of disease free equilibrium (DFE)

To establish the global stability of the disease free equilibrium of the model using the theorem by Castillo-chavez et al. The conditions H_1 and H_2 must be satisfied.

$H_1 : \frac{dx}{dt} = H(X,0), \mathrm{x}^0$ is globally asymptotically stable (GAS)

$H_2 : G(X,Z) = pz - \hat{G}(X,Z), \hat{G}(X,Z) \geq 0$ for $(x,z) \in \Omega$, where $P = \Delta_z G(X^0,0)$ is an $M - matrix$ (the off diagonal elements of P are non-negative) and is also jacobian of $\mathrm{G}(X,Z)$

We write the model equation given by (3.7-3.16) as

$$\frac{dx}{dt} = H(X,Z)$$
$$\frac{dz}{dt} = G(X,Z), G(X,\mathrm{o}) = 0$$

$$E_0(X^0,0) = \left(\frac{\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}, \frac{\theta\phi}{(\psi+\mu)(\theta+\mu)-\psi\theta}, \frac{\pi}{\mu_b} \right)$$

Where $X = (S_H, V_H, R_H, S_b) \in \mathbb{R}^4$ denotes the number of un-infected individuals and $Z = (E_H, Q_H, J_H, J_H, D_E, I_b) \in \mathbb{R}^6$ denotes the number of infected individuals.

$E_o = (X^0, 0)$ denotes the DFE of the system.

Take $(\mathrm{E}_H, \mathrm{Q}_H, \mathrm{I}_H, \mathrm{J}_b, \mathrm{D}_H, \mathrm{I}_\mathrm{b})$ and evaluated at

$$E_O\left(S_H, V_H, S_b\right) = \left(\frac{\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)-\psi\theta}, \frac{\theta\phi}{(\psi+\mu)(\theta+\mu)-\psi\theta}, \frac{\pi}{\mu_b} \right) \quad (4.75)$$

If the model satisfies the conditions H_1 and H_2 above, then according to Castillo-Chaves (2002), the following theorems hold.

Theorem 5: The fixed point $E_0 = (x^0, 0)$ is a globally asymptotic stable (GAS) provided that $R_0 < 1$(L.A.S) and that assumptions H_1 and H_2 are satisfied.

Proof: from, the two functions $H(X,Z)$ and $\mathrm{G}(X,Z)$ are given by

$$H(X,Z) = \begin{bmatrix} \phi - \dfrac{\alpha S_H}{N_H}(I_H + D_H + I_b) + \psi V_H + \varepsilon Q_H - (\theta + \mu)S_H + \gamma R_H \\ \theta S_H - (\phi + \mu)V_H \\ \beta J_H - (\gamma + \mu)R_H \\ \pi - \dfrac{\eta S_b}{N_b}(I_H + I_b) - \mu_b S_b \end{bmatrix}$$

$$\mathrm{G}(X,Z) = \begin{bmatrix} \dfrac{\alpha S_H}{N_H}(\mathrm{I}_H + D_H + I_b) - (\sigma + \mu + \varepsilon)\mathrm{E}_H \\ \sigma E_H - \varepsilon Q_H - (\tau + \mu)Q_H \\ \tau Q_H - (\delta_1 + \omega + \mu)I_H \\ \omega I_H - (\beta + \mu + \delta_2)J_H \\ \delta_1 I_H + \delta_2 J_H - \rho D_H \\ \dfrac{\eta S_b}{N_b}(I_H + I_b) - \chi I_b - \mu_b I_b \end{bmatrix}$$

We then consider the reduced system $\dfrac{dx}{dt} = H(X,0)$ from condition (1)

$$H(x,0) = \begin{cases} \phi - (\theta + \mu)S_H + \psi V_H \\ \theta S_H - (\psi + \mu)V_H \\ 0 \\ \pi - \mu_b S_b \end{cases} \tag{4.76}$$

From the first equation of the system (4.76)

$$\frac{dS_H}{dt} = \phi - (\theta + \mu)S_H + \psi V_H$$

$$\frac{dS_H}{dt} + (\theta + \mu)S_H = \phi + \psi V_H \tag{4.77}$$

Integrate (4.77) by integrating factor;

$$\Rightarrow S_H If = (\phi + \psi V_H)\int Ifdt$$

$$\therefore If = \ell^{\int(\theta+\mu)dt} = \ell^{(\theta+\mu)t}$$

$$S_H \ell^{(\theta+\mu)dt} = \phi\int \ell^{(\theta+\mu)tdt}$$

$$S_H(\mathrm{t}) = \frac{\phi + \psi V_H}{(\theta+\mu)} + C\ell^{-(\theta+\mu)t} \tag{4.78}$$

$$at\ t \to 0$$

$$S_H(0) = \frac{\phi + \psi V_H}{(\theta+\mu)} + C \Rightarrow C = S_H(0) - \frac{\phi + \psi V_H}{(\theta+\mu)}$$

By putting the value of C into (4.78)

$$\Rightarrow S_H(\mathrm{t}) = \frac{\phi + \psi V_H}{(\theta+\mu)} + \left(S_H(0) - \frac{\phi + \psi V_H}{(\theta+\mu)} \right)\ell^{-(\theta+\mu)\mathrm{t}}$$

As $t \to \infty$

$$S_H(\mathrm{t}) \to \frac{\phi + \psi V_H}{(\theta+\mu)} \tag{4.79}$$

By putting the value of V_H (4.79)

$$\Rightarrow S_H(\mathrm{t}) = \frac{\phi^2\theta + \psi\theta\phi}{(\theta+\mu)^2(\psi+\mu)\psi\theta} \tag{4.80}$$

$\therefore$ from the second equation of the system (4.76),

$$\frac{dV_H}{dt} = \theta S_H - (\psi+\mu)V_H$$

$$\Rightarrow \frac{dV_H}{dt} + (\psi+\mu)V_H = \theta S_H \tag{4.81}$$

$\therefore$ let integrate (4.81) using integrating factor

$$V_H \cdot If = \theta S_H \int Ifdt$$

$$If = \ell^{(\psi+\mu)dt} = \ell^{(\psi+\mu)t}$$

$$V_H \ell^{(\psi+\mu)t} = \theta S_H \int \ell^{(\psi+\mu)t} dt$$

$$V_H \ell^{(\psi+\mu)t} = \theta S_H \frac{\ell^{(\psi+\mu)t}}{(\psi+\mu)} + C$$

$$V_H(t) = \frac{\theta S_H}{(\psi+\mu)} + C\ell^{-(\psi+\mu)t} \tag{4.82}$$

$$\text{At } t \to 0 \Rightarrow V_H(0) = \frac{\theta S_H}{(\psi+\mu)} + C$$

$$\therefore C = V_H(0) - \frac{\theta S_H}{\psi+\mu}$$

By substituting the value of C into (4.82)

$$\Rightarrow V_H(t) = \frac{\theta S_H}{(\psi+\mu)} + (V_H(0) - \frac{\theta S_H}{\psi+\mu})\ell^{-(\psi+\mu)t}$$

$$t \to \infty, V_H(\mathrm{t}) = \frac{\theta S_H}{(\psi+\mu)} \tag{4.83}$$

By putting back the value of S_H in (4.83)

$$\Rightarrow V_H(t) = \frac{\theta\phi}{(\psi+\mu)(\theta+\mu)\psi\theta}$$

From the last equation of the system (4.76)

$$\frac{dS_b}{dt} = \pi - \mu_b S_b$$

$$\Rightarrow \frac{dS_b}{dt} + \mu_b S_b = \pi \tag{4.84}$$

$\therefore$ let integrate (4.84) using integrating factor

$$S_b \cdot If = \pi \int Ifdt$$

$$If = \ell^{S\mu_b S_b dt} = \ell^{\mu_b S_b t}$$

$$S_b \ell^{\mu_b S_b t} = \pi \int \ell^{\mu_b S_b t} dt$$

$$S_b \ell^{\mu_b S_b t} = \pi \frac{\ell^{\mu_b S_b t}}{\mu_b S_b} + C$$

$$S_b(t) = \frac{\pi}{\mu_b S_b} + C\ell^{-\mu_b S_b t} \tag{4.85}$$

$$\text{At } t \to 0 \Rightarrow S_b(0) = \frac{\pi}{\pi_b S_b} + C$$

$$\therefore C = S_b(0) - \frac{\pi}{\pi_b S_b}$$

By substituting the value of C into (4.85)

$$\Rightarrow S_b(t) = \frac{\pi}{\mu_b S_b} + (\mathrm{S}_b(0) - \frac{\pi}{\pi_b S_b})\ell^{-\mu_b S_b t}$$

$$t \to \infty, S_b(\mathrm{t}) \leftrightarrow \frac{\pi}{\pi_b S_b}$$

Convergence of x^0 is therefore global in Ω . This implies $x^0 = \left(= \frac{\phi^2\theta + \psi\theta\phi}{(\theta+\mu)^2(\psi+\mu)\psi\theta}, \frac{\theta\phi}{(\psi+\mu)(\theta+\mu)\psi\theta}, 0, \frac{\pi}{\pi_b S_b} \right)$ is g.a.s equilibrium of

$$\frac{dx}{dt} = H(x,0)$$

Next, we compute $H(X,Z) = pz - \hat{G}(X,Z)$ and show that $\hat{G}(X,Z) \geq 0$

$$\therefore J[G(X,Z)] = \mathrm{p} \tag{4.86}$$

Where j is the Jacobean of $\mathrm{G}(X,Z)$ taken in $(\mathrm{E}_H, \mathrm{Q}_H, \mathrm{I}_H, \mathrm{J}_b, \mathrm{D}_H, \mathrm{I}_b)$ and evaluated at

$$E_O = \left(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b\right)$$

$$= \left(\frac{\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu) - \psi\theta}, \frac{\theta\phi}{(\psi+\mu)(\theta+\mu) - \psi\theta}, 0,0,0,0,0,0, \frac{\pi}{\mu_b}, 0 \right)$$

Therefore (4.86) gives

$$P = \begin{bmatrix} E_H & Q_H & I_H & J_H & D_H & I_b \\ -(\sigma+\mu) & 0 & \frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} & 0 & \frac{\alpha\phi}{(\theta+\mu)} \frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} & \frac{\alpha\phi}{(\theta+\mu)} \frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} \\ \sigma & -(\tau+\mu) & 0 & 0 & 0 & 0 \\ 0 & \tau & -(\delta_1+\omega+\mu) & 0 & 0 & 0 \\ 0 & 0 & \omega & -(\beta+\mu+\delta_2) & 0 & 0 \\ 0 & 0 & \delta_1 & \delta_2 & -\rho & 0 \\ 0 & 0 & \frac{\eta\pi}{\mu_b} & 0 & 0 & \frac{\eta\pi}{\mu_b} - \mu_b \end{bmatrix}$$

$$PZ=\begin{bmatrix} E_H & Q_H & I_H & J_H & D_H & I_b \\ -\sigma+\mu) & 0 & \frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} & 0 & \frac{\alpha\phi}{(\theta+\mu)}\frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} & \frac{\alpha\phi}{(\theta+\mu)}\frac{\alpha\phi(\psi+\mu)}{(\psi+\mu)(\theta+\mu)\psi\theta} \\ \sigma & -(\tau+\mu) & 0 & 0 & 0 & 0 \\ 0 & \tau & -(\delta_1+\omega+\mu) & 0 & 0 & 0 \\ 0 & 0 & \omega & -(\beta+\mu+\delta_2) & 0 & 0 \\ 0 & 0 & \delta_1 & \delta_2 & -\rho & 0 \\ 0 & 0 & \frac{\eta\pi}{\mu_b} & 0 & 0 & \frac{\eta\pi}{\mu_b}-\mu_b \end{bmatrix}\begin{bmatrix} \\ \underline{E_H} \\ \\ Q_H \\ I_H \\ J_H \\ D_H \\ \underline{I_b} \end{bmatrix}$$

$$\Rightarrow PZ=\begin{bmatrix} -(\sigma+\mu-\kappa)E_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}I_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}D_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}I_b \\ \sigma E_H-(\tau+\mu+\varepsilon)Q_H \\ \tau Q_H-(\delta_1+\omega+\mu)I_H+\kappa E_H \\ \omega I_H-(\beta+\mu+\delta_2)J_H \\ \delta_1 I_H+\delta_2 J_H-\rho D_H \\ (\frac{\eta\pi}{\mu_b}I_H+\frac{\eta\pi}{\mu_b}-\mu_b)I_b \end{bmatrix}-$$

$$\begin{bmatrix} -(\sigma+\mu-\kappa)E_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}I_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}D_H+\frac{\alpha\phi(\psi+\mu)}{(\theta+\mu)(\psi+\mu)\psi\theta}I_b \\ \sigma E_H-(\tau+\mu+\varepsilon)Q_H \\ \tau Q_H-(\delta_1+\omega+\mu)I_H+\kappa E_H \\ \omega I_H-(\beta+\mu+\delta_2)J_H \\ \delta_1 I_H+\delta_2 J_H-\rho D_H \\ (\frac{\eta\pi}{\mu_b}I_H+\frac{\eta\pi}{\mu_b}-\mu_b)I_b \end{bmatrix}$$

$$\therefore G(X,Z)=PZ-G(X,Z)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e G(X,Z) = [0 0 0 0 0 0]T. This shows that G(X,Z) = 0. Hence, the DFE is globally asymptotically stable.

Table 4.1: Variables values used for numerical simulations

Variables	Values	Reference
$S_H(0)$	800	Assumed
$V_H(0)$	500	Assumed
$E_H(0)$	250	Assumed
$Q_H(0)$	150	Assumed
$I_H(0)$	10	Assumed
$J_H(0)$	8	Assumed
$R_H(0)$	6	Assumed
$D_H(0)$	4	Assumed
$S_b(0)$	200	Assumed
$I_b(0)$	10	Assumed

Table 4.2: Parameters values used for numerical simulation

Parameters	Values	Reference
ϕ	500	Rachah & Defim (2018)
ɵ	0.5	Durojaye & Ajie (2017)
Ψ	0.03	Durojaye & Ajie (2017)
α	0.2	Amenaghawon & Abubakary (2015)
σ	0.4	Assumed
Κ	0.6	Assumed
τ	0.3	Amenaghawon & Abubakary (2015)
ε	0.7	Assumed
ω	0.8	Amenaghawon & Abubakary (2015)
μ	0.02	Assumed
β	0.27	Durojaye & Ajie (2017)
ρ	0.4	Rachah & Defim (2018)
δ_1	0.6	Danny & Robert (2016)

δ_2	0.2	Assumed
χ	0.4	Danny & Robert (2016)
γ	0.23	Durojaye & Ajie (2017)
π	2000	Danny & Robert (2016)
η	0.65	Assumed
μ_b	0.025	Assumed

4.1.8 Computational results

The following figures 4.1 to 4.16 are graphical representations showing the behavior of infected individuals in various stages of Ebola virus Disease over a period of time.

4.1.8.1 Effect of vaccine on infected human population

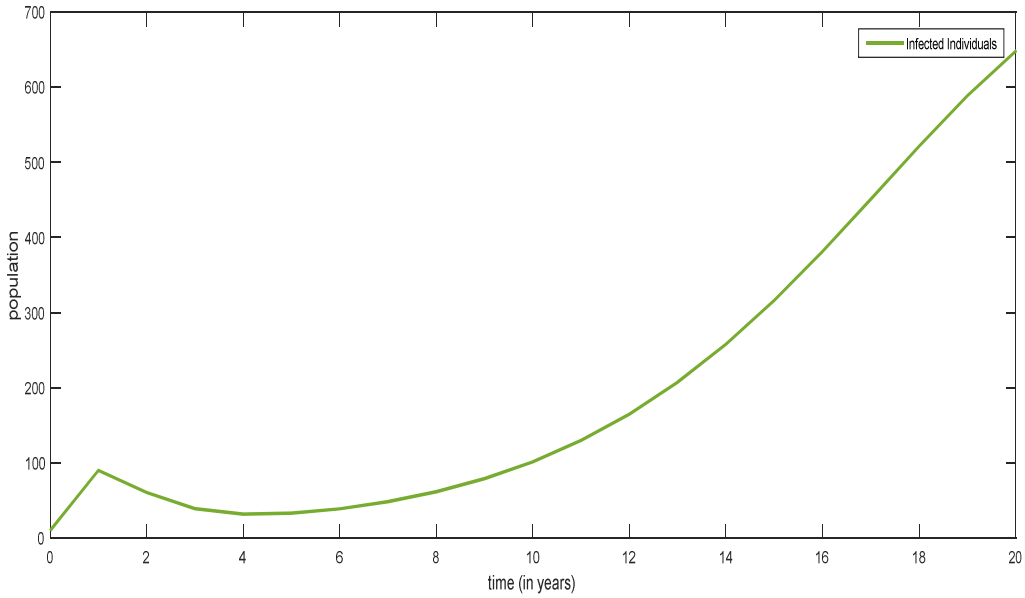

Figure 4.1: the simulation result of infected human without vaccination $(\theta = 0)$

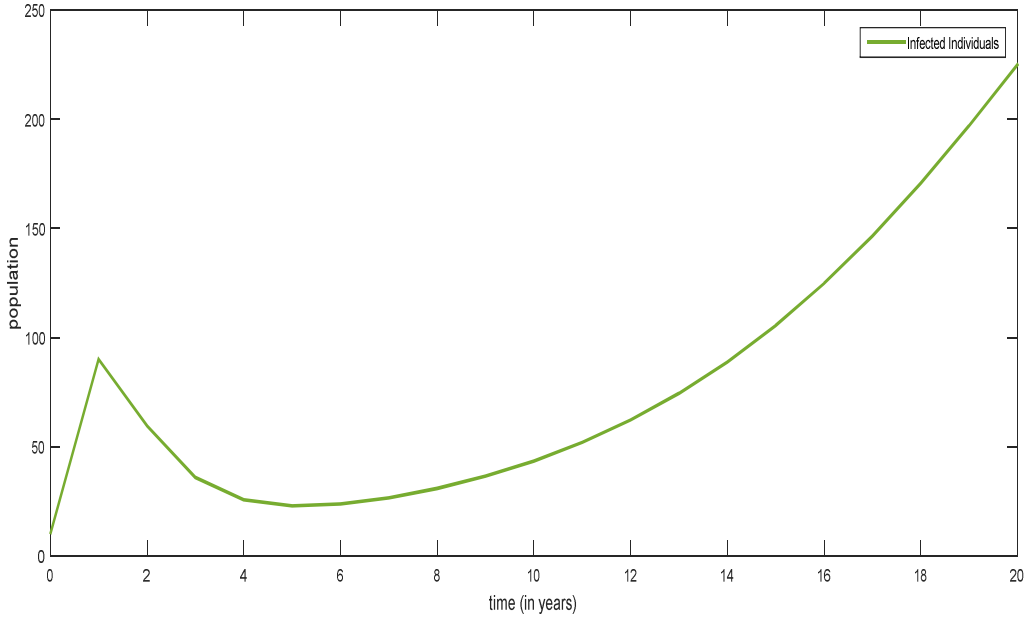

Figure 4.2: the simulation result of infected human with vaccination ($\theta = 0.2$)

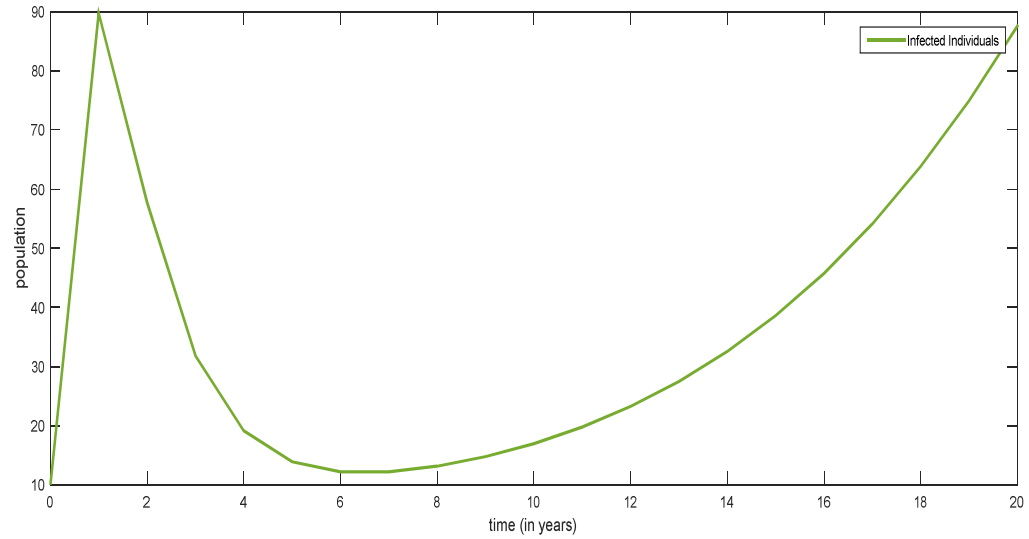

Figure 4.3: the simulation result of infected human with vaccination ($\theta = 0.6$)

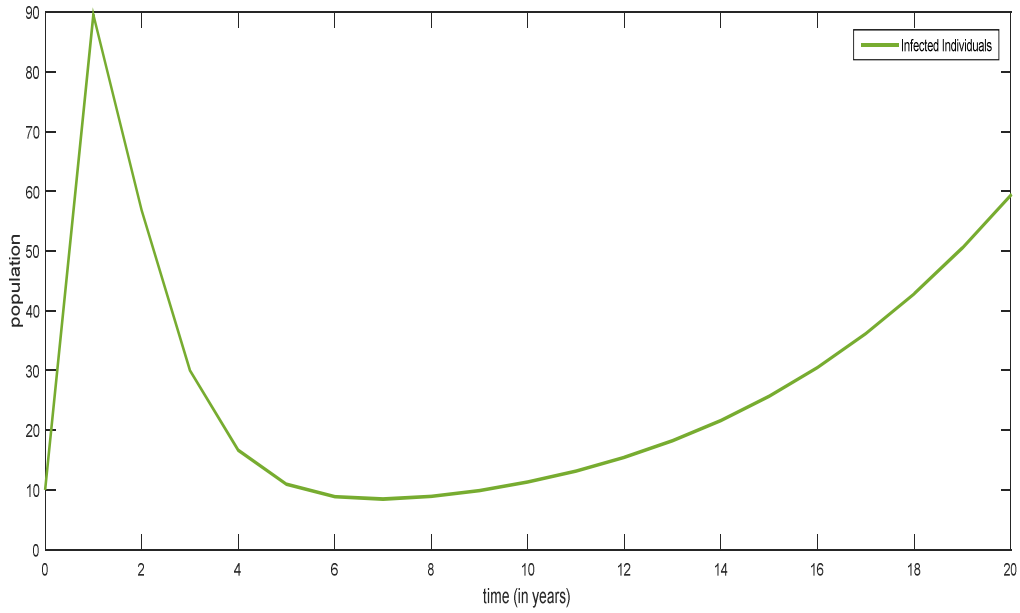

Figure 4.4: the simulation result of infected human with vaccination ($\theta = 0.9$)

4.1.8.2 Effect of quarantine on infected human population

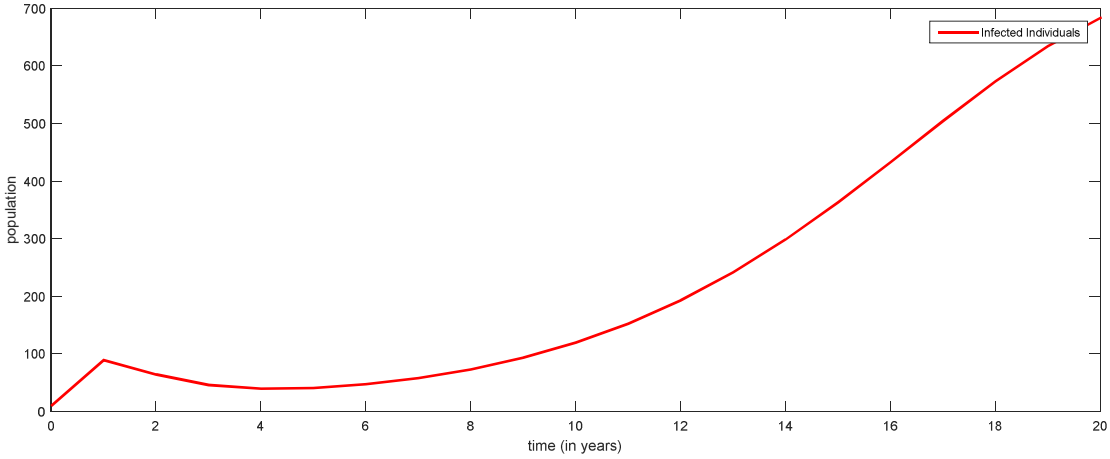

Figure 4.5: the simulation result of infected human without quarantine ($\sigma = 0$)

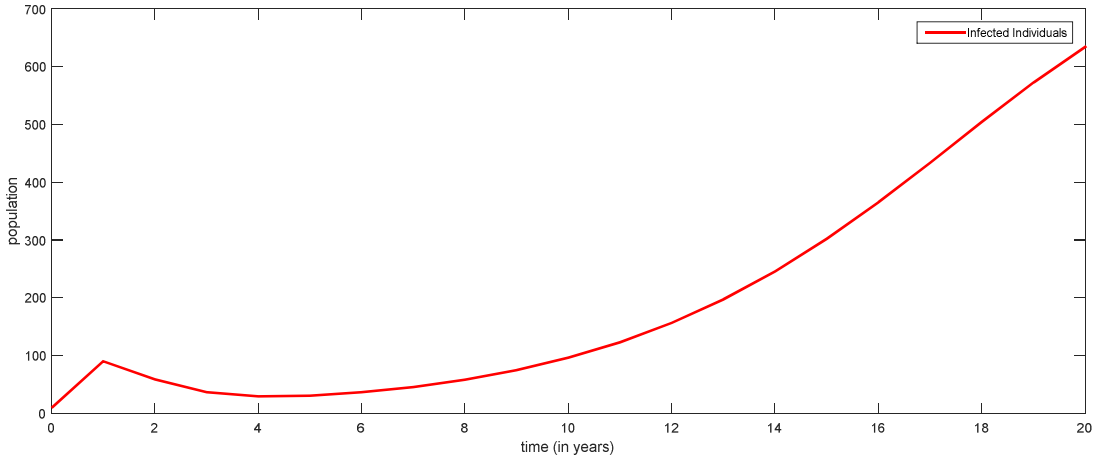

Figure 4.6: the simulation result of infected human witt quarantine ($\sigma = 0.2$)

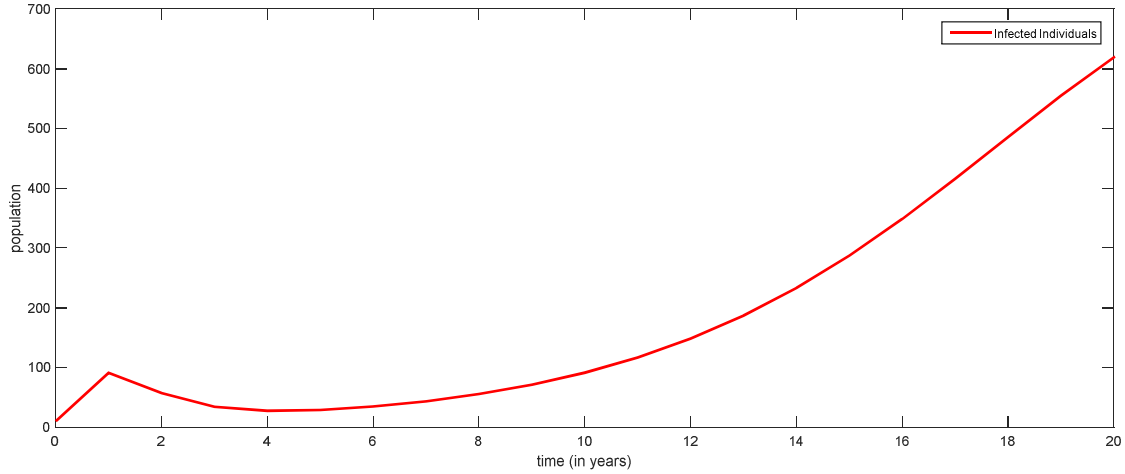

Figure 4.7: the simulation result of infected human with quarantine ($\sigma = 0.6$)

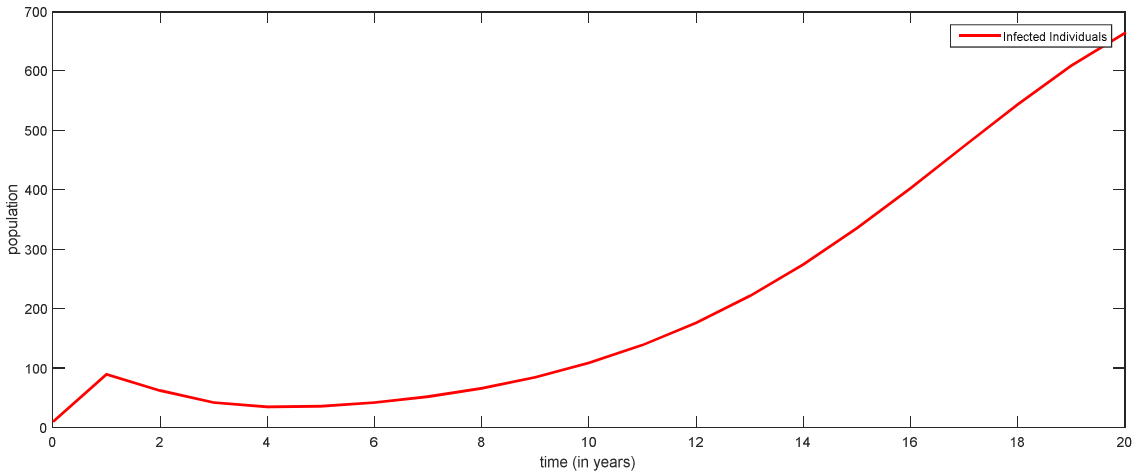

Figure 4.8: the simulation result of infected human with quarantine ($\sigma = 0.9$)

4.1.8.3 Effect of isolation on infected human population

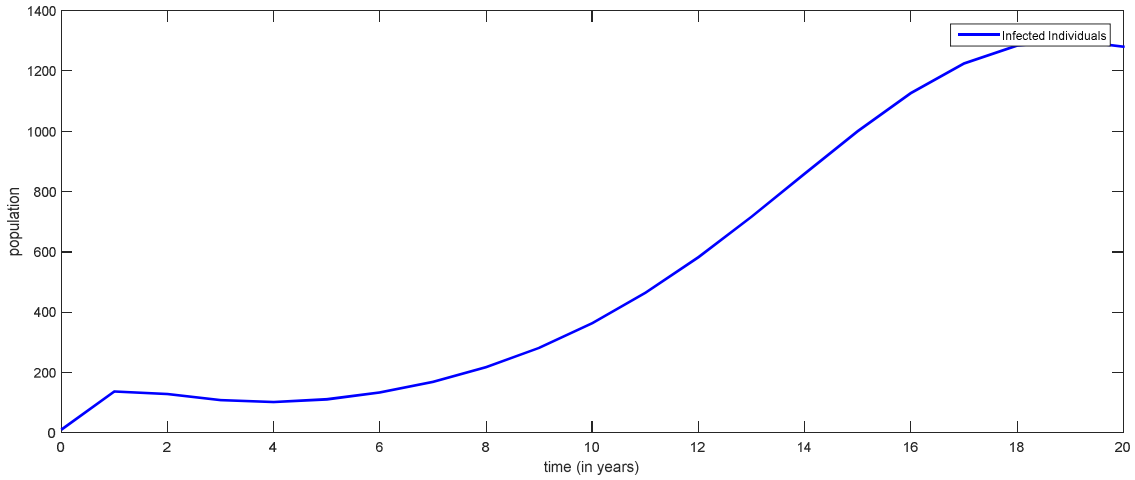

Figure 4.9: the simulation result of infected human without isolation ($\omega = 0$)

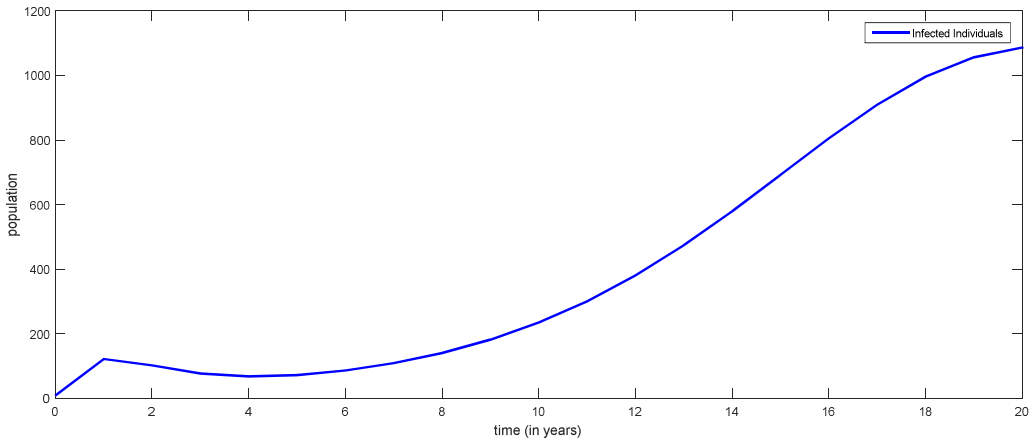

Figure 4.10: the simulation result of infected human with isolation ($\omega = 0.2$)

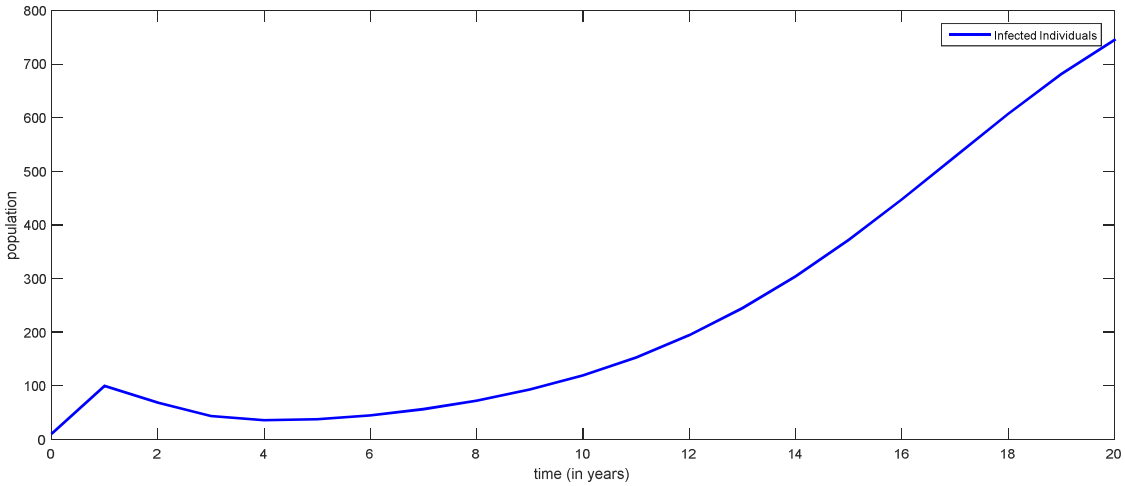

Figure 4.11: the simulation result of infected human with isolation ($\omega = 0.6$)

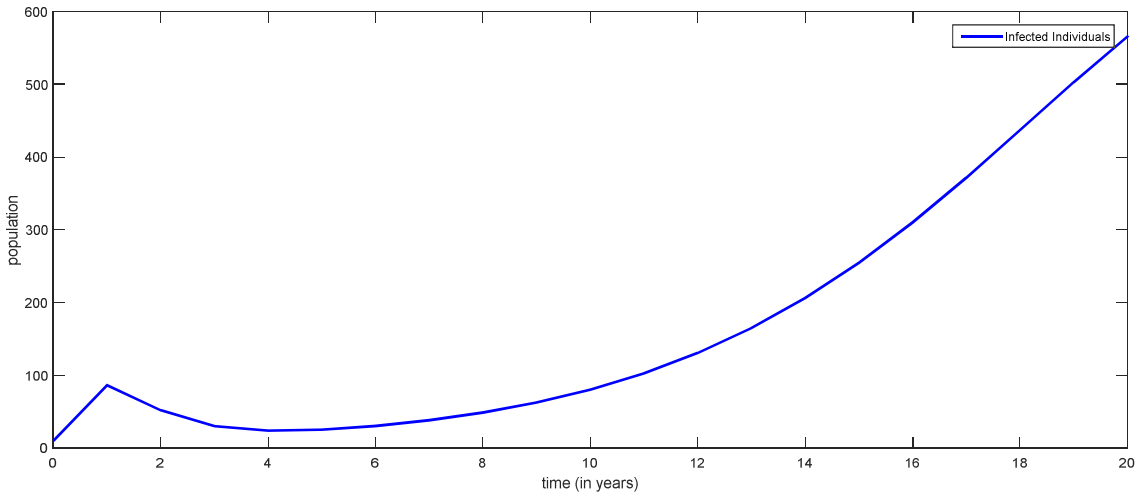

Figure 4.12: the simulation result of infected human with isolation ($\omega = 0.9$)

4.1.8.4 Effect of vaccination, quarantine and isolation *on infected human population*

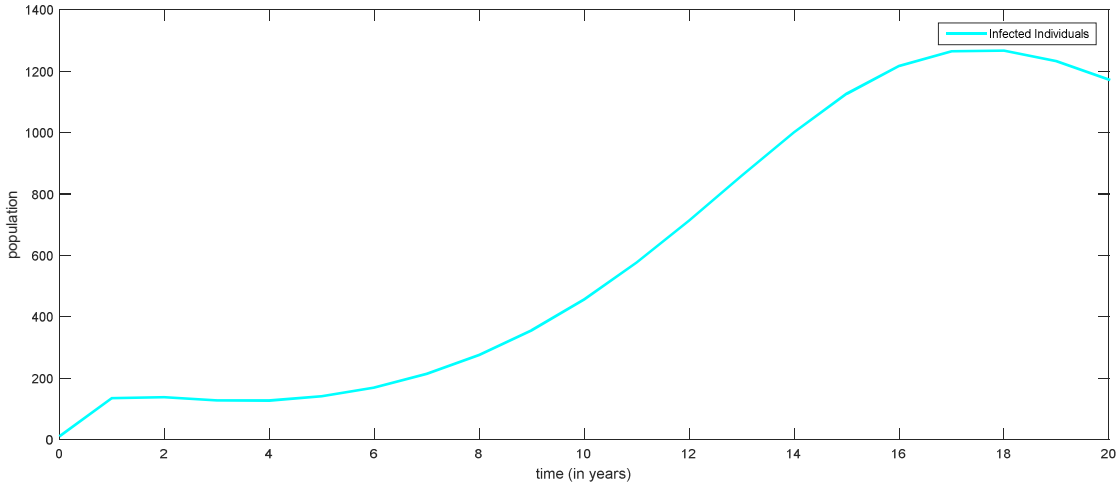

Figure 4.13: the simulation result of infected human without vaccination, quarantine and isolation ($\theta = 0,\ \sigma = 0$ and $\omega = 0$)

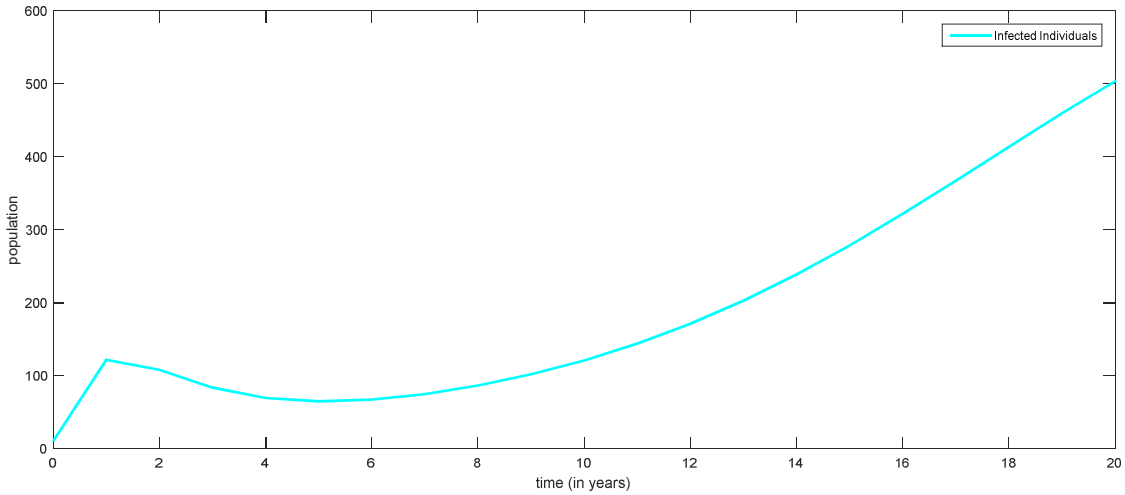

Figure 4.14: the simulation result of infected human with vaccination, quarantine and isolation ($\theta = 0.2,\ \sigma = 0.2$ and $\omega = 0.2$)

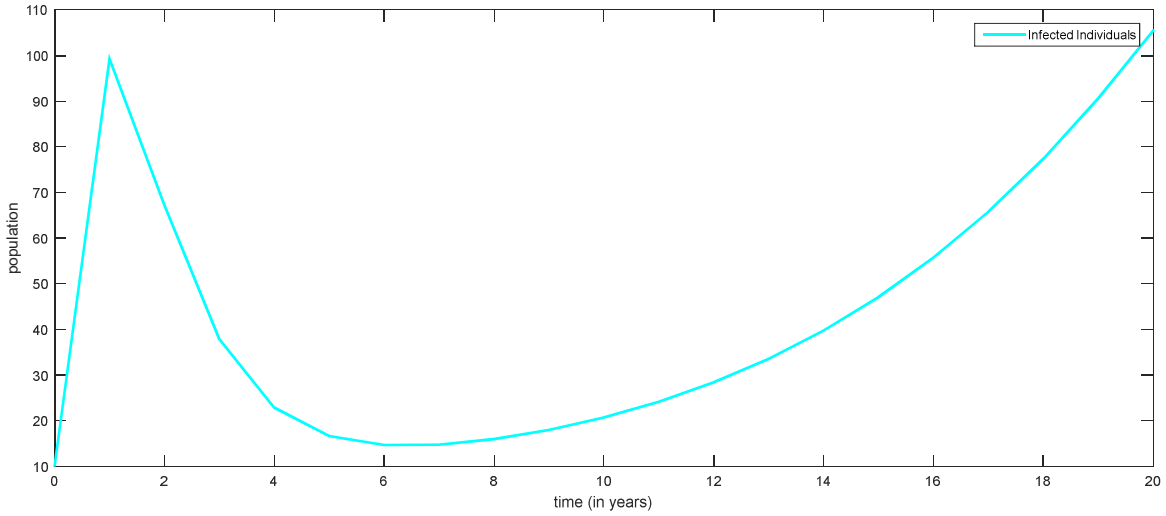

Figure 4.15: the simulation result of infected human with vaccination, quarantine and isolation ($\theta = 0.6,\ \sigma = 0.6$ and $\omega = 0.6$)

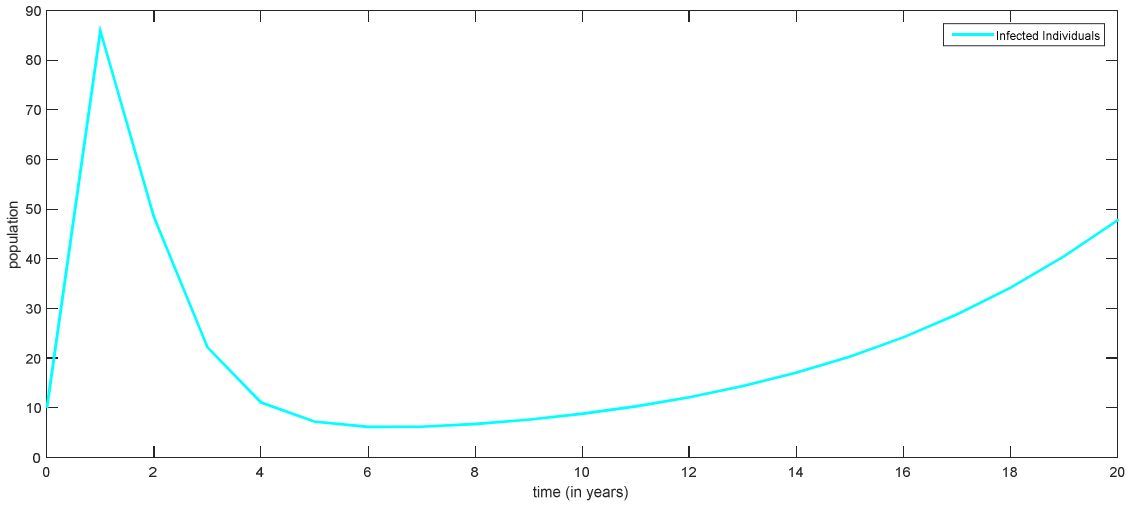

Figure 4.16: the simulation result of infected human with vaccination, quarantine and isolation ($\theta = 0.9,\ \sigma = 0.9$ and $\omega = 0.9$)

4.2 Discussion of Results

In this section, we present the discussion of both analytical and the numerical solutions results.

4.2.1 Discussion of analytical results

We established the existence and uniqueness of the solution, positivity of the solution, invariant region and disease free equilibrium for Ebola disease dynamics model. We also obtained the model's basic reproduction using the next generation matrix technique. We also established local stability of the DFE point. Lastly we obtained the global stability Of the DFE using the method of Castillo-Chavez et-al,. and is found to be globally asymptotically stable. This means the disease can be eradicated in a stable equilibrium.

4.2.2 Discussion of numerical results

The discussion of our numerical simulation is categorized in to four sections, namely: effect of vaccine on the infected human's population, effect of quarantine on the infected human's populations, effect of isolation/treatment on the infected human's population, and effect of vaccination, quarantine and isolation/treatment on the infected human's population.

4.2.3 **Discussion** *of simulation results*

Figures (4.1 – 4.4) show the effects of vaccination of humans on infected humans. The population of the infected humans is reducing when the vaccination rate is increase, in figure 4.1 the vaccination rate is 0 and the population of the infected individual is 650, in figure 4.2 the vaccination rate is 0.2 and the population of the infected individual is 250,in figure 4.3 the vaccination rate is 0.6 and the population of the infected individual is 86 and in figure 4.4 the vaccination rate is 0.9 and the population of the infected individual is 60. As the simulation results indicate, the human's vaccination plays a vital role in reduction and eradication of Ebola Virus Disease.

Figures (4.5 – 4.8) show the effects of quarantine on the infected human's population. As the simulation result indicated when the quarantine rate is increasing it reduces the number of the infected individuals as shown below;

in figure 4.5 the quarantine rate is 0 and the population of the infected individual is 680, in figure 4.6 the quarantine rate is 0.2 and the population of the infected individual is 630 ,in figure 4.7 the quarantine rate is 0.6 and the population of the infected individual is 610 and in figure 4.8 the quarantine rate is 0.9 and the population of the infected individual is 608.

Figures (4.9 – 4.12) show the effects of both isolation on infected humans population. The infected human's population grows rapidly with no treatment. But as a result of increment of isolation/treatment you can see that the infected class is reducing.in figure 4.9 the isolation rate is 0 and the population of the infected individual is 1200, in figure 4.10 the oiisolation rate is 0.2 and the population of the infected individual is 1100, in figure 4.11 the isolation rate is 0.6 and the population of the infected individual is 750 and in figure 4.12 the isolation rate is 0.9 and the population of the infected individual is 560.

Finally in the combination of all of them, i.e when the vaccination, quarantine and isolation are all zeros the infected humans population grows rapidly up to 1190, but it is reducing when the rete of vaccine, quarantine and isolation is increase to 0.9, the infected humans population is reduced to 48.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this research work, we modified the models proposed by Durojaye and Ajie (2017) by incorporating vaccination compartment for vaccinated human and additional compartments for susceptible and infected non-human vector. The Analytical studies were also carried out for the transmission dynamics of ebola virus disease model. The model consists of ten compartments with corresponding model equations. We obtained Disease free equilibrium, invariant region, positivity of the solution, and the basic reproduction number using next generation matrix method while the local and global stability of the disease free equilibrium was obtained using the Castillo-Chavez conditions. MATLAB R2015, was used to carry out numerical simulation and some of the results discused show the significant of Ebola virus disease (EVD) treatment in the population.

5.2 Conclusion

This research work is a modification of the spread and control of ebola virus disease models by combining their properties to establish transmission dynamics of Ebola virus diseases in the present of non-human vector model. The model subdivides the human population into ten compartments namely; Susceptible individuals S_H, Vaccination V_H, Exposed E_H, Quarantine Q_H, infected I_H, Isolation J_H, Recovered R_H, Decease D_H, Susceptible bat S_b, and Infected bat I_b. The analytical studies were carried out which revealed that the disease free equilibrium of the model, model is locally asymptotically stable if $R_0 < 0$. We obtained Endemic equilibrium and Global stability of disease free equilibrium. The numerical simulation carried out shows that treatment of Ebola virus diseases increase the human population. The diseases also exhibit synergetic relation where the infection of the disease speeds progressively.

5.3 Recommendations

In view of the findings of this study, we recommend that health Authorities should shade more light on the control strategies that will help to reduce the effect of Ebola virus diseases.

5.4 Contribution to Knowledge

i. The basic reproduction numbers were obtained
ii. The invariant region was obtained
iii. Positivity of the solution was obtained
iv. The stability analysis was carried out.
v. The disease free equilibrium point was obtained.
vi. Numerical simulations were also obtained.

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