OSCILLATORY PROPERTIES AND ASYMPTOTIC BEHAVIOUR OF NEUTRAL DELAY IMPULSIVE DIFFERENTIAL EQUATIONS

BY

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CERTIFICATION

I, ABASIEKWERE, UBON AKPAN with registration number MTH/Ph.D/09/002, hereby certify that this thesis on "Oscillatory Properties and Asymptotic Behaviour of Neutral Delay Impulsive Differential Equations" is original, and has been written by me. It is a record of my research work and has not been presented before in any previous publication.

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DECLARATION

We declare that this thesis entitled "*OSCILLATORY PROPERTIES AND ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF NEUTRAL DELAY IMPULSIVE DIFFERENTIAL EQUATIONS*" by Abasiekwere, Ubon Akpan (Reg. Number MTH/PhD/09/002) carried out under our supervision, has been found to have met the regulations of the University of Calabar. We, therefore recommend the work for the award of the Doctor of Philiosophy degree in Mathematics.

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[illegible] and many warm-hearted people,

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ABSTRACT

Since Sturm's famous memoir in the 17th century, it is observed that a great deal of interest has been focused on the behaviour of solutions of ordinary and delay differential equations in spite of the existence of extensive literature in these fields. Still more interesting, the theory of impulsive differential equations has brought in yet another dimension to the whole scenario and has helped to usher in a new body of knowledge for further considerations. The effects of these new inputs can be observed in the study of oscillatory properties of impulsive differential equations with deviating arguments as well as the investigation of neutral impulsive differential equations which have recently captured the attention of many applied mathematicians as well as other scientists around the world. This work considers second order neutral delay impulsive differential equations and investigates the oscillatory properties and asymptotic behaviour of its solutions. Here, we demonstrate how well known mathematical techniques and methods can be extended in the prove of theorems for the oscillation and non-oscillation of all solutions of linear and nonlinear neutral differential equations with constant and variable coefficients and retarded arguments, prove of theorems for the oscillation and non-oscillation of bounded solutions of unstable type neutral delay impulsive differential equations with constant and variable coefficients, prove of the existence of positive solutions for stable and unstable type neutral delay impulsive differential equations, prove of theorems for the oscillation of all solutions of impulsive differential equations with advanced arguments and classification of non-oscillatory solutions of the generalized form of non-linear neutral delay impulsive differential equations. all within the framework of impulsive differential equations.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

Oscillation theory of the solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. Its essence is to establish conditions for existence of oscillating (non-oscillating) solutions, to study the maxima and minima of the solutions, to obtain estimates of the distance between the neighbouring zeros and the number of zeros in a given interval, to describe the relationship between the oscillatory and other basic properties of the solutions of various classes of differential equations, etc.

The development of oscillation theory for ordinary differential equations dates back to the 1840s when the classical work of Sturm (1836) appeared. In the said work, the theorems of oscillation and comparison of the solutions of second order linear homogeneous ordinary differential equations were proved. The first oscillation results for differential equations with a translated argument were obtained by Fite (1921). He paid attention to the great differences between the oscillatory properties of the solutions of differential equations with a translated argument and of the corresponding equations without a translation of the argument.

Differential equations of neutral type, to whose aspect the present research work is devoted, play an important role in the theory of functional differential equations. In recent years, the theory of this class of equations has become an independent entity. Unfortunately, the number of research results on this subject continues to be elusive. Neutral equations find numerous applications in natural sciences and technology but, as a rule, they enjoy specific properties which make their study difficult but interesting both in aspects of ideas and techniques.

These difficulties explain the relatively small number of results devoted to the investigation of the oscillatory properties of the solutions of neutral equations. Norkin (1977) published a paper concerning the oscillation theory of neutral functional differential equations. The heavy restrictions imposed on it, however, practically eliminate the influence of the neutral member. The first work in which a criterion for oscillation of the solutions of neutral equations that proved essentially different from the classical criteria, was published by Zahariev and Bainov (1980). Some general approaches to the investigation of the oscillatory and asymptotic properties of the solutions of the neutral equations were later given by Bainov, Myshkis and Zahariev (1987, 1989), Myshkis, Bainov and Zahariev (1984), Gyori (1989) and Ntouyas and Sficas (1983).

The development of the theory of impulsive differential equations is yet another mile-stone in the history of qualitative theory of differential equations (Bainov, Dimitrova and Dishliev, 2000; Bainov and Simeonov, 1985; Bainov and Simeonov, 1986; Chen and Feng, 1997;Dishliev and Bainov, 1990; Gurgula, 1982; Krishna, Vasundlara and Satyavani, 1991; Kulev and Bainov, 1989; Kulev and Bainov, 1991; Lakshmikantham, Bainov and Simeonov, 1989; Lakshmikantham and Liu, 1989; Peng and Ge, 2000; Samoilenko and Perestyuk, 1977; Zabreiko, Bainov and Kostadinov, 1988; Zhang, Zhao and Yan, 1997; Isaac, Lipscey and Ibok, 2014). There are many monographs related to this subject (Bainov and Simeonov, 1998; Samoilenko and Perestyuk, 1995; Agarwal, Benchohra, O'Regan and Ouahab, 2004; Deo and Pandit, 1982), etc. In this direction, credit must be given to Professor Drumi Bainov, Lakshmikantham and Pavel Simeonov, to mention just a few, for their contributions in the development of the oscillatory and non-oscillatory properties for various classes of impulsive differential equations with delay and with advanced arguments.

It is worthy to note here that the theory of impulsive differential equations in general, and that of impulsive neutral differential equations in particular, were first brought into the Department of Mathematics, University of Calabar, by Isaac (2008) while presenting his Ph.D. dissertation.

The pioneering efforts of Isaac and Lipcsey over here in identifying some of the essential oscillatory and non-oscillatory conditions of neutral impulsive differential equations of the first order is also worth commending (*Oscillations in Systems of Neutral Impulsive Differential Equations*, Isaac and Lipcsey, 2009d; *Oscillations in Non-Autonomous Neutral Impulsive Differential Equations with Several Delays*, Isaac and Lipcsey, 2009c; *Linearized Oscillations in Nonlinear Neutral Delay Impulsive Differential Equations*, Isaac and Lipcsey, 2009a; *Oscillations in Neutral Impulsive Logistic Differential Equations,* Isaac and Lipcsey, 2009b; *Oscillations in Neutral Impulsive Differential Equations with Variable Coefficients*, Isaac and Lipcsey, 2010b; *Oscillations in Linear Neutral Delay Impulsive Differential Equations with Constant Coefficients,* Isaac and Lipcsey, 2010a; *Nonoscillatory and Oscillatory Criteria for First Order Nonlinear Neutral Impulsive Differential Equations*, Isaac, Lipcsey and Ibok, 2011a; *Oscillatory Conditions on Both Directions for a Nonlinear Impulsive Differential Equation with Deviating Arguments*, Isaac, Lipcsey and Ibok, 2011b; *Linearized Oscillations in Autonomous Delay Impulsive Differential Equations*, Isaac and Lipcsey, 2007). The results of their subsequent investigations reveal that neutral impulsive differential equations are dependable tools, not only in applied mathematics, but also in science in general. For example, in the present drive to improve information and computer technology (ICT), neutral impulsive differential equations remain at the center stage. Indeed, neutral impulsive differential equations appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching). They are involved in the study of vibrating masses attached to an elastic bar, and also as Euler equation in some problems of variation (Gyori and Ladas, 1991).Therefore, through the study of oscillations of neutral impulsive differential equations, one gets deeper insight into the dynamics of solutions to equations modelling applied problems in engineering, technology and natural sciences.

There is no doubt that this study will further improve upon the existing results in view of its role and timing of the oscillatory and asymptotic properties of neutral impulsive differential equations of the second order.

At this point we define a general second order neutral delay impulsive differential equation as follows:

$$\begin{cases} [y(t)+p(t)y(t-\tau)]''+q(t)y(t-\sigma)=0, & t\neq t_k \\ \Delta\left[y(t_k)+p_k y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0, & t=t_k, \quad 1\leq k\leq\infty. \end{cases}$$

This is an equation with the impulsive conditions in which the second order derivative of the unknown function appears in the equation, both with and without delay. It is worth mentioning here that our aim is not to find the unknown function or solution $y(t)$, but to determine its nature and behaviour in oscillatory sense.

1.2 Basic definitions

Let E be our set of subscripts which can be the set of natural numbers $\mathbb{N}$ or the set of integers $\mathbb{Z}$. Except otherwise stated, we will assume that the elements of the sequence $S := \{t_k\}_{k\in E}$ are the moments of impulsive effects and satisfy the following properties:

H1.2.1: If t_k is defined $\forall\, k \in \mathbb{N}$ then $0 < t_k < t_{k+1}$, $\forall k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} t_k = \infty$;

H1.2.2: If t_k is defined $\forall\, k \in \mathbb{Z}$ then $t_k < t_{k+1}$, $\forall k \in \mathbb{Z}$ and $\lim\limits_{k\to\pm\infty} t_k = \pm\infty$.

Definition 1.1. The differential equation

$$\begin{cases} y^{(n)}(t)=f(t,y(t),y'(t),y''(t),...,y^{(n-1)}(t)), & t\notin S \\ \Delta y^{(n-1)}(t_k)=f_k(y(t_k),y''(t_k),...,y^{(n-1)}(t_k)), & t_k\in S, \end{cases} \tag{1}$$

where

$$\Delta y(t_k) = y(t_k+0) - y(t_k-0)$$

and

$$y(t_k+0)=\lim_{\varepsilon\searrow 0} y(t_k+\varepsilon) \text{ and } y(t_k-0)=\lim_{\varepsilon\nearrow 0} y(t_k+\varepsilon)$$

is called an nth order impulsive differential equation.

Definition 1.2. The function $y(t)$ is said to be the solution of equation (1) in the interval $J=(\alpha,\beta)\subset R$ if

i) the function $y(t)$ admits nth order derivative $y^{(n)}(t)$ and satisfies the equation $y^{(n)}(t)=f(t,y(t),y'(t),y''(t),...,y^{(n-1)}(t))$ for $t\in J,\ t\notin S$;

ii) the functions $y(t)$ and $y'(t),...,y^{(n-1)}(t)$ satisfy the relations

$$y^{(n-1)}(t_k^+)-y^{(n-1)}(t_k^-)=f_k(y(t_k^-),y'(t_k^-),y''(t_k^-),...,y^{(n-1)}(t_k^-)),$$

$$y^j(t_k^+)=y^{j-1}((t_k^-),\ \ 0\le j\le n-1 \ \text{ for } \ t_k\in J\cap S.$$

Usually, the solution $y(t)$ for $t\in J,\quad t\notin S$ of the impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k,\ \ t_k\in J\cap S$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r\in N,\ D:=[T,\infty)\subset R$ and let the set S be fixed. We denote by $PC(D,R)$ the set of all values $\psi:D\to R$ which is continuous for all $t\in D,\ t\notin S$. They are functions from the left and have discontinuity of the first kind at the points for $t\in S$. By $PC^r(D,R)$, we denote the set of functions $\psi:D\to R$ having derivative $\dfrac{d^j\psi}{dt^j}\in PC(D,R),\quad 0\le j\le r$. To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S),\ \ r\in N$.

1.3 Scope and objectives of the study

Whereas Isaac's work (2008) focused on first order neutral impulsive differential equations, the present investigation is deeply concerned with the second order neutral delay impulsive differential equations and is intended to investigate the oscillatory properties and asymptotic behaviour of their solutions. The reason for the choice of this topic is that the area presently attracts very little attention, and as such, very few results are known. Also, one of Isaac's recommendations was the extension of his oscillation results to second and higher order neutral delay impulsive differential equations. In view of these, we systematically present the results and demonstrate how well known mathematical techniques and methods can be extended in the

i) Prove of theorems for the oscillation of all solutions of linear neutral impulsive differential equations with constant and variable coefficients and retarded arguments;

ii) Prove of theorems for the oscillation of all solutions of nonlinear neutral impulsive differential equations with constant and variable coefficients and retarded arguments;

iii) Prove of theorems for the oscillation and non-oscillation of bounded solutions of unstable type neutral delay impulsive differential equations with constant and variable coefficients;

iv) Prove of theorems for the existence of positive solutions for stable and unstable type neutral delay impulsive differential equations;

v) Classification of non-oscillatory solutions of the generalized form of non-linear neutral delay impulsive differential equations.

vi) Prove of theorems for the oscillation of all solutions of neutral delay impulsive differential equations with nonlinear neutral term;

vii) Prove of theorems for the oscillation of all solutions of neutral delay impulsive differential equations with a forcing term;

viii) Prove of theorems for the oscillation of all solutions of neutral impulsive differential equations with advanced arguments;

within the framework of impulsive differential equations. *Applications will be* considered in those cases where they are possible and needed to drive home the understanding of the expected results.

CHAPTER TWO

LITERATURE REVIEW

In this chapter, we present a review of literature on various studies carried out in the fields of related disciplines, namely, the oscillation theory for ordinary differential equations with delay and the oscillation theory of neutral ordinary differential equations.

2.1 Introduction

In chapter one, we remarked that since Sturm's famous memoir in 1836, oscillation theory has become an important area of research in the qualitative theory of ordinary differential equations (Angelova and Bainov, 1981, 1982a,b; Brands, 1978; Burkowski, 1971; Burton and Haddock, 1976; Chen, 1977, 1978; Foster and Grimmer, 1979; Garner, 1975; Graef, 1983; Grammatikopoulos, 1977; Hino, 1974; Isaac, 2008; Ivanov and Shevelo, 1981; Kartsatos and Manougian, 1976; Kung, 1971; Lillo, 1969; Lim, 1976; Liossatos, 1970; Onose, 1982; Graef, Katamura, Kusano and Spikes, 1979).

Oscillation theory of ordinary differential equations with delay is a natural extension of oscillation theory of ordinary differential equations, being that some results from oscillation theory of ordinary differential equations carry over to the said differential equations with delay. By this, some fundamental knowledge in oscillation theory for ordinary differential equations is essential for an understanding of the oscillation theory of ordinary differential equations with delay.

Some facts about ordinary differential equations are now presented here.

2.2 Definitions of oscillation

There are various definitions of the oscillation of solutions of ordinary differential equations (with or without delay). Here, we shall list some definitions used most extensively in this context and which are similar to those most frequently used in literature.

To achieve our goal, we shall restrict our discussion to those solutions $y(t)$ of the equation

$$y''(t) + a(t)y(t - \tau(t)) = 0 \tag{2}$$

which exist on some interval $[T_y, \infty)$, $T_y \geq 0$ and satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for every $T \geq T_y$. In other words, $|y(t)| \neq 0$ on any unbounded interval $[T, \infty)$. Such a solution sometimes is said to be a regular solution.

We shall assume that $a(t) \geq 0$ or $a(t) \leq 0$ in equation (2), and in doing so we imply that $a(t) \neq 0$ on any unbounded interval $[T, \infty)$.

Definition 2.1. A nontrivial solution $y(t)$ (implying a regular solution always) is said to be oscillatory if and only if it has arbitrary large zeros for $t \geq t_0$, that is, there exist a sequence of zeros $\{t_n\}_{n=1}^{\infty}$ $[y(t_n) = 0]$ of $y(t)$ such that $\lim_{n\to\infty} t_n = +\infty$, otherwise $y(t)$ is said to be non-oscillatory (Isaac, 2008).

For non-oscillatory solutions there exist a t_1 such that $y(t) \neq 0$, for all $t \geq t_1$. This means that throughout the range, $y(t)$ must be eventually positive or eventually negative. That is, $y(t)$ is positive for all $t \geq t_1$ or is negative for all $t \geq t_1$.

Definition 2.2. A nontrivial solution $y(t)$ is said to be oscillatory if it changes sign on (T, ∞), where T is any number (Isaac, 2008).

Notice that when $\tau(t) \equiv 0$ and $a(t)$ is continuous in equation (2), the two definitions are equivalent. However, for higher order equations where the possibility of multiple zeros of non-trivial solutions is likely, it becomes increasingly difficult to sustain this balance. The above definitions can be extended to include systems of equations with delays. Let us see how this is demonstrated in the case of two-dimensional first order systems.

Consider the first order system of equations with deviating arguments

$$\begin{cases} x'(t) = f_1(t, x(t), x(\tau_1(t)), y(t), y(\tau_2(t))) \\ y'(t) = f_2(t, x(t), x(\tau_1(t)), y(t), y(\tau_2(t))). \end{cases} \tag{3}$$

The solution $(x(t), y(t))$ is said to be strongly oscillatory if each of its components is oscillatory and weakly oscillatory if at least one of its components is oscillatory.

2.3 Oscillation theory for ordinary differential equations

As mentioned earlier, we shall recall only those facts concerning oscillation theory of ordinary differential equations that will be useful in our discussion.

We consider a second order linear ordinary differential equation

$$y''(t) + a(t)y(t) = 0. \tag{4}$$

Sturm's comparison theorem for equation (4) is very important in oscillation theory (Leighton, 1981). Using this comparison theorem, it is easy to draw the following conclusions:

i) For the linear ordinary differential equation (4), solutions are either all oscillatory or all non-oscillatory. Equation (4) is said to be oscillatory if every solution of it is oscillatory and it is said to be non-oscillatory otherwise.

ii) We consider another second order linear ordinary differential equation

$$y''(t) + b(t)y(t) = 0. \tag{5}$$

If $a(t) \leq b(t)$ for all $t \geq t_0$ and equation (4) is oscillatory, then so is equation (5). Moreover, from (i), if equation (5) is oscillatory, then so is equation (4)

(Domshlak, 1982; Grace and Lalli, 1989; Sibgatullin, 1980; True, 1975).

Using Sturm's comparison theorem, we can obtain the oscillatory property of an ordinary differential equation from some other ordinary differential equation with known oscillatory behaviour. In fact, many good oscillatory criteria have been obtained from Sturm's comparison theorem.

iii) Assume that $a(t) \leq 0$, then equation (4) is non-oscillatory. This follows from (ii).

The comparison method is one of the important methods in oscillatory theory of second order linear ordinary differential equations (Barrett, 1969; Swanson, 1968; Willet, 1969). There is much literature on the extension of the comparison method to nonlinear and higher order differential equations. Most relevant among them include studies by Atkinson (1955), Butler (1979), Macki and Wong (1968), Wong (1968), Wong (1975), Philos (1984) and Isaac (2008).

Now we consider a second order non-linear ordinary differential equation

$$y''(t) + a(t)f(y(t)) = 0. \tag{6}$$

The interest in nonlinear oscillation problems for equations of this type began with the publication of the pioneering work by (Atkinson, 1955). We would like to point out that the nonlinearity of equation (6) may generate both oscillatory and non-oscillatory solutions (Liossatos, 1970; Lovelady, 1975; Macki and Wong, 1968; Yan, 1983; Yeh, 1980; Yoshizawa, 1970; Zhang, 1980; Zhang, Ding, Feng, Wu and Wang, 1982).

A special case of equation (6) is represented as

$$y''(t) + a(t)y^{\alpha}(t) = 0. \tag{7}$$

Equation (7) is said to be superlinear if $\alpha > 1$ and sublinear if $\alpha < 1$. (Ladde, Lakshmikantham and Zhang, 1987; Burkowski, 1971). We usually need to distinguish between these cases in our study because of the difference in the type of results that are known (Isaac, 2008; Kusano and Onose, 1974; Kusauo and Onose, 1973; Sficas and Stavroulakis, 1987;Graef, Grammatikopoulos and Spikes, 1980 ; Grammatikopoulos, Sficas and Staikos, 1979)

For instance, consider the equation

$$y''(t) + a(t)\left|y(t)\right|^{\alpha} sgn\, y(t) = 0, \tag{8}$$

where $a(t) \in C(R_{+})$ and $a(t) \geq 0$. Then, for $\alpha \neq 1$ (superlinear), equation (7) is oscillatory if and only if $\int\limits_{0}^{\infty} sa(s)ds = \infty$.

2.4 Second order linear differential equations with delay

Mathematical modeling of several real-world problems leads to differential equations that depend on the past history rather than only the current state. The models may have discrete time lags or delays.

In recent years, there has been much research activity concerning the oscillation of solutions of delay differential equations and, to a large extent, this is due to the realization that delay differential equations are important in applications. New applications which involve delay differential equations continue to arise with increasing frequency in the modeling of diverse phenomena in physics, biology, ecology and physiology.

Much of the work in the theory of oscillations center on second order or higher order ordinary differential equations, but in this section, we'll be looking at second order linear differential equations with delay. The oscillatory behaviour of functional differential equations with delay has been the subject of intense study in the last three decades (Dosly and Rehak, 2005; Gyori and Ladas, 1991; Agarwal, Grace and O'Regan, 2002)

The oscillatory behaviour of a functional differential equation with delay and of the associated ordinary differential equation are not always the same. Indeed, the delay differential equation

$$y''(t) + y(t - \pi) = 0$$

admits $\sin t$ and $\cos t$ as oscillatory solutions. On the other hand, the associated ordinary differential equation

$$y''(t) - y(t) = 0$$

has the non-oscillatory solutions e^{-t} and e^{t}. Conversely, we see that the delay differential equation

$$y''(t) - \frac{1}{2t^2} y\left(\frac{t}{4}\right) = 0, \ t > 0$$

has a non-oscillatory solution $y(t) = \sqrt{t}$, while the associated ordinary differential equation $y''(t) - \frac{1}{2t^2} y(t) = 0$ admits $t \cos \ln t$ and $t \sin \ln t$ as oscillatory solutions. Such a change in the oscillatory behaviour of a differential equation is obviously generated or disrupted by the delay, and so the study of oscillatory solutions of differential equations with delay is very important in applications. As an example, oscillations caused by delays should be seriously taken into account in studying the motion of a controlled craft moving with increasing velocities, where it is possible to have a sudden release of oscillations leading to instability (Minorsky, 1962).

In this section, we attempt to uncover relevant literatures and as well present the state of the art in this rapidly growing area. We shall begin with second order linear ordinary differential equations with delay and proceed to the non-linear equivalent of it exposing, where possible, the various techniques of extracting the oscillatory properties of the solutions.

2.4.1 Classification of solutions of linear equations

Consider the second order linear differential equation with delay in the general form

$$f(t, y(t), y'(t), y''(t), y[t-\tau(t)], y'[t-\tau(t)], y''[t-\tau(t)]) = 0, \tag{9}$$

where $\tau(t) > 0$. Let t_0 be the given initial point. The delay $\tau(t)$ defines the initial set E_{t_0} given by

$$E_{t_0} = \{t_0\} \bigcup \{t - \tau(t) < t \text{ for } t > t_0\}.$$

On E_{t_0} we shall assume that continuous functions $\varphi_k(t)$, $k = 0, 1$ are given. Furthermore, for equation (9), we shall assume that the initial values $y_0^{(k)}(t)$, $k = 0, 1$ are known, and $\varphi_0(t_0) = y_0^{(0)}$. Also, for equation (9), the basic initial value problem consist of finding a continuously differentiable function y that satisfies it for $t \geq t_0$ and conditions

$$y^{(k)}(t_0 + 0) = y_0^{(k)}, \; k = 0, 1$$

and

$$y^{(k)}[t - \tau(t)] = \varphi_k[t - \tau(t)] \text{ if } t - \tau(t) < t_0, \; k = 0, 1.$$

In oscillation theory, we study solutions which are defined on an half open interval $[t_0, \infty)$. Therefore, we are interested only in those equations for which global existence theorems can be established.

A non-trivial solution $y(t)$ of equation (9) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, $y(t)$ is said to be non-oscillatory, i.e., $y(t)$ is non-oscillatory if there exist a $t_1 \geq t_0$ such that $y(t) \neq 0$ for $t \geq t_1$. In other words, a non-oscillatory solution must be eventually positive or negative. Equation (9) itself is said to be oscillatory if all its solutions are oscillatory (Agarwal, Grace and O'Regan, 2003).

In this section, we extend the results of Norkin (1972) in the classification of solutions of initial value problems of the type

$$(r(t)y'(t))' = p(t)y(g(t)), \tag{10}$$

where $p(t),\ \ g(t),\ \ r(t) \in C(R_+, R_+),\ \ g(t) \leq t,\ \ r(t) > 0$ and $g(t)$ is the general delay function.

Initial conditions are given as follows:

$$y(s) = \varphi(s) \text{ for } s \in E_{t_0},\ \ y(t_0) = y_0,\ \ y'(t_0) = y_0',$$

where

$$E_{t_0} = \{t_0\} \cup \{g(t) < t_0,\ \ t > t_0\}, \quad \varphi \in C(E_{t_0}).$$

Definition 2.3. Let S denote the set of all solutions of equation (10). We define the following subsets of S :

$$S^{+\infty} = \{y \in S : \lim_{t\to\infty} y(t) = \infty\},$$

$$S^{-\infty} = \{y \in S : \lim_{t\to\infty} y(t) = -\infty\},$$

$$S^{k} = \{y \in S : 0 < \lim_{t\to\infty} y(t) < \infty\},$$

$$S^{-k} = \{y \in S : -\infty < \lim_{t\to\infty} y(t) < 0\},$$

$$S^{0} = \{y \in S : y(t) \neq 0 \text{ and } \lim_{t\to\infty} y(t) = 0 \text{ monotonically }\},$$

$$S^{\sim} = \{y \in S : y(t) \text{ is oscillatory }\}.$$

We now present some sufficient conditions for the qualitative behaviour of the solutions of equation (10). We begin by considering the following lemma:

Lemma 2.1. Assume that

i) $p \geq 0, \ \ r > 0$ are continuous;

ii) $g \in C(R_+, R_+), \ \ g(t)$ is non-decreasing, $g(t) \leq t$ and $\lim_{t\to\infty} g(t) = \infty$;

iii) $\lim_{t\to\infty} \int_{t_n}^{t} \frac{ds}{r(s)} = \infty$.

Then it can be shown that

i) $\varphi(t) \geq 0$ on E_{t_0} and $y_0' > 0$ imply $y(t, \varphi, y_0') \in S^{\infty}$;

ii) $\varphi(t) \leq 0$ on E_{t_0} and $y_0' < 0$ imply $y(t, \varphi, y_0') \in S^{-\infty}$.

Again, let conditions (i) and (ii) of Lemma 2.1 be satisfied and further assume that

$$\int_0^{\infty} (R(t) - R(s))p(s)ds = \infty,$$

where $R(t) = \int_{t_n}^{t} \dfrac{ds}{r(s)}$,
then, it is readily seen that

i) $\varphi(t) \geq 0$ on E_{t_0}, $\varphi(t) \not\equiv 0$ and $y_0' \geq 0$ imply $y \in S^{+\infty}$;

ii) $\varphi(t) \leq 0$ on $E_{t_0}, \varphi(t) \not\equiv 0$ and $y_0' \leq 0$ imply $y \in S^{-\infty}$ (Ladde, Lakshmikantham and Zhang, 1987).

A very strong condition ensuring the correctness of the above statement is the fact that

$$\lim_{t\to\infty} \int_{t_n}^{t} \frac{ds}{r(s)} = \infty.$$

Note that here we have imposed the continuity condition on $g(t)$, $r(t)$ and $p(t)$, where $g(t)$ is further restricted to be non-decreasing.

Now, assuming that Lemma 2.1 holds and assuming that

$$\int_{0}^{\infty} R(s)p(s)ds = \infty,$$

it can also be seen that every solution of equation (9) belongs to either S^0 *or* $S^{\sim}$.

In a related development, Professor Agarwal observed something similar. He considered the second order linear delay differential equation

$$y''(t) + p_1 y'(t-\tau_1) + p_2 y'(t-\tau_2) + q_1 y'(t-\sigma_1) + q_2 y'(t-\sigma_2) = 0, \qquad (11)$$

where the coefficients $p_1,\ p_2,\ q_1,\ q_2$ and the delays $\tau_1,\ \tau_2,\ \sigma_1,\ \sigma_2$ are non-negative numbers. The characteristic equation of equation (11)is

$$F(\lambda) = \lambda^2 + p_1\lambda e^{-\lambda\tau_1} + p_2\lambda e^{-\lambda\tau_2} + q_1 e^{-\lambda\sigma_1} + q_2 e^{-\lambda\sigma_2} = 0. \qquad (12)$$

The oscillatory behaviour of solutions of equation (11) depends on the location of the roots of equation (12). In fact, the following theorem provides necessary and sufficient condition for the oscillation of equation (11).

Theorem 2.1. *The following statements are equivalent:*

i) Every solution of equation (11) *is oscillatory.*

ii) The characteristic equation (12) *has no real roots (Agarwal et al., 2003).*

2.4.2 Existence of bounded oscillatory solutions

The methods of ordinary differential equations are adapted to delay differential equations to obtain oscillation and non-oscillation criteria for linear delay differential equations which are similar to known criteria for ordinary differential equations (Bradley, 1970; G., 1971; Gollwitzer, 1969; Odaric and Sevelo, 1971; Sevelo and Odaric, 1968; Shere, 1973; Staikos, 1970; Staikos and Petsoulas, 1970; Travis, 1972; Waltman, 1968). In particular, the second order linear delay differential equation

$$y''(t) + q(t)y(g(t)) = 0$$

with $q(t) > 0$, $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$ has been investigated by a number of authors (Bradley, 1970; G., 1971; Gollwitzer, 1969; Odaric and Sevelo, 1971; Sevelo and Odaric, 1968; Shere, 1973; Staikos, 1970; Staikos and Petsoulas, 1970; Travis, 1972; Waltman, 1968).

In the study of oscillation and non-oscillation of differential equations the restriction on the solution to be continuous is required. Consider the equation

$$y''(t) - y(t) = 0 \tag{13}$$

As we know, equation (13) has no oscillatory solution. On the other hand, consider the same equation with delay π, so that we have

$$y''(t) - y(t - \pi) = 0. \tag{14}$$

It is easy to check that $y_1 = \sin t$, $y_2 = \cos t$ are oscillatory solutions of equation (14).

Now, let us consider the general linear equation

$$y''(t) - p(t)(t - \tau(t)) = 0; \;\; p(t) \geq 0, \;\; t \geq t_0 \geq 0. \tag{15}$$

We pose the problem: What conditions guarantee the presence of oscillatory solutions for equation (15)? It can be immediately noticed that this problem has no meaning when $\tau(t) \equiv 0$. Ladde et al. (1987) obtained some sufficient conditions for every bounded solution to be oscillatory. The following results were obtained.

Theorem 2.2. *Assume that the hypothesis (i),(ii) and (iii) of Lemma 2.1 are satisfied. Further assume that*

$$\lim_{t\to\infty} sup \frac{1}{r(t)} \int_{g(t)}^{t} (z - g(t))p(z)dz > 1. \qquad (16)$$

Then every bounded solution of equation (10) *is oscillatory.*

Gustafson (1974) verified this by proving a contradiction. The solution $y(t) > 0$ is assumed unbounded, and on integrating equation (10) by parts from s to t and applying the monotonicity condition of $y(t)$, we arrive at a contradiction to equation (16); i.e., $y(t)$ is a bounded oscillatory solution.

Closely related to this are the following results:

Corollary 2.1. If $\tau \geq 0$, $p(t) \geq 0$ is continuous, and $\tau^2 p(t) \geq 2$ for $t \geq 0$, then bounded solutions of the equation

$$y''(t) - p(t)y(t - \tau) = 0$$

are oscillatory (Ladas and Lakshmikantham, 1974).

Corollary 2.2. If $k > 1$, $p(t) \geq 0$ is continuous and

$$p(t) \geq \frac{2k^2}{((1-k)t)^2}$$

for large t, then bounded solutions of

$$y''(t) - p(t)y\left(\frac{t}{k}\right) = 0$$

are oscillatory.

The results of these two corollaries can be made clearer by considering these examples:

Example 2.1. The equation

$$\left(\frac{1}{t}y'\right)' - 4ty\sqrt{t^2-\pi} = 0; \;\; t \geq 2$$

satisfies the condition of Theorem 2.2. Therefore, all bounded solutions are oscillatory. In particular, $y(t) = \cos t^2$ is a bounded oscillatory solution.

Example 2.2. Again, the equation

$$y''(t) - p(t)y(t-\pi) = 0, \;\; 0 \leq \tau \geq 2e^{-1} \tag{17}$$

does not satisfy the conditions of Theorem 2.2 as expected. Equation (17) has a bounded non-oscillatory solution. Indeed, the characteristic equation $F(\lambda) = \lambda^2 - e^{-i\lambda\tau} = 0$ has negative real root λ, and hence $y(t) = e^{\lambda t}$ is a bounded non-oscillatory solution. Note that if we do not require that $\int\limits_0^\infty \frac{ds}{r(s)} = \infty$, but ensure that $r(t)$ is non-decreasing and equation (17) is satisfied, then the conclusion of Theorem 2.2 remains valid.

Now, there exist many relevant interesting conditions for the theory of oscillation associated with differential equations with several delays. Let us consider the linear equation with several delays

$$y''(t) - \sum_{i=1}^{n} p_i(t)y(g_i(t)) = 0. \tag{18}$$

The following result is obtainable.

Theorem 2.3. *Assume that*

1. $p_i, \; g_i \in C([0,\infty), R), \quad p_i \geq 0, \;\; i = 1, \; 2, \; \cdots, \; n,$ *and for some index* $i_0, \; 1 \leq i_0 \leq n, \;\; p_{i_0}(t) > 0$ *for* $t \geq 0$;
2. $g_i(t) \leq t$ *and* $\lim\limits_{t\to\infty} g_i(t) = \infty$ *for* $i = 1, \; 2, \; \cdots, \; n$;
3. *There exist a non-empty set of indices* $K = \{k_1, \; k_2, \; \cdots, \; k_\ell\}, \;\; 1 \leq k_1 <$

$k_2 < \cdots < k_\ell < n$, such that for $t \geq t_0$, $g_k(t) < t$ and $g_k'(t) \geq 0$ for $k \in K$ and

$$\lim_{t\to\infty} \sup \sum_{k\in K} \int_{g^*(t)}^{t} [g_k(t) - g_k(s)]p_k(s)ds > 1,$$

where $g^*(t) \equiv \max_{k\in K} g_k(t)$.
Then every bounded solution of equation (18) is oscillatory (Ladas, Ladde and Papadakis, 1972).

This verification is done by assuming the non-boundedness of the solution $y(t)$ of equation (16), and without loss of generality we can say that $y(t) > 0$. Due to the condition on $g_i(t)$, there exist a $t_1 \geq t_0$ such that $y(g_i(t)) > 0$ for $t \geq t_1$ and for $i = 1, 2, \cdots, n$. In view of equation (16), we have that $y''(t) > 0$ for $t \geq t_1$. From the boundedness condition on $y(t)$, it can be seen that there exist a $t_2 \geq t_1$ such that $y'(t) < 0$ for $t \geq t_2$. From these observations and knowing that $y(t)$ is concave up and decreasing for $t \geq t_2$, we finally arrive at a contradiction to equation (16) which shows that every bounded solution $y(t)$ is bounded.

One must note that the result of Theorem 2.3 can be extended to a more general equation of the form

$$(r(t)y'(t))' = \sum_{i=1}^{n} p_i(t)y(g_i(t)) = 0,$$

where $r(t) > 0$ and $\int_0^\infty \frac{dz}{r(z)} = \infty$.

Now, consider the general second order differential equation

$$(a(t)y'(t))' + p(t)y'(t) + c(t)y(t) + q(t)f(y[g(t)]) = e(t) \tag{19}$$

under the assumption that

i) $c, e, p, q \in C([t_0, \infty), R)$, $t_0 \geq 0$, $f \in C(R, R)$ and $yf(y) > 0$ for

$y \neq 0;$

ii) $a,\ g \in C^1([t_0,\infty), R_+),\ \ a'(t) \geq 0;$

iii) $g(t) \leq t,\ \ g'(t) \geq 0$ and $\lim_{t\to\infty} g(t) = \infty;$

iv) there exist a number k such that $f(y)sgn\,y \leq k\,|y|$ for $y \neq 0$, and $f(y)$ is increasing in y.

Agarwal et al. (2003), in their work titled *Oscillation Theory of Second order Dynamic Equations*, provided sufficient conditions for all solutions of equation (19) to be continuous or bounded. The following results are valid.

Theorem 2.4. *Let conditions (i)–(iv) of equation* (19) *hold. Then any non-trivial solution of equation* (19) *can be continued indefinitely on* R_+ *(Singh, 1980).*

Do note that condition (iv) in equation (19)can be replaced by

$$f(y)sgn\,y \leq k\,|y|^{\gamma},\ \ 0 \leq \gamma \leq 1.$$

The following result provides a bound on the growth of non-oscillatory solutions of equation (18).

Lemma 2.2. In addition to conditions (i)–(iv) of equation (19), suppose that

$$p(t) \geq 0,$$

$$c(t) - p'(t) \geq 0 \; for \; t \geq t_0, \tag{20}$$

$$\int^{\infty} \frac{p(s)}{a(s)} ds < \infty,$$

$$\int^{\infty} |e(s)|\, ds < \infty \tag{21}$$

$$\int^{\infty} \frac{ds}{a(s)} < \infty$$

and

$$\int^{\infty} c(s)ds < \infty.$$

Then all oscillatory solutions of equation (19) are bounded above.

Next, we now use Lemma 2.4 to find a criterion so that all solutions of equation (19) are non-oscillatory.

Theorem 2.5. *In addition to the conditions of Lemma 2.4, suppose that* $p(t) \equiv 0$ *and* $\int^{\infty} e(s)ds = \infty$,
then all solutions of equation (19) *are non-oscillatory (Singh, 1977).*

This is illustrated by the following example.

Example 2.3. Let us consider the equation

$$(e^t y'(t))' + 2e^{-3t+\pi} y(t-\pi) = 4\cosh 2t, \quad t \geq 0.$$

It is easy to verify that the conditions of Theorem 2.5 are satisfied. All solutions of this equation are non-oscillatory. In fact, $y(t) = e^t$ is one of such equations.

2.5 Second order nonlinear differential equations with delay

We wish to extend in this section some results of section 2.4.1 to the nonlinear equation

$$y''(t) - f(t, y(t), y(g(t))) = 0 \tag{24}$$

subject to the following conditions:

i) $f \in C[R_+ \times R \times R, R]$ and $f(t,u,v)$ is non-decreasing in u and v for fixed large t,

ii) $f(t, u, v)u > 0$ if $u \cdot v > 0$,

iii) $g \in C[R_+ \times R]$, $g(t) < t$, $g'(t) > 0$ and $\lim_{t\to\infty} g(t) = \infty$,

iv) for any constant $c \neq 0$, $\int^{\infty} f(s, g(s)c, g(s)c)\, ds = \pm\infty$.

As a nonlinear delay differential equation in this form, we examine the conditions for the oscillation of the solutions via the following theorems provided by Ladde (1972, 1973).

Theorem 2.6. *Assume that equation* (24) *satisfies its conditions (i),(ii),(iii). Furthermore, let $y(t)$ be a bounded solution of equation* (24), *with $|y(t)| \leq \beta$ for large t, and $\beta > 0$. Let us assume that there exist a functionG$_\beta \in C[R_+, R_+]$ such that*

$$z^2 G_\beta(t) \leq z f(t, x, z) \tag{25}$$

for $sgnx = sgnz$, $x \cdot sgnx \leq z \cdot sgnz \leq \beta$, and sufficiently large t. Further assume that

$$\lim_{t\to\infty} \sup \int_{g(t)}^{t} [g(t) - g(s)]\, G_\beta(s) ds > 1. \tag{26}$$

Then $y(t)$ is oscillatory.

Corollary 2.3. Assume that equation (24) satisfies its conditions (i), (ii), (iii). Furthermore, assume that for any $\beta > 0$, there exist a function $G_\beta \in C[R_+, R_+]$ such that inequalities (25) and (26) hold, then every bounded solution of equation (24) is oscillatory.

A similar result to this is the following:

Corollary 2.4. Consider the equation

$$y''(t) - p_1(t)y(g(t)) - p_2(t)y(t) = 0, \tag{27}$$

where $p_1(t),\ p_2(t) \geq 0$ and are continuous on R_+, and

$$\lim_{t\to\infty} \int_{g(t)}^{t} [g(t) - g(s)]\, p_1(s)ds > 1. \tag{28}$$

Then every bounded solution of (27) is oscillatory.

This can be made clearer to the reader with the following illustration:

Example 2.4. Consider the equation

$$y''(t) - y(t-\pi)[(k+1) + ky^{2n}(t-\pi)] - ky(t)[1 + y^{2n}(t)] = 0, \tag{29}$$

where $k \geq 0$, for any integer $n > 0$. For any $\beta > 0$, $G_\beta(t) = (k+1)$ satisfies the condition (25). Then equation (26) reduces to

$$\int_{t-\pi}^{t} (k+1)(t-s)ds = \frac{k+1}{2}\pi^2 > 1,$$

and by Theorem 2.6, every bounded solution of equation (29) is oscillatory. In fact, equation (29) has bounded oscillatory solutions $A\cos t + B\sin t$, where A and B are any arbitrary constants.

2.5.1 Nonlinear equations with $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$

We consider the second order nonlinear delay differential equation expressed in the form

$$(r(t)y'(t))' + f(t, y(t), y(g(t)), y'(t), y'(h(t)) = 0. \tag{30}$$

Now, with respect to equation (30), the following results follow the development of Zhang (1981).

Theorem 2.7. *Assume that*

i) the conditions

$$r \in C[R_+, R_+],\ \ r(t) > 0\ \ for\ \ t \geq t_0,\ \ t_0 \in R_+,\ \ \lim_{t\to\infty} R(t) = \infty \qquad (31)$$

hold,

where $R(t)$ is defined by $R(t) = \int_{t_0}^{\infty} \frac{ds}{r(s)}$,

ii) $g, h \in C[R_+, R_+],\ \ g(t) \leq t,\ \ \lim_{t\to\infty} g(t) = \infty,$

iii) $f \in C[R_+ \times R^4, R]$ *and* $uf(t, u, v, w, z) > 0$ *for* $u \cdot v > 0,\ t \geq t_0,$

iv) there exist a constant β such that $0 < \beta < 1$ and

$$\int_{t_0}^{\infty} R^{\beta}(g(t)) \frac{|f(t, y(t), y(g(t)), y'(t), y'(h(t))|}{|y(g(t))|^{\beta}} dt = +\infty \qquad (32)$$

for every positive non-decreasing or negative non-increasing function $y(t)$. Then every solution of equation (30) *oscillates.*

One must note that the strict inequality in condition (iii) of Theorem 2.7 can be relaxed. Again, Theorem 2.7 remains valid if the argument $g(t)$ is of mixed type, that is, it is advanced or retarded for certain values of t.

Example 2.5. To understand Theorem 2.7 better, we consider

$$(ty'(t))' + p(t)y^{\frac{1}{3}}(\ln t)(1 + y'^2(\sqrt{t})) = 0, \qquad (33)$$

where $p(t) \geq 0$ and $\int_t^{\infty} p(s)ds > 0$.
According to Theorem 2.7, if $\int^{\infty} (\ln \ln t)^{\frac{1}{3}} p(t)dt = \infty$, then every solution to equation (30) is oscillatory.

Consequently, in equation (30),if

$$f(t, y(t), y(g(t)), y'(t), y'(h(t))) = \sum_{i=1}^{n} p_i(t)\, |y(g(t))|^{\alpha_i}\, sgny(g(t)), \qquad (34)$$

then by Theorem 2.7, we have the following results:

Corollary 2.5. Assume that $r(t)$, $g(t)$ satisfy conditions (i) and (ii) of Theorem 2.7, and $p_i \in C[R_+, R_+]$, $0 < \alpha_i < 1$, $i \in I_n$. Further assume that

$$\int^{\infty} R^{\alpha_k}(g(t))p_k(t)dt = \infty \text{ for some } k \in I_n. \tag{35}$$

Then every solution of equation (34) is oscillatory.

Theorem 2.8. *Assume that conditions (i),(ii),(iii) of Theorem 2.7 hold. Further assume that there exist a positive number ε such that $0 < \varepsilon < 1$ and*

$$\int_{t_0}^{\infty} R^{1-\varepsilon}(g(t))\frac{f(t, y(t), y(g(t)), y'(t), y'(h(t)))}{y(g(t))}dt = \infty \tag{36}$$

for every positive non-decreasing or negative non-increasing function $y(t)$. Then every solution of equation (30) *is oscillatory.*

Now consider the equation

$$(r(t)y'(t))' + \sum_{i=0}^{n} p_i(t)y^{2i+1}(g(t)) = 0. \tag{37}$$

The following result is valid.

Corollary 2.6. Assume that $r(t)$ and $g(t)$ satisfy the conditions (i) and (ii) of Theorem 2.7, and $p_i \in C[R_+, R_+]$ for $i \in I_n$. Furthermore,

$$\int^{\infty} R^{1-\varepsilon}(g(t))(\sum_{i=0}^{n} p_i(t))dt = \infty, \quad 0 < \varepsilon < 1. \tag{38}$$

Then every solution of equation (37) oscillates.

It is easy to see that equation (37) satisfies the conditions of Theorem 2.8. In particular, for $n = 0$, equation (37) becomes

$$(r(t)y'(t))' + p(t)y(g(t)) = 0 \tag{39}$$

and condition (38) becomes

$$\int^{\infty} R^{1-\varepsilon}(g(t))p(t)dt = \infty, \quad 0 < \varepsilon < 1.$$

We note that εcannot be equal to zero. In fact, this is seen in the equation

$$y''(t) + \frac{1}{2\sqrt{2}}\frac{1}{t^2}y\left(\frac{t}{2}\right) = 0$$

which satisfies the condition $\int^{\infty} R(g(t))p(t)dt = \infty$, but it has a non-oscillatory solution $y(t) = t^{\frac{1}{3}}$.

Theorem 2.9. *Assume that conditions (i),(ii),(iii) of Theorem 2.7 hold. Further assume that there is a constant $\beta > 1$ such that*

$$\int_{t_0}^{\infty} R(g(t))\frac{f(t, y(t), y(g(t)), y'(t), y'(h(t)))}{|y(g(t))|^{\beta}}dt = \infty, \tag{40}$$

for every positive non-decreasing or negative non-increasing function $y(t)$. Then every solution of equation (30) *is oscillatory.*

Corollary 2.7. Consider the equation

$$(r(t)y'(t))' + \sum_{i=0}^{n} p_i(t)y^{2i+1}(g(t)) = 0. \tag{41}$$

Assume that $r(t)$ and $g(t)$ satisfy all conditions of Theorem 2.9,

$$p_i(t) \geq 0, \quad (i = 1, 2, ..., n)$$

and

$$\int^{\infty} R(g(t))(\sum_{i=1}^{n} p_i(t))dt = \infty. \tag{42}$$

Then every solution of equation (41) is oscillatory.

We observe that the condition (42) cannot be improved. Equation (41), including the equation $y''(t) + p(t)y^{2n+1}(t) = 0$, $n \geq 1$, was discussed by Atkinson (1955), but condition (42) is a necessary and sufficient condition for the oscillation of Atkinson's equation.

These illustrations are of importance to help in the understanding of these concepts.

Example 2.6. Consider the equation

$$y''(t) + \frac{1}{4a^2t^2}y^3(t) = 0. \tag{43}$$

It is well known that every solution of equation (43) oscillates.

Example 2.7. The second equation,

$$y''(t) + \frac{1}{4a^2t^2}y^3(t^{\frac{1}{3}}) = 0 \tag{44}$$

has a non-oscillatory solution $y(t) = at^{\frac{1}{2}}$, but for the equation

$$y''(t) + \frac{1}{4a^2t^2}y^3(\lambda t) = 0, \quad 0 < \lambda < 1, \tag{45}$$

every solution of equation (45) oscillates (according to Corollary 2.7). These examples show that the order of the deviating argument $g(t)$ is very important for the oscillation of the solutions. If $g(t)$ is of the same order as t, then we can obtain a necessary and sufficient condition for the oscillation of a functional differential equation. The following result is based on the above idea.

Theorem 2.10. *Assume that conditions (i),(ii) and (iii) of Theorem 2.7 hold, and further assume that* $\lim_{t\to\infty} g'(t) = c, \ \ c > 0, \ \ r(t)$ *and* $r(g(t))$ *are of the same order if* $t \to \infty$, *and*

$$\int_{t_0}^{\infty} R(t)\frac{|f(t,y(t),y(g(t)),y'(t),y'(h(t)))|}{|y(g(t))|^{\beta}}dt = \infty \tag{46}$$

for some $\beta > 1$ *and every positive non-decreasing or negative non-increasing function* $y(t)$. *Then every solution of equation* (30) *is oscillatory.*

Theorem 2.11. *Consider the equation*

$$(r(t)y'(t))' + p(t)(y(t) + y(g(t)))^{2n+1} = 0. \tag{47}$$

Assume that $p(t) \geq 0, \ \ r(t), \ \ g(t)$ *satisfy the conditions of Theorem 2.10. Then, a necessary and sufficient condition for* (47) *to be oscillatory is that*

$$\int^{\infty} R(t)p(t)dt = \infty.$$

Corollary 2.8. Under the conditions of Theorem 2.11, equation (47) has a bounded oscillatory solution if and only if

$$\int^{\infty} R(t)p(t)dt < \infty. \tag{48}$$

Corollary 2.8 gives birth to the following theorem.

Theorem 2.12. *Assume that* $r(t)$ *and* $g(t)$ *of equation* (41) *satisfy the conditions of Theorem 2.10 and* $p_i(t) > 0, \ \ i = 1, \ 2, \ \cdots, \ n$. *Then a necessary and sufficient condition for equation* (41) *to be oscillatory is that*

$$\int^{\infty} R(t)(\sum_{i=1}^{n} p_i(t))dt = \infty. \tag{49}$$

Corollary 2.9. Under the conditions of Theorem 2.12, equation (41) has a bounded oscillatory solution $y(t)$ if and only if

$$\int^{\infty} R(t)(\sum_{i=1}^{n} p_i(t))dt < \infty. \tag{50}$$

Now, observe that for the equation

$$(r(t)y'(t))' + p(t)y(t)^{2n+1}(g(t)) = 0, \tag{51}$$

where n is a positive integer, if $p(t) \geq 0,\ r(t)$ and $g(t)$ of equation (51) satisfy the conditions of Theorem 2.12, then a necessary and sufficient condition for equation (50)to be oscillatory is that

$$\int^{\infty} R(t)p(t)dt = \infty. \tag{52}$$

This is an extension of Atkinson's theorem.

The following results are also known to be valid.

Theorem 2.13. *Every solution of equation* (30) *is oscillatory if and only if*

$$\int^{\infty} R(t)p(t)dt = \infty. \tag{53}$$

Corollary 2.10. Under the conditions of Theorem 2.13, equation (30) has a bounded non-oscillatory solution if and only if

$$\int^{\infty} R(t)p(t)dt < \infty.$$

Theorems 2.4.7 -2.4.13 are quite recent and are the results of Ladde's work (Ladde et al., 1987).

2.5.2 Nonlinear equations with $\int_{t_0}^{\infty} \frac{ds}{r(s)} < \infty$

In this section, we shall discuss the case where $\int_{t_0}^{\infty} \frac{ds}{r(s)} < \infty$ relative to equation(30). For simplicity we shall restrict our discussion to the equation

$$(r(t)y'(t))' + f(y(g(t)), t) = 0. \tag{54}$$

Definition 2.4. Equation (54) is called:

i) Superlinear if, for each fixed t, $\frac{f(y,t)}{y}$ is non-decreasing in y for $y > 0$ and non-increasing in y for $y > 0$;

ii) Strongly superlinear if there exist a number $\sigma > 1$ such that, for each fixed t, $\frac{f(y,t)}{|y|^{\sigma}} sgny$ is non-decreasing in y for $y > 0$ and non-increasing in y for $y > 0$;

iii) Sublinear if, for each fixed t, $\frac{f(y,t)}{y}$ is non-increasing in y for $y > 0$ and non-decreasing in y for $y > 0$;

iv) Strongly sublinear if there exist a number $\tau < 1$ such that, for each t, $\frac{f(y,t)}{|y|^{\tau}} sgny$ is non-increasing in y for $y > 0$.

Let us look at some important results obtained for equation (54) under the stated condition.

Lemma 2.5. Assume that

i) $r(t)$ is positive continuous for $t \geq \alpha$ and $\int_{t_0}^{\infty} \frac{ds}{r(s)} < \infty$;

ii) $g(t)$ is continuous for $t \geq \alpha$ and $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$;

iii) $f(y,t)$is continuous for $|y| < \infty$, $t \geq \alpha$ and $y \cdot f(t,y)$ for $y \neq 0$, $t \geq \alpha$.

If $y(t)$ is a positive solution of equation (54), then it is bounded above and satisfies

$$y(t) \geq -r(t)y'(t)\rho(t) \tag{55}$$

for all sufficiently large t, where $\rho(t) = \int_t^\infty \frac{ds}{r(s)}$ for all $t > 0$.

A follow up to this are the following theorems.

Theorem 2.14. *Assume that equation* (54) *satisfies assumptions (i) to (iii) of Lemma 2.5 and furthermore that it is either superlinear or sublinear. A necessary and sufficient condition for equation* (54) *to have a non-oscillatory solution which is asymptotic to a non-zero constant is*

$$\int^\infty \rho(t)\,|f(c,t)|\,dt < \infty \text{ for some } c \neq 0. \tag{56}$$

Theorem 2.15. *Assume that equation* (54) *satisfies assumptions (i) to (iii) of Lemma 2.5 and furthermore that it is either superlinear or sublinear. A necessary and sufficient condition for equation* (54) *to have a non-oscillatory solution which is asymptotic to* $a \cdot \rho(t)$ *as* $t \to \infty$ *for* $a \neq 0$ *is that*

$$\int^\infty f(c\rho(g(t)), t)dt < \infty \text{ for some } c. \tag{57}$$

Theorem 2.16. *Assume that conditions (i),(ii) and (iii) of Lemma 2.5 hold and let equation* (54) *be strongly superlinear. A sufficient condition for equation* (54) *to be oscillatory is that*

$$\int^\infty f(c\rho(t), t)dt = \infty \text{ for all } c > 0. \tag{58}$$

One must note that from Theorem 2.15, it follows that if equation (54), whether superlinear or sublinear, is oscillatory, then

$$\int_{t_1}^\infty f(c\rho(t), t)dt = \infty \ \text{ for all } \ c > 0, \ \ t \geq t_1 \tag{59}$$

In some cases, conditions (58) and (59) are equivalent. Indeed, such cases can be made clear from the following illustration.

Example 2.8. Let us rewrite the coefficient $r(t)$ and the general delay function $g(t)$ of equation (54) in the following forms:

i) $r(t) = ct(\log t)^{\alpha}$, $g(t) = t^{\beta}$ or $g(t) = \nu t$, where $c > 0$, $\alpha > 1$, $0 < \beta < 1$ and $0 < \nu < 1$;

ii) $r(t) = ct^{\rho}$, $g(t) = \nu t$, where $c > 0$, $\rho > 1$ and $0 < \nu < 1$;

iii) $r(t) = ce^{qt}$, $g(t) = t - \tau(t)$, $0 \leq \tau(t) \leq M$, where $c > 0$, $q > 0$ and $M > 0$.

Under the above considerations, equation (58) is certainly a necessary and sufficient condition for the oscillation of equation (54).

However, there exist some cases where there are disparities between both conditions and this is made clearer in the following illustration.

Example 2.9. Consider the delay equation

$$[t^3 y'(t)]' + t[y(t^{\frac{1}{3}})]^3 = 0. \tag{60}$$

It satisfies equation (59) but does not satisfy condition (58). In fact, this equation has a non-oscillatory solution $y(t) = \dfrac{1}{t}$.

The just mentioned equivalence and non-equivalence of equations (58) and (59) are by the way. Let us return to the subject matter of the context. The following theorem is also valid for the solutions of equation (54) to be oscillatory.

Theorem 2.17. *Assume that conditions (i),(ii) and (iii) of Lemma 2.5 hold and let equation* (54) *be strongly sublinear. A sufficient condition for equation* (54) *to be oscillatory is that*

$$\int^{\infty} \rho(t) f(c,t)dt = \infty \quad for\ all\ \ c > 0 \tag{61}$$

By combining Theorems 2.14 and 2.17, we now have the following theorem.

Theorem 2.18. *Assume that conditions (i),(ii) and (iii) of Lemma 2.5 hold and let equation* (54) *be strongly sublinear. A necessary and sufficient condition for equation* (54) *to be oscillatory is that equation* (61) *remains valid (Kusano and Naito, 1976; Kusano and Onose, 1977).*

2.6 Oscillations of neutral differential equations

A neutral delay differential equation is a differential equation in which the highest order derivative appears in the equation both with and without delay (Gyori and Ladas, 1991). These equations find numerous applications in natural sciences and technology. In contrast with delay differential equations, neutral equations inherit special structure which makes their study more difficult, but interesting. However, we are not going to delve into the details of this issue for now, but may touch its peripheries as we progress. For a better understanding, we begin the study of the concept of neutral delay differential equations with that of the first order.

A neutral delay differential equation of the first order is an equation of the form

$$[y(t) + p(t)y(t-\tau)]' + q(t)y(t-\sigma) = 0, \tag{62}$$

where

$$p,\ q \in C([t_0, \infty), R) \ \text{ and } \ \tau,\ \sigma \in [0, \infty). \tag{63}$$

Let $\gamma = \max\{\tau, \sigma\}$ and let $t_1 \geq t_0$. By a solution of equation (62) on $[t_1, \infty)$, we mean a function $y \in C([t_1 - \gamma, \infty), R)$ such that $y(t) + p(t)y(t-\tau)$ is continuously differentiable for $t \geq t_1$ and such that equation (62) is satisfied for $t \geq t_1$.

Let $t_1 \geq t_0$ be a given initial point and let $\varphi \in C([t_1 - \gamma, t_1], R)$ be a given initial function. Then, as can be proved by method of steps (Grammatikopoulos and Marusiak, 1995; Gyori and Ladas, 1991), equation (62) has a unique solution on $[t_1, \infty)$ satisfying the given initial condition

$$y(t) = \varphi(t), \text{ for } t_1 - \gamma \leq t \leq t_1. \tag{64}$$

As usual, when we say that each solution of the first order neutral delay differential equation (62) oscillates, we mean that for every initial point $t_1 \geq t_0$ and for every initial function $\varphi \in C([t_1 - \gamma, t_1], R)$, the unique solutions of equations (62) and (64) on $[t_1, \infty)$ has arbitrary large zeros. If it is false, then there exist a $t_1 \geq t_0$, an initial function $\varphi \in C([t_1 - \gamma, t_1], R)$ and a$T \geq t$such that the solutions of equations (62) and (64) are either eventually positive or negative for $t \geq T$ (Isaac, 2008).

As earlier remarked, the theory of neutral delay differential equations, in general, presents a lot of very interesting complications. In one of such cases, it is observed that there exist some results which are true for non-neutral equations, but are not necessarily true for neutral equations. One then wonders what stands as the easiest methods for finding their solutions and how they are likely to behave in the entire process after all. Snow (1965) makes it clear, for example, that even though the characteristic roots of a neutral differential equation may all have negative real parts, it is still possible for some solutions to be unbounded. Similarly, Slemrod and Infante (1972) arrived at the same conclusion.

In spite of these limitations, the oscillatory behaviour of the solutions of neutral systems is important both in theory and applications, such as the motion of retarding electrons, population growth, the spread of epidemics and networks containing lossless transmission lines (Driver, 1984; Gyori and Ladas, 1991; Hale, 1977; Krisztin and Wu, 1996).

The aim of this section is to present a review of some recent literature on neutral equations.

2.6.1 Oscillations of neutral delay equations with constant coefficients

Consider the neutral equation of the form

$$\frac{d^2}{dt^2}(y(t) + py[t-\tau]) + qy[t-\sigma] = 0, \tag{65}$$

where $p,\ q,\ \tau,\ \sigma \in R$. The main results are the following theorems which give necessary and sufficient conditions for the oscillation of all unbounded and bounded solutions of equation (65) by means of its characteristic equation (Gopalsamy and Zhang, 1990; Grammatikopoulos et al., 1986; Ladas and Partheniadis, 1989; Ladas et al., 1988; Ladas and Sficas, 1986; Li and Liu, 1996; Li, 1997; Philos, 1989; Wong, 2000).

$$F(\lambda) = \lambda^2 + \lambda^2 p e^{-\lambda\tau} + q e^{-\lambda\sigma} = 0. \tag{66}$$

Theorem 2.19. *Assume that* $p,\ q,\ \tau$ *and* σ *are real numbers, then the following statements are equivalent:*

i) Every unbounded solution of equation (65) *oscillates;*

ii) The characteristic equation (66) *has no roots in* $[0,\infty)$ *(Ladas et al., 1992).*

Theorem 2.20. *Assume that* $p,\ q,\ \tau$ *and* σ *are real numbers, then the following statements are equivalent:*

i) Every bounded solution of equation (65) *oscillates;*

ii) The characteristic equation (66) *has no roots in* $(-\infty, 0]$.

One can observe here that zero cannot be a multiple root of equation (66), therefore it is necessary to assume in condition (ii) of Theorem 2.19 that zero is not a root of equation (66). It is well known that all solutions of equation (65) oscillate if and only if equation (66) has no real roots. These results can easily be established by using Laplace transforms. However, the method of Laplace transforms cannot be applied to unbounded solutions of equations when the deviating arguments are not all delays. In fact, the Laplace transforms of such solutions may not exist (Farrel, 1990; Ladas and Schultz, 1989; Grove, Kulevonic and Ladas,1987 ; Grove, Ladas and Schinas ,1988a ; Grove, Ladas and Schultz, 1988b ; Kulenovic, Ladas and Meimaridou, 1987 ; Agarwal and Saker, 2001; Bainov and Mishev, 1991; Gopalsamy et al., 1992; Sficas and Stavroulakis, 1987; Graef, Grammatikopoulos and Spikes, 1991;1991a;1993).

2.6.2 Oscillations of neutral delay equations with variable coefficients

Consider the neutral differential equation of the form

$$\frac{d^2}{dt^2}(y(t)+py[t-\tau])+qy[t-\sigma]=0, t\geq t_0, \tag{67}$$

where $p(t),\ q(t)\in C([t_0,\infty),R)$ and the delays τ and σ are non-negative real numbers.

Let $\varphi(t)\in C([t_0-\rho,t_0],R)$, where $\rho=\max\{\tau,\sigma\}$ is a given function and let z_1 be a given constant.

Definition 2.5. The function $y(t)\in C([t_0-\rho,\infty],R)$ is said to be a solution of equation (67) if

$$y(t)=\varphi(t),\quad t\in[t_0-\rho,t_0];$$

$$\frac{d^2}{dt^2}[y(t)+p(t)y(t-\tau)]|_{t=t_0}=z_1;$$

The function $y(t)+p(t)y(t-\tau)$ is twice differentiable for $t\geq t_0$ and $y(t)$ satisfies equation (67) for $t\geq t_0$.

We shall note that theorems of existence and uniqueness of the solutions of neutral differential equations were obtained by Driver (1965, 1984), Bellman and Cooke (1963) and Hale (1977). The results in this section are primarily those of

Grammatikopoulos, Ladas and Meimaridou (1985,1987).

We shall investigate the oscillatory properties of the solution of equation (67). The following results remain valid.

Theorem 2.21. *Assume that*

i) $p(t) \in C([t_0, \infty), R)$, $p_1 \le p(t) \le p_2$ *for* $t \in [t_0, \infty)$, *where* p_1 *and* p_2 *are constants;*

ii) $q(t) \in C([t_0, \infty), R)$, $q(t) \ge Q > 0$ *for* $t \in [t_0, \infty)$*;*

iii) $-1 < p_1 \le p_2 < 0$.

Then each non-oscillating solution $y(t)$ *of equation* (67) *tends to zero as* $t \to \infty$.

A careful analysis of this illustrates that if condition (ii) of Theorem 2.21 is violated, the result may not be true. Consider the neutral delay differential equation

$$\frac{d^2}{dt^2}\left[y(t) + \left(-\frac{1}{2} + (t-1)^{-\frac{1}{2}}\right) y(t-1)\right] \\ + \frac{1}{4}(t-2)^{-\frac{1}{2}}\left(t^{-\frac{3}{2}} - \frac{1}{2}(t-1)^{-\frac{3}{2}}\right) y(t-2) = 0, \quad t \ge 2.$$

All conditions of Theorem 2.21, except for condition (ii), are satisfied. Note that the function $y(t) = \sqrt{t}$ is a solution with $\lim_{t\to\infty} y(t) = \infty$.

In the subsequent theorems, sufficient conditions are given for oscillation of the solutions of equation (67).

Theorem 2.22. *Consider the neutral delay differential equation* (67) *and assume that conditions (i) and (ii) of Theorem 2.21 hold. Furthermore, assume* $p(t)$ *is not eventually negative, then each solution of equation* (67) *oscillates.*

Theorem 2.23. *Consider the neutral differential equation* (67) *and assume that conditions (i) and (ii) of Theorem 2.21 are satisfied with*

$$-1 \le p_1 \le p_2 < 0. \tag{68}$$

Suppose also that there exist a positive constant r such that

$$\frac{q(t)}{p(t+\tau-\sigma)} \leq -r \tag{69}$$

and

$$r^{\frac{1}{2}}\frac{\sigma-\tau}{2} > \frac{1}{e}. \tag{70}$$

Then each solution of equation (67) *oscillates.*

Theorem 2.24. *Consider the neutral differential equation* (67) *and assume that conditions (i) and (ii) of Theorem 2.21 are satisfied with*

$$p_2 < 0. \tag{71}$$

Suppose also that there exist a positive constant r such that

$$\frac{q(t)}{p(t+\tau-\sigma)} \leq -r \tag{72}$$

and

$$r^{\frac{1}{2}}\frac{\sigma-\tau}{2} > \frac{1}{e}. \tag{73}$$

Then each bounded solution of equation (67) *oscillates.*

The following illustration gives a better understanding of Theorem 2.24.

Example 2.9. Consider the neutral delay differential equation

$$\frac{d^2}{dt^2}\left[y(t) - \frac{\sqrt{2}}{2}y\left(t-\frac{\pi}{4}\right)\right] + \frac{\sqrt{2}}{2}y\left(t-\frac{7\pi}{4}\right) = 0, \quad t \geq 0.$$

All conditions of Theorem 2.24 are fulfilled. Therefore, each bounded solution of this equation oscillates. For instance, $y(t) = \sin t$ is such a solution.

In Theorems 2.25 and 2.26 given below, condition (ii) of Theorem 2.21 is not required.

Theorem 2.25. *Consider the neutral differential equation* (67) *and assume that the following conditions are eventually fulfilled:*

$$q(t) \geq 0, \quad -1 \leq p(t) \leq 0 \quad and \quad \int_{t_0}^{\infty} q(s)ds = \infty.$$

Then each unbounded solution of equation (67) *oscillates.*

Theorem 2.26. *Consider the neutral differential equation* (67) *and assume that the following conditions are eventually fulfilled:*

$0 < p(t) \equiv P$ *is constant;*

$q(t) \geq 0, \ q(t) \not\equiv 0$ *and* τ *- periodic.*

Then every solution of equation (67) *oscillates.*

2.6.3 Oscillations of Nonlinear Neutral Delay Equations

In this section, the oscillatory properties and asymptotic behaviour of the solutions of nonlinear neutral differential equations of the form

$$\frac{d^2}{dt^2}[y(t) + p(t)y(t-\tau)] + q(t)f(y(t-\sigma)) = 0, \quad t \geq t_0 \tag{74}$$

are investigated, where $p(t), q(t) \in C([t_0, \infty), R)$, $f \in C(R, R)$ and the delays are non-negative constants. The results of this section are due to Graef, Grammatikopoulos and Spikes (1988). We shall note that the first oscillation criterion for second order equations, valid for both linear and nonlinear neutral differential equations, was obtained by Zahariev and Bainov (1980).

Consider the following conditions:

H2.6.1: $p(t),\ q(t) \in C([t_0, \infty), R), \ \ f(u) \in C(R, R)$;

H2.6.2: $q(t) \geq 0$ for $t \in [t_0, \infty)$, $p(t) \not\equiv 0$, q(t) $\not\equiv 0$;

H2.6.3: $uf(u) > 0$ for $u \neq 0.$;

H2.6.4: If eventually, the inequality $y(t) \geq a > 0$ holds, where $a \in R$, then there exist a constant A such that eventually we have $f(y(t)) \geq A > 0$;

H2.6.5: $\int_{t_0}^{\infty} q(s)ds = \infty$;

H2.6.6: There exist a continuous function $b(t)$ such that $b(t) = o(t), \quad t \to \infty$ and $b(t) \leq p(t) \leq 0$.

We shall say that conditions (H2.6) are met if conditions (H2.6.1)-(H2.6.6) hold.

First, we consider the asymptotic behaviour of the non-oscillating solutions of equation (74) contained in the following Lemma. Note that sufficient conditions for the oscillation and asymptotic behaviour of the solutions of second order nonlinear neutral differential equations were obtained by Erbe and Zhang (1989), Grace and Lalli (1987, 1989).

Lemma 2.6. Let $y(t)$ be a non-oscillating solution of equation(74). Then the following statements are valid for

$$z(t) = y(t) + p(t)y(t - \tau).$$

i) Assume conditions (H2.6) are fulfilled. If $y(t)$ is eventually positive, then the functions $z(t)$ and $z'(t)$ are either both decreasing with

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = -\infty \tag{75}$$

or $z'(t)$ is decreasing with

$$\lim_{t\to\infty} z'(t) = 0, \quad z'(t) > 0 \text{ and } z(t) < 0. \tag{76}$$

ii) Assume conditions (H2.6) are fulfilled. If $y(t)$ is eventually negative, then the functions $z(t)$ and $z'(t)$ are either both increasing with

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = \infty \tag{77}$$

or $z'(t)$ is increasing with

$$\lim_{t\to\infty} z'(t) = 0, \;\; z'(t) < 0 \;\text{ and }\; z(t) > 0. \tag{78}$$

iii) Assume conditions (H2.6.1)-(H2.6.5) are fulfilled and that there exist a constant $p_1 < 0$ such that

$$p_1 \le p(t) \le 0. \tag{79}$$

If $y(t)$ is eventually positive, then either equation (75) holds or $z'(t)$ is decreasing with

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0, \;\; z'(t) > 0 \;\text{ and }\; z(t) < 0. \tag{80}$$

iv) Assume conditions (H2.6.1)-(H2.6.5) are fulfilled in addition to condition (79). If $y(t)$ is eventually negative, then either equation (77) holds or $z'(t)$ is increasing with

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0, \;\; z'(t) < 0 \;\text{ and }\; z(t) > 0. \tag{81}$$

v) Assume conditions (H2.6.1)-(H2.6.5) are fulfilled in addition to condition (79). If $p_1 \ge -1$, then equation (80) holds when $y(t)$ is eventually positive and equation (81) holds when $y(t)$ is eventually negative.

The following theorems are consequences of Lemma 2.6.

Theorem 2.27. *Assume conditions (H2.6.1)-(H2.6.5) are fulfilled. If equation* (79) *holds with $p_1 > -1$, that is*

$$-1 < p_1 \le p(t) \le 0, \tag{82}$$

then each non-oscillating solution $y(t)$ of equation (74) *satisfies*

$y(t) \to 0$ *as* $t \to \infty$.

Theorem 2.28. *Let* $p(t) \geq 0$. *Then each non-oscillating solution* $y(t)$ *of equation* (74) *satisfies the following:*

i) $|y(t)| \leq b_1 t$ *for some constant* $b_1 > 0$ *and all* $t \geq \max\{1, t_0\}$;

ii) If $t(p(t))^{-1}$ *is bounded, then* $y(t)$ *is bounded;*

iii) If $t(p(t))^{-1} \to 0$ *as* $t \to \infty$, *then* $y(t) \to 0$ *as* $t \to \infty$.

In the subsequent theorems, results concerning the oscillatory behaviour of solutions of equation (74) were obtained by Zahariev and Bainov (1988). The first result in this direction is an immediate consequence of Lemma 2.6.

Theorem 2.29. *Assume conditions (H2.6.1)-(H2.6.5) are fulfilled in addition to condition* (79) *with* $p_1 \geq -1$, *that is,*

$$-1 \leq p(t) \leq 0. \tag{83}$$

Then each unbounded solution $y(t)$ *of equation* (74) *is oscillatory.*

We need to observe here that under the present hypothesis, part (iv) of Lemma 2.6 implies that all non-oscillating solutions of equation (74) are bounded. It can also be observed that Theorem 2.29 reduces to Theorem 2.25 in section 2.6.1 and to the second order version of Theorem 12 in the monograph by Grammatikopoulos, Sficas and Stavroulakis (1988) when $f(u) \equiv u$.

In the next theorem, we obtain the conclusion of Theorem 2.29 without requiring condition H2.6.5, but with more restrictive condition on f(u).

Theorem 2.30. *Assume conditions (H2.6.1)-(H2.6.3) are fulfilled in addition to condition* (83) *and that* f *is increasing,*

$$\int_t^\infty \int_s^\infty q(v)dvds = \infty \tag{84}$$

and

$$\int_c^\infty \frac{1}{f(u)}du < \infty \text{ and } \int_c^{-\infty} \frac{1}{f(u)}du < \infty \tag{85}$$

for every constant $c > 0$. Then every unbounded solution of equation (74) *is oscillatory.*

We now give sufficient conditions for all solutions of equation (74) to be oscillatory.

Theorem 2.31. *Assume conditions (H2.6.1)-(H2.6.3) are fulfilled, that f is increasing,*

$$0 \le p(t) \le 1 \tag{86}$$

and

$$\int_{t_0}^\infty q(s)f([1-p(s-\sigma)]c)ds = \infty \tag{87}$$

for any positive constant c. Then all solutions of equation (74) *oscillate.*

Careful survey shows that Theorem 2.31 extends Theorem 1 in the monograph by Grammatikopoulos et al. (1985) and reduces to the second order version of Theorem 10 in the monograph by Grammatikopoulos, Ladas and Meimaridou (1988) when $f(u) \equiv u$. When $\sigma = 0$ and $p(t) \equiv 0$, Theorem 2.31 reduces to a well-known oscillation result for ordinary differential equations.

Theorem 2.32. *Assume conditions (H2.6.1)-(H2.6.5) are fulfilled and that $p(t)$ is not eventually negative. Then any solution $y(t)$ of equation* (74) *either oscillates or satisfies*

$$\lim_{t\to\infty} \inf |y(t)| = 0.$$

Next we obtain as a corollary to the verification of Theorem 2.32, a necessary condition for equation (74) to have a non-oscillating solution.

Corollary 2.11: Assume that

i) $q(t) \geq q > 0$;

ii) $p_1 \leq p(t) \leq p_2$;

iii) there exist a constant $A > 0$ such that

$$|f(u)| \geq A\,|u| \quad \text{for all } u; \tag{88}$$

iv) $p(t)$ is not eventually negative.

Then all solutions of equation (74) are oscillatory.

Note that Corollary 2.11 is an extension of Theorem 2.22 in section 2.6.2, the second order version of Theorem 7 in the monograph by Grammatikopoulos et al. (1988a) and Theorem 4 in the article by Ladas and Sficas (1986). Theorem 1 in Zahariev and Bainov (1980) includes Corollary 2.11 when $p(t) \equiv p > 0$ and $\tau = \sigma$. However, their method of proof does not appear to carry over under the hypothesis of Corollary 2.11. A similar remark can be made about the second order versions of the result in the work by (Zahariev and Bainov, 1986).

It seems reasonable to ask if the conclusion of Corollary 2.11 can be obtained with equation (88) replaced by either condition H2.6.4 or requiring f to be increasing. Another interesting question is whether this corollary can be verified without the requirement that $p(t)$ is not eventually negative. Theorem 2.29 may be considered a partial answer to the last question in case $p(t)$ is eventually negative and bounded from below by -1.

The next theorem shows that if $p(t)$ is bounded, with upper bound less than -1, then conditions H2.6.4 and H2.6.5 are sufficient to ensure that bounded non-oscillating solutions of equation (74) tend to zero as $t \to \infty$.

Theorem 2.33. *Assume conditions (H2.6.1)-(H2.6.5) are fulfilled and there exist constants* p_1 *and* p_3 *such that*

$$p_1 \le p(t) \le p_3 < -1. \tag{89}$$

Then each bounded solution $y(t)$ *of equation* (74) *either oscillates or satisfies* $y(t) \to 0 \ \ as \ \ t \to \infty.$

We conclude with an oscillation theorem for equation (74) when $q(t)$ is τ-periodic.

Theorem 2.34. *Assume conditions (H2.6.1)-(H2.6.3) are fulfilled, that* $p(t) \equiv p > 0,$ *q(t) is* τ*-periodic and that f is increasing and satisfies*

$$f(u+v) \le f(u) + f(v) \ \ if \ \ u, v > 0;$$

$$f(u+v) \ge f(u) + f(v) \ \ if \ \ u, v < 0;$$

$$f(ku) \le kf(u) \ \ if \ \ k > 0 \ \ and \ \ u > 0 \tag{90}$$

and

$$f(ku) \ge kf(u) \ \ if \ \ k > 0 \ \ and \ \ u < 0. \tag{91}$$

Then each solution of equation (74) *is oscillatory.*

Theorem 2.34 includes Theorem 2.26 in section 2.6.1 and the second order version of Theorem 9 in the monograph by (Grammatikopoulos et al., 1988b) as special cases.

2.6.3 Linearized oscillation

We consider the second order non-linear delay differential equations

$$y''(t) + A(t)y'(t) + B(t)f(y(t-\tau)) = 0, \;\; t \geq 0 \tag{92}$$

and

$$y''(t) - A(t)y'(t) + B(t)f(y(t-\tau)) = 0, \;\; t \geq 0 \tag{93}$$

where

$$\tau > 0, \;\; A, \; B \in C(R_+, (0, \infty)), \quad f \in C(R, R),$$

$$\lim_{t\to\infty} A(t) = a \in (0, \infty), \quad \lim_{t\to\infty} B(t) = b \in (0, \infty). \tag{94}$$

The sunflowing equation

$$y''(t) + \frac{a}{\tau}y'(t) + \frac{b}{\tau}\sin y(t-\tau) = 0, \;\; t \geq 0 \tag{95}$$

is a special case of equation (92). Under some assumptions, the following equations are called the linearized limiting equations of equations (92) and (93) respectively:

$$y''(t) + ay'(t) + by(t-\tau) = 0, \;\; t \geq 0 \tag{96}$$

and

$$y''(t) - ay'(t) + by(t-\tau) = 0, \;\; t \geq 0 \tag{97}$$

respectively (Kulenovic et al., 1987a,b).

We establish the relations between the oscillations of equations (92), (93) and that of their linearized limiting equations, (96) and (97) respectively.

The following lemmas are useful for the formulation of the theorems on oscillation.

Lemma 2.7. Assume that $a,\ b,\ \tau \in (0,\infty)$ and every solution of equation (96) is oscillatory. Then there exist an $\varepsilon \in (0,b)$ such that every solution of the equation

$$z''(t) + (a+\varepsilon)z'(t) + (b-\varepsilon)z(t-\tau) = 0 \tag{98}$$

is oscillatory also.

Lemma 2.8. Assume that $A,\ B \in C\left(R_{+},(0,\infty)\right),\quad f \in C(R,R),\quad u\cdot f(u) > 0 \ \ as\ \ u \neq 0\ \ and\ \ |u| \leq H,$

where $H \in (0,\infty)$ and f is non-decreasing in [-H,H]. If

$$x(t) \geq \int_t^{\infty}\int_T^{s} B(u)f(y(u-\tau))\exp(-\int_u^{s} A(v)dv)duds,\ \ t \geq T \tag{99}$$

has a positive solution $y(t) : [T-\tau,\infty) \to (0,H]$, then

$$z(t) \geq \int_t^{\infty}\int_T^{s} B(u)f(z(u-\tau))\exp(-\int_u^{s} A(v)dv)duds,\ \ t \geq T \tag{100}$$

has a positive solution $z(t)\ \ on\ \ [T-\tau,\infty)$and

$$0 < z(t) \leq x(t). \tag{101}$$

The following important theorems are direct consequences of the above.

Theorem 2.35. *Assume that*

i) $uf(u) > 0$ *for* $u \neq 0, \ |u| \leq H$, *where* $H \in (0, \infty)$ *and* $\lim\limits_{u \to 0} \dfrac{f(u)}{u} = 1$,

ii) The characteristic equation of equation (96)

$$f(\lambda) = \lambda^2 + a\lambda + be^{-\lambda\tau} = 0 \tag{102}$$

has no negative roots.

Then every solution of equation (92) *whose graph lies eventually in the strip* $R_+ \times [-H, H]$ *is oscillatory.*

Theorem 2.36. *Assume that*

i) $u \cdot f(u) > 0$ *for* $u \neq 0, \ \lim\limits_{|u| \to \infty} \dfrac{f(u)}{u} = 1$,

ii) The characteristic equation of equation (97)

$$f(\lambda) = \lambda^2 - a\lambda + be^{-\lambda\tau} = 0 \tag{103}$$

has no positive roots. Then every solution of equation (93) *is oscillatory.*

The following results are about the existence of non-oscillatory solutions, where condition (94) is no longer required.

Theorem 2.37. *Assume that*

i) there exist $a > 0, \ b > 0$ *such that* $A(t) \geq a, \ B(t) \geq b, \ t \geq 0$;

ii) there exist an $H > 0$ *such that* $u \cdot f(u) > 0$, *for* $u \in (0, H]$, $f(u) \leq u$ *for* $u \in [0, H]$, *and* f *is non-decreasing on* $[0, H]$;

iii) The characteristic equation (102) *has a real root.*

Then equation (92) *has an eventually positive solution* $y(t)$ *lying in the strip* $R_+ \times (0, H]$ *eventually.*

Theorem 2.38. *Assume that*

i) $u \cdot f(u) > 0$ *for* $u \neq 0$;

ii) there exist an $M > 0$ *such that* $f(u) \leq u,$ *for* $u \geq M$;

iii) there exist $a > 0,\ \ b > 0$ *such that* $A(t) \geq a,\ \ B(t) \leq b,\ \ t \geq 0$ *and* f *is non-decreasing on* $[0, \infty)$; $f(u) \leq u$ *for* $u \in [0, H]$;

iv) The characteristic equation (103) *has a real root.*

Then equation (93) *has an eventually positive solution* $y(t)$ *lying in the strip* $R_+ \times (0, H]$ *eventually.*

CHAPTER THREE

METHODOLOGY

3.1 Introduction

The theory of impulsive differential equations is based on the behaviour of processes under the influence of short-time but intensive perturbations. The duration of these perturbations are extremely small and can be ignored compared to the total duration of the process itself. Therefore, they are regarded as 'momentary', that is, the perturbations are of impulsive type.

In ordinary differential equations,the solutions are continuously differentiable at least once or more, whereas impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play a fundamental role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differentiations are basically different, including the definitions of some of the basic concepts. Such concepts as the positive (negative) solutions defined on the interval $[t_0, \infty)$, the oscillatory behaviour of some solutions and the existence of solutions on the given interval are some of the concepts most affected (Isaac, 2008).

In this chapter, we will visit some of the regularly used concepts which are clearly different from those of ordinary differential equations. Moreover, we will provide some basic lemmas used in establishing the oscillatory behaviour of the solutions of the differential equations in question.

3.2 Existence of solution

Let$\Omega \subset R^n$ be an open set and let $D = R_+ \times \Omega$. Let us assume that for each $k = 1, 2, \cdots, \tau_k \in C[, (0, \infty)]$, $\tau_k(y) < \tau_{k+1}(y)$ and $\lim\limits_{k \to \infty} \tau_k(y) = \infty$ for $y \in \Omega$ For convenience of notation, we shall assume that $\tau_0 \equiv 0$ and that k always runs from 1 to ∞. Also, let $S := \{t : t = \tau_k(y), y \in R^n\}$ which are surfaces $\forall\ k,\ 1 \leq k \leq \infty$.

In addition let $y : (a, b) \subset R \to \Omega$ and let $\Delta y(t) = y(t + 0) - y(t - 0)$.

Let $f : D \rightarrow R^n$ be a continuous (differentiable, local or global Lipschitz continuous function). Let $l_k(y) : \Omega \rightarrow R^+, \forall\ k \in \mathsf{N}$ be a piece-wise continuous function.

Consider the initial value problem of the impulsive differential system

$$\begin{cases} y' = f(t,y), \ \ t \neq \tau_k(y) \\ \Delta y(t) = l_k(y), \ \ t = \tau_k(y) \\ y(t_0^+) = y_0, \ \ t_0 \geq 0, \end{cases} \tag{104}$$

where $f : D \rightarrow R^n$ and $l_k : \Omega \rightarrow R^n$, $\forall\, k \in \mathbb{N}$.

Definition 3.1. A function $y : (t_0, t_0 + a) \rightarrow R^n, \ \ t_0 > 0, \ \ a > 0$ is said to be a solution of system (104) if

i) $y(t_0^+) = y_0$ and $(t, y(t)) \in D$ for $t \in (t_0, t_0 + a)$;

ii) $y(t)$ is continuously differentiable and satisfies $y'(t) = f(t, y(t))$ for $t \in (t_0, t_0 + a)$ and $t \neq \tau_k(y(t))$;

iii) If $t \in (t_0, t_0 + a)$ and $t = \tau_k(y(t))$, *then* $y(t^+) = y(t) + l_k(y(t))$, and for such t's we always assume that $y(t)$ is left continuous.

Lemma 3.1. *The solution y as defined in definition 3.1 fulfils: $\exists \delta > 0$ such that $s \neq \tau_j(y(s))$, $\tau_k(y(t)) = t < s < \delta\ \&\ \forall j \in \mathbb{N}$.*

Proof. The proof follows from the definition of $\tau_k, k \in \mathbb{N}$ and the properties of the solution.

a) From the properties of $\tau_k, k \in \mathsf{N}$: Since $\tau_k(y) < \tau_{k+1}(y), \ \forall y \in \Omega, \ \forall k \in \mathsf{N}$ follows that $\Phi_k(y_1, y_2) := \tau_{k+1}(y_1) - \tau_k(y_2), \forall\ (y_1, y_2) \in \Omega \times \Omega$ is strictly positive on the diagonal. By the continuity of Φ at $(y, y) \in \Omega \times \Omega$, $\exists \delta(y) > 0$ such that $\Phi_k(y_1, y_2) > \dfrac{2\Phi_k(y, y)}{3}$, $\forall\ (y_1, y_2) \in B_{\delta(y)}(y) \times B_{\delta(y)}(y)$.

b) By the same property $\tau_k(y) < \tau_{k+1}(y), \ \forall y \in \Omega, \ \forall k \in \mathbb{N}$ follows that $\Phi_k(y_1, y_2) = \tau_{k+1}(y_1) - \tau_k(y_2) < \tau_t(y_1) - \tau_{k+1}(y_1) + \Phi_k(y_1, y_2) = \tau_t(y_1) - \tau_k(y_2)$, $\forall\, t > k$.

c) Since $\tau_k(y(t)) = t < s$ is investigated, $s \neq \tau_k(y(t)) = t$. y is continuous on an interval $t < s < \delta_0$. Hence $\exists\ \eta > 0$ such that $y(s) \in B_{\delta(y(t))}(y(t))$, $\forall\ t < s < t + \eta$. Moreover let $\delta := \min\{\eta, \dfrac{2\Phi_k(y(t), y(t))}{3}\}$. Then by (a), (b) and definition of $\delta, t < s < \delta, s \neq \tau_{k+1}(y(s))$. Hence by (b) the statement stands for all $k \in \mathbb{N}$.

□

It should be observed that instead of the usual initial condition $y(t_0) = y_0$, we have imposed the limiting condition $y(t_0^+) = y_0$ which, in general, is natural for system (104) since (t_0, y_0) may be such that $t_0 = \tau_k(y_0)$ for some k. Whenever $t_0 \neq \tau_k(y_0)$ for any k, $y(t_0^+) = y_0$ will be understood in the usual sense of initial condition $y(t_0) = y_0$.

Unlike ordinary differential systems, the impulsive system (104) may not have any solution at all even if f is continuous (or continuously differentiable) since the only solution $y(t)$ of the problem $y' = f(t, y)$, $y(t_0) = y_0$ may totally lie on a surface S. Hence we need some extra conditions on τ_k and/or f besides continuity in order to establish any general existence theorem for system (104).

Consequently, we state the following theorem:

Theorem 3.1. *Assume that*

i) $f : D \to R^n$ *is continuous at* $t \neq \tau_k(y)$, $\forall\ k \in \mathbb{N}$ *and for each* $(t, y) \in D$, *there exist an* l *such that, in a neighbourhood of* (t, y),

$$|f(s, z)| \leq l(s); \tag{105}$$

ii) If $\exists\, k \in \mathbb{N}$, $t_1 = \tau_k(y_1)$ *implies the existence of a* $\delta > 0$ *such that*

$$t \neq \tau_k(y) \tag{106}$$

for any $0 < t - t_1 < \delta$ *and* $|y - y_1| < \delta$.

Then for each $(t_0, y_0) \in D$*, there exist a solution* $y : (t_0, t_0 + \alpha) \to R^n$ *of the initial value problem* (104) *for some* $\alpha > 0$.

It is obvious that condition (106) is reasonable only for irregular functions $\tau_k(y)$ since the theory of implicit functions implies that if τ_k is differentiable at y_0 and $\tau_k'(y_0) \neq 0$, then condition (106) can never hold. However, we have the following theorem where some regularity conditions on $\tau_k(y)$ are required.

Theorem 3.2. *Assume that*

i) $f : D \to R^n$ *is continuous*

ii) $\tau_k : \Omega \to (0, \infty)$ *are differentiable*

iii) If $t_1 = \tau_k(y_1)$ *for some* $(t_1, y_1) \in D$ *and* $k \geq 1$*, then there is a* $\delta > 0$ *such that*

$$\left(\frac{d\tau_k(y)}{dy}, f(t, y)\right) \neq 1, \tag{107}$$

for $(t, y) \in D$ *such that* $|y - y_1| < \delta$ *and* $0 < t - t_1 < \delta$.

Notice that the left hand side of relation(107) represents a scalar product. Then for each $(t_0, y_0) \in D$, there exist a solution $y : (t_0, t_0 + \alpha) \to R^n$ of the system (104) for some $\alpha > 0$.

Here we limited ourselves to the simplest conditions and theorems only. When delay is introduced, the situation becomes much more complicated because on the right side of the system (104) there may be more discontinuities than in the case without delay. When $t \notin S$, $y(t)$ should be continuous.The right side of system (104) may, however, contain a delay point $y(t - \tau_k)$ such that $t - t_k \in S$, thus forcing the right side to be continuous (Isaac, 2008).

3.3 Qualitative behaviour

In this section, we will formulate some basic concepts about the qualitative behaviour of impulsive differential equations. But first, we will consider an ordinary differential equation given as follows:

Let $f : [0, \infty) \times \Omega \to \Omega$ be a continuous function and let f fulfil Lipschitz condition in the spatial variable for each fixed t. We also assume that f is continuously differentiable with respect to the spatial variables.

The qualitative analysis examines the solution of an initial value problem of the form

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

Here we select an arbitrary solution of the initial value problem and desire to see how the other solutions behave. In other words we investigate the behaviour of the difference between our selected solution $y(t)$ and another solution $z(t)$. We therefore need the equation describing the difference $z(t) - y(t)$, where $y(t)$ is 'known' while $z(t)$ varies. This leads to the following equation:

$$z'(t) - y'(t) = f(t, z(t)) - f(t, y(t)).$$

We let $\varphi(t) = z(t) - y(t), \ \forall t$ belonging to the specified domain and have that

$$\frac{d\varphi(t)}{dt} = f(t, \varphi(t)) + y(t)) - f(t, y(t)).$$

If f is differentiable then

$$\frac{d\varphi(t)}{dt} = \frac{\partial f(t, y(t))}{\partial y}\varphi(t) + r(t, \varphi(t)),$$

where the difference between $y(t)$ and $z(t)$ is described by a non-homogeneous linear differential equation of

$$\frac{d\varphi(t)}{dt} = A(t)\varphi(t) + r(t, \varphi(t)).$$

Here $\lim_{\|h\| \to 0} \frac{\|r(t,h)\|}{\|h\|} = 0$ holds, hence the identically zero function is a solution.

Having discussed this, we now assume $y(t)$ to be the solution of an arbitrary impulsive differential equation.

Definition 3.2. The solution $y(t)$ is said to be regular if it is defined on a half line $[T_x, \infty)$ for some $T_x \in R$ and $\sup\{|y(t)| : t \geq T\} > 0 \ \forall \ T > T_x$.

The oscillatory solutions will be defined in a way different from the classical theory since the solutions are piece-wise continuous only.

Let us begin with the non-oscillatory behaviour of the solution.

Definition 3.3. The solution $y(t)$ is said to be

i) finally positive, if there exist $T \geq 0$ such that $y(t)$ is defined and is strictly positive for $t \geq T$;

ii) finally negative, if there exist $T \geq 0$ such that $y(t)$ is defined and is strictly negative for $t \geq T$ (Isaac et al., 2011b).

Definition 3.4. The solution $y(t)$ is said to be non-oscillatory, if it is either finally positive or finally negative.

Definition 3.5. The solution $y(t)$ is said to be oscillatory, if it is neither finally positive nor finally negative.

It can be seen that finally positive or finally negative solutions are regular solutions. Moreover, regular oscillatory solutions are the real oscillatory solutions because Definition 3.5 is fulfilled by an identically zero solution.

Having defined these concepts, let us examine some theories that will serve as tools for the results of the main work and the determination of the oscillatory (or non-oscillatory) behaviour of solutions.

3.4 Fixed point theory

Over the last 50 years the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena, especially in problems related with the existence and uniqueness of solutions of differential equations. In fact, fixed point methods are most important in solving non-linear differential problems. There are several ways to reduce a non-linear existence problem to a fixed point problem (for a mapping in function space). The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.

On the other hand, fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. In mathematics, their applications abound in the theory of existence of solutions for differential,integral and other equations in the diverse areas of mathematics.

In order to fully understand the concept of fixed point theory and its application to the obtainability of sufficient conditions for the existence of solutions of differential equations, we will begin by giving some definitions of associated terms.

3.5 Some basic definitions

Definition 3.6. Given a vector space X over a subfield F of the complex numbers, a norm on X is a real-valued function $\rho(x) : X \longrightarrow R$ with the following properties:

i) $\rho(x) = 0, \Leftrightarrow \rho(x) = 0, \forall x \in X$;

ii) $\rho(\alpha x) = |\alpha| \, \rho(x), \ \ \forall \alpha \in F, \forall x \in X$;

iii) $\rho(x + y) \leq p(x) + p(y), \forall x, \, y \in X$.

Definition 3.7. A vector space X on which a norm $\|\cdot\|$ is defined is called a normed vector space.

Definition 3.8. A subset S of a normed vector space X is said to be bounded if there is a number M such that $\| x \| \leq M$ for all $x \in S$.

Definition 3.9. A subset S of a normed vector space X is called convex if, for any $x, \ y \in S, \ \ ax + (1 - a)y \in S$ for all $a \in [0, 1]$.

Definition 3.10. A sequence $\{x_n\}$ in a normed vector space X is said to converge to the vector $x \in X$ if and only if the sequence $\|x_n - x\|$ converges to zero as $n \to \infty$.

Definition 3.11. A sequence $\{x_n\}$ in a normed vector space X is a Cauchy sequence in X if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, \ m \geq N\,(\varepsilon)$.

Remark 3.1: Clearly, a convergent subsequence is a Cauchy sequence, but the converse may not be true.

Definition 3.12. A space X where every Cauchy sequence of elements of X converges to an element of X is called a complete space. A complete normed vector space is said to be a Banach space.

Definition 3.13. Let M be a subset of a Banach space X. A point $x \in X$ is said to be a limit point of M if there exists a sequence of vectors in M which converges to x.

Definition 3.14. We say a subset M is closed if M contains all of its limit points. The union of M and its limit points is called the closure of M and will be denoted by $\overline{M}$.

Definition 3.15. Let N, M be normed spaces, and X, a subset of N. A mapping $T : X \to M$ is continuous at a point $x \in X$ if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon$ for all $y \in X$ such that $\|x - y\| < \delta$.

Remark 3.2: T is continuous on X, or simply continuous, if it is continuous at all points of X.

The following result is worth knowing.

Theorem 3.3. *Every continuous mapping of a closed bounded convex set in R^n into itself has a fixed point.*

Definition 3.16. A subset S of a Banach space X is compact, if every infinite sequence of elements of S has a subsequence which converges to an element of S. We say M is relatively compact if every infinite sequence in S contains a subsequence which converges to an element in X. That is, M is relatively compact, if $\overline{M}$ is compact.

Definition 3.17. A family S in $C([a, b], R)$ is called uniformly bounded if there exists a positive number M such that $|f(t)| \leq M$ for all $t \in [a, b]$ and all $f \in S$.

Definition 3.18. S is called equicontinuous if for every *varepsilon* > 0 there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ for all $t_1,\ t_2 \in [a, b]$ with $|t_1 - t_2| < \delta$ and for all $f \in S$.

Theorem 3.4. *(Arzela-Ascoli Theorem) A subset S in $C([a, b], R)$ with norm*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.

Definition 3.19. A topology $\mathbf{T}$ on a linear space E is called locally convex if every neighborhood of the element zero includes a convex neighborhood of zero.

Definition 3.20. A real valued function $p(x)$ defined on a linear space X is called a semi-norm on X if the following conditions are satisfied:

i) $p(x) \geq 0$, $x = 0 \Rightarrow p(x) = 0$, $\forall x \in X$;

ii) $p(\alpha x) = |\alpha|\, p(x)$, $\forall \alpha \in R$, $\forall x \in X$;

iii) $p(x+y) \leq p(x) + p(y)$, $\forall x,\, y \in X$.

Remark 3.3: From this definition, we can prove that a semi-norm $p(x)$ satisfies

$$p(x_1 - x_2) \geq |p(x_1) - p(x_2)|\,.$$

However, in contrast to norms, it may happen that $p(x) = 0$ for $x \neq 0$.

Definition 3.21. A family P of semi-norms on X is said to be separating if to each $x \neq 0$ there exists at least one $p \in P$ with $p(x) \neq 0$.

Remark 3.4. For a separating semi-norm family P, if $p(x) = 0$ for every $p \in P$, then $x = 0$.

A locally convex topology $\mathbf{T}$ on a linear space is determined by a family of semi norms $\{p_\alpha : \alpha \in I\}$, I being the index set.

Let E be a locally convex space, and x, $\{x_n\}_{n=1}^{\infty} \in E$. Then $x_n \to x$ in E if and only if $p_a(x_n \to x) \to 0$ *as* $n \to \infty$, for every $\alpha \in I$.

Definition 3.22. A set $S \subset E$ is bounded if the set of numbers $\{p_\alpha(x),\ x \in S\}$ is bounded for every $\alpha \in I$.

Definition 3.23. A complete metrizable locally convex space is called a Frechet space.

Example 3.1. The space of functions $C([t_0, \infty), R)$ is a locally convex space consisting of the set of all continuous functions. The topology of the space is the topology of uniform convergence on every compact interval of $[t_0, \infty)$. The semi-norm of the space $C([t_0, \infty), R)$ is defined by $\rho_\alpha(x) = \max_{x \in [t_0, \alpha)} |x(t)|$, where $x \in C$, $\alpha \in [t_0, \infty)$.

Definition 3.24. Let X be any set. A metric on X is a function $d : X \times X \to R$ having the following properties for all $x, y, z \in X$:

i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$

ii) $d(y, x) = d(x, y)$

iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space is a set X together with a given metric on X.

Definition 3.25. A complete metric space is a metric space X in which every Cauchy sequence converges to a point in X.

Definition 3.26. Let (X, d) be a metric space and let $T : X \to X$. If there exists a number $r \in [0, 1)$ such that $d(Tx, Ty) \leq r \cdot d(x, y)$ for every $x, y \in X$, then we say T is a contraction mapping on X.

Having established some background knowledge of necessary topological concepts for the understanding of fixed point theory, we now give a list of some well-known fixed point theorems. They include, but are not limited to the following: Banach fixed point theorem (Contraction mapping principle), Brouwer fixed point theorem, Knaster–Tarskifixed-point theorem, Atiyah–Bott fixed-point theorem, Borel fixed-point theorem, Caristi fixed-point theorem, Kakutani fixed-point theorem, Kleene fixed-point theorem, Lefschetz fixed-point theorem, Nielsen fixed-point theorem, Woods Hole fixed-point theorem, Schauder fixed point theorem, Tychonoff fixed-point theorem, Krasnoselkii fixed point theorem and Schauder-Tychonoff fixed point theorem.

We will examine the last four fixed point theorems due to their direct application to the analysis of solutions of nonlinear functional equations.

Theorem 3.5. *(Schauder's fixed point theorem) Let S be a closed convex and nonempty subset of a Banach space X. Let $T : S \to S$ be a continuous mapping such that T(S) is a relatively compact subset of X. Then T has at least one fixed point in S. That is, there exists an $x \in X$ such that $Tx = x$.*

One observes here that in oscillation theory we usually want to prove that the family of functions is uniformly bounded and equicontinuous on $[t_0, +\infty)$. According to Levitan (1947), the family S is equicontinuous on$[t_0, \infty)$ if for any given$\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family S have oscillations less than ε.

Theorem 3.6. *(Tychonoff fixed point theorem) Let X be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f : K \to K$ there exists $x \in K$ such that $f(x) = x$.*

Remark 3.5. Notice that a normed vector space is a locally convex topological vector space, therefore this theorem extends the Schauder fixed point theorem.

In 1935, the Soviet mathematician H. Tychonoff gave a generalization of the Schauder fixed point theorem for locally convex vector spaces (Tychonoff, 1935). This result is usually termed the Schauder-Tychonoff theorem.

Theorem 3.7. *(Schauder-Tychonoff fixed point theorem) Let X be a locally convex linear space, S a compact convex subset of X, and let $T : S \to S$ be a continuous mapping with $T(S)$ compact. Then T has a fixed point in S.*

Theorem 3.8. *(Krasnoselskii's Fixed Point Theorem). Let X be a Banach space, Ω a bounded closed convex subset of X and A, B be maps of Ω into X such that $Ax + By \in \Omega$ for every pair $x, y \in \Omega$. If A is a contraction and B is completely continuous, then the equation $Ax + Bx = x$ has a solution in Ω.*

3.6 Nagumo condition

A Nagumo condition for ordinary differential equations is a given condition which guarantees that each solution of the nth order ordinary differential equation

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$$

either extends or becomes unbounded on its maximal interval of existence.In particular, the classical Nagumo condition for the second order ordinary differential equation

$$y'' = f(x, y, y') \tag{108}$$

is a growth condition on $f(x, y, y')$ which implies that solutions of equation (108) either extend or become unbounded on their maximal intervals of existence. Nagumo (1937) used this growth condition on $f(x, y, y')$ to prove the existence of solutions of boundary-value problems, assuming that $f(x, y, y')$ is continuous. One formulation of the condition is contained in the following theorem.

Theorem 3.9. *Assume that equation* (108) *is a scalar equation with $f(x, y, y')$ continuous on $(a, b) \times R^2$. If for each $M > 0$ and each compact interval $[c, d] \subset (a, b)$ there is a corresponding positive continuous function $\phi(s)$ on $[0, \infty)$ such that $|f(x, y, y')| \leq \phi(|y'|)$ for all (x, y, y') satisfying $c \leq x \leq d$, $|y| \leq M$ and such that $\int_0^\infty \frac{s}{\phi(s)} ds = +\infty$, then each solution of equation* (108) *either extends to (a,b) or becomes unbounded on its maximal interval of existence.*

Other formulations from which Theorem 3.8 follows may be found in Hartman (1964) and Jackson (1968).

This property of solutions which is stated as the conclusion in Theorem 3.8 along with the assumed existence of solutions of certain types of differential inequalities plays an important role in demonstrating the existence of solutions of boundary value problems, not only for second order equations but for higher order equations as well (Hartman, 1964; Kelley, 1975; Klaasen, 1971; Schrader, 1969, Bebernes, Gaines and Schmitt, 1974).

We conclude this section by stating the following simple but very important theorem.

Theorem 3.10. *Assume that for each $b > a \cdot f(x,y,y')$ satisfies a Nagumo condition on $[a,b]$ with respect to the pair $\alpha(x), \beta(x) \in C^1[a,\infty)$, where $\alpha(x) \leq \beta(x)$ on $[a,\infty)$, and $\alpha(x)$ and $\beta(x)$ are, respectively, lower and upper solutions on $[a,\infty)$. Then for any $\alpha(a) \leq c \leq \beta(a)$, the boundary value problem*

$$y'' = f(x,y,y'), \quad y(a) = c \tag{109}$$

has a solution $y(x) \in C^2[a,\infty)$ with $\alpha(x) \leq y(x) \leq \beta(x)$ on $[a,\infty)$.

There is yet another concept that will play an important role in the discussion of the main work. This is Sturm's Comparison Theorem. In what follows, we present a brief discussion of the concept.

3.7 Sturm's comparison theorem

We consider the second order linear delay equations of the form

$$(p(t)x'(t))' + \sum_{i=1}^{n} q_i(t)x\left(\tau_i(t)\right) + \int_{\tau(a)}^{t} k(s,t)x(s)ds = 0, \;\; t \in [a,b) \tag{110}$$

and

$$(p(t)y'(t))' + \sum_{i=1}^{n} Q_i(t)y\left(\tau_i(t)\right) + \int_{\tau(a)}^{t} K(s,t)y(s)ds = 0, \;\; t \in [a,b), \tag{111}$$

where $a < b \leq \infty$, $p \in C^1([a,b),(0,\infty))$, q_i, Q_i, $K(s,t)$, $k(s,t)$ are continuous functions over $[a,b)$ and $\{(s,t) : s \leq t,\ a \leq t < b\}$, respectively. Also, $\tau_i(t) \leq t$, where τ_i is continuous, $i = 1,\ 2,\ \cdots,\ n$, and

$$\tau(t) = \min\ \{\tau_i(s),\ s \geq t,\ i = 1,\ 2,\ \cdots,\ n\}. \tag{112}$$

For a given initial function $\phi \in C[\tau(a),a]$, there exists a unique solution $x(t)$ to equation (110) in $[a,b)$ with

$$x(t) = \phi(t),\ \ t \in [\tau(a),a] \text{ and } x'(a^+) = \phi'(a^-). \tag{113}$$

Let $\psi(t) \in C[\tau(a),a]$ be an initial function for equation (111) and $y(t)$ be the corresponding solution to equation (111) with the initial condition given by $\psi(t)$.

For equations (110) and (111) assume the following comparison conditions hold.

(A_1) $Q_i(t) \geq |q_i(t)|,\ \ i = 1,\ 2,\ \cdots,\ n,$

(A_2) $K(s,t) \geq |k(s,t)|,\ \ \ s,\ t \in \{a,b),$

(A_3) $\dfrac{\psi(t)}{\psi(a)} \geq \left|\dfrac{\phi(t)}{\phi(a)}\right|,\ \ t \in [\tau(a),a].$

If we assume that all of the $q_i(t)$ and $k(s,t)$ are nonnegative, then we can relax condition (A_3) to get the following conditions:

(B_1) $Q_i(t) \geq q_i(t) \geq 0,\ \ i = 1,\ 2,\ \cdots,\ n,$

(B_2) $K(s,t) \geq k(s,t) \geq 0,\ \ \ s,\ t \in \{a,b),$

(B_3) $\dfrac{\psi(t)}{\psi(a)} \geq 0,\ \ \ \dfrac{\psi(t)}{\psi(a)} \geq \dfrac{\phi(t)}{\phi(a)},\ \ t \in [\tau(a),a].$

Likewise, if we assume that $\frac{\phi(t)}{\phi(a)} \geq 0$, we can relax conditions (A_1) and (A_2) to get the following conditions:

(C_1) $Q_i(t) \geq 0,\ \ Q_i(t) \geq q_i(t),\ \ i = 1,\ 2,\ \cdots,\ n,$

(C_2) $K(s,t) \geq 0,\ \ K(s,t) \geq k(s,t),\ \ \ s,\ t \in \{a,b),$

(C_3) $\dfrac{\psi(t)}{\psi(a)} \geq \dfrac{\phi(t)}{\phi(a)} \geq 0,\ \ t \in [\tau(a), a].$

In the following, we will use the conditions:

(D_1) $Q_i(t) \geq q_i(t) \geq 0,\ \ i = 1,\ 2,\ \cdots,\ n,$

(D_2) $K(s,t) \geq k(s,t) \geq 0,\ \ \ s,\ t \in \{a,b),$

(D_3) $\psi(a) \neq 0$ and $\psi(t)$ does not change sign in $[\tau(a), a]$, $\phi(a) = 0,\ \ \psi'(a) \neq 0$, and $\phi(t)$ does not change sign in $[\tau(a), a]$.

From conditions (A_3), (B_3) and (C_3), we obtain

$$\frac{\psi'(a^-)}{\psi(a)} \leq \frac{\phi'(a^-)}{\phi(a)}. \tag{114}$$

Conditions $(D_1) - (D_3)$ imply conditions $(B_1) - (B_3)$. In fact, from condition (D_3), we see that $\frac{x'(t)}{x(t)} \to \infty\ \ as\ \ t \to a^+$. A new initial point a^- can be chosen so that with the shifting of the initial interval to $[\tau(a^-), a^-]$, the conditions $(B_1)-(B_3)$ now hold.

Theorem 3.11. *Assume that one of the sets of comparison conditions* $(A_1/B_1/C_1/D_1)-(A_3/B_3/C_3/D_3)$*holds, and that the solution* $y(t)$ *of equation* (105) *does not vanish in* $[a, b)$. *Then, for all* $t \in [a, b)$

$$\frac{y'(t)}{y(t)} \leq \frac{x'(t)}{x(t)} \tag{115}$$

and

$$\frac{y(t)}{y(a)} \leq \frac{x(t)}{x(a)}. \tag{116}$$

As a consequence, $x(t)$ does not vanish in (a, b).

Now, in order to understand the application of Sturm's Comparison Theorems to delay differential equations, we associate equation (111) with the delay equation

$$\left(p(t)z'(t)\right)' + \sum_{i=1}^{n} Q_i(t)z\left(\bar{\tau}_i(t)\right) + \int_{\tau(a)}^{t} K(s,t)z(s)ds = 0, \tag{117}$$

where

$$\bar{\tau}_i(t) \leq \tau_i(t) \leq t, \quad i = 1,\ 2,\ \cdots,\ n. \tag{118}$$

We assume that the initial condition

$$z'(a^+) = \psi'(a^-) \tag{119}$$

holds. Furthermore, assume that

$$Q_i(t), \;\; K(s,t) \geq 0 \tag{120}$$

and

$$\psi'(t) < 0 \;\; or \;\; \psi'(t) \geq 0 \;\; in \;\; [\tau(a), a]. \tag{121}$$

Theorem 3.12. *Let $y(t)$ and $z(t)$ be, respectively, positive solutions of equations(111) and (117)in $[a, b)$, with the same initial value given by $\psi(t)$. Suppose that equations (118) and (120) hold and that $\psi'(t) \leq 0 \;\; in \;\; [\tau(a), a]$.*

Then for all $t \in [a, b)$,

$$\frac{z'(t)}{z(t)} \leq \frac{y'(t)}{y(t)} \tag{122}$$

and

$$z(t) \leq y(t). \tag{123}$$

On the other hand, if $\psi'(t) \geq 0 \;\; in \;\; [\tau(a), a]$ and both $z'(t)$ *and* $y'(t)$ are non-negative in $[a, b]$, then the reverse inequalities hold in (122) and (123).

Theorem 3.11 asserts that for a decreasing solution, a 'shorter memory' slows down oscillation, whereas for an increasing solution, it speeds up oscillation (in the sense that the solution reaches its maximum or rebounces faster, and not that the solution becomes zero faster).

We now apply Theorem 3.11 to an oscillation problem. We consider delay equations of the form

$$\left(p(t)x'(t)\right)' + \sum_{i=1}^{n} q_i(t)x\left(\tau_i(t)\right) + \int_{\tau(t)}^{t} k(s,t)x(s)ds = 0, \;\; t \geq a \tag{124}$$

with the assumptions that

$$q_i(t) \geq 0, \;\; k(s,t) \geq 0 \tag{125}$$

and

$$r(t) = \min_{t \to \infty} \; \{\tau_i(t) : \; i = 1, \; 2, \; \cdots, \; n\} \to \infty, \;\; t \to \infty. \tag{126}$$

We compare equation (124) with another delay equation

$$\left(p(t)y'(t)\right)' + \sum_{i=1}^{n} Q_i(t)y\left(\bar{\tau}_i(t)\right) + \int_{\bar{\tau}(t)}^{t} K(s,t)ds = 0, \;\; t \geq a \tag{127}$$

The following results can be easily deduced from Theorems (113) and (114).

Theorem 3.13. *Suppose that for sufficiently large s and t,*

$$Q_i(t) \geq q_i(t) \geq 0, \;\; i = 1,\, 2,\, \cdots,\, n, \tag{128}$$

$$K(s,t) \geq k(s,t) \geq 0, \;\; s < t, \tag{129}$$

$$\bar{\tau}_i(t) \geq \tau_i(t), \;\; i = 0,\, 1,\, \cdots,\, n. \tag{130}$$

If equation (124) *is oscillatory, so is equation* (127).

Theorem 3.14. *Suppose that* $p(t) \equiv 1$ *and equations* (125), (126) *hold, and the ordinary differential equation*

$$y''(t) + \theta(t)y(t) = 0 \tag{131}$$

is oscillatory. Then so is the delay equation (124), *where*

$$\theta(t) = \frac{1}{t}\left[\sum_{i=1}^{n} q_i(t)\tau_i(t) + \int_{\tau(t)}^{t} sk(s,t)dt\right]. \tag{132}$$

The complete proof of the above theorem is unfortunately outside the scope of this work. However, we will highlight some salient facts necessary to understand and apply the theorem appropriately. A close examination of the theorem reveals that the author approaches the proof by supposing the contrary and assuming the eventual positivity of the solution $x(t)$ of equation (124). This immediately implies that the derivative of the solution $x'(t)$ is also eventually positive. Consequently, by the convergence of $x'(t)$ implying the integrability of the second derivative of the solution $x''(t)$, we arrive at the finiteness of the integral of the quantity $t\theta(t)$, that is,

$$\int_a^\infty t\theta(t)dt < \infty.$$

But it is well known that the above integral implies the non-oscillation of equation (131) which contradicts the initial hypothesis. By taking a point $(t_1, x(t_1))$ on the solution curve and denoting by L the straight line joining this point and the origin $(0,0)$, we arrive at the condition of concavity, implying that L may intersect the solution curve in at most two points. In the case that there are two points of intersection, say, t_1 and t_2, the part of the straight line between these two points lies below the curve. Without loss of generality, we assume that $t_1 < t_2$. Now, let $t_3 \geq t_2$ be so large that $\tau(t) > t_1$ for all $t > t_3$. For any $t > t_3$, the line joining $(0,0)$ and any arbitrary point $(t, x(t))$ lies below L. Hence the part of this line between $\tau(t)$ and t lies below the solution curve. This implies that

$$x(\tau(t)) \geq \frac{\tau(t)}{t}x(t), \text{ for all } t > t_3. \tag{133}$$

We conclude this discussion by examining the case in which L is tangent to the solution curve at t_1. This can be treated as the degenerate case with $t_1 = t_2$. Now suppose the point $(t_1, x(t_1))$ is the only point of intersection. If L lies below the solution curve in the interval $[a, t_1]$, then equation (133) actually holds for $t > a$. Finally, let us note that the remaining case is void since the conditions of concavity and $\lim_{t\to\infty} x'(t) = 0$ dictate that the curve must meet L again.

At this juncture, we may rewrite equation (124) in the form

$$(p(t)x'(t)) + \left(\sum_{i=1}^{n} q_i(t)\frac{x(\tau_i(t))}{x(t)} + \int_{\tau_0(t)}^{t} k(s,t)\frac{x(s)}{x(t)}ds\right)x(t) = 0 \tag{134}$$

and regard it as a linear equation without delay. By equation (133) the coefficient is larger than $\theta(t)$. Therefore, from the classical Sturmian theory, equation(134) or equivalently, equation (124) oscillates faster than equation (131), and so we have a contradiction.

It would be unfair to round up this chapter without mentioning these rather unavoidable concepts, namely, convergence theorems, piece-wise continuity and quasi-equicontinuity.

3.8 Convergence theorems

Convergence theorems are concerned with the analysis of the dynamics of integrabilty in the case when sequences of measurable functions are considered. Roughly speaking, a "convergence theorem" states that integrability is preserved under the limit operator. In other words, if one has a sequence $\{f_n\}_{n=1}^{\infty}$ of integrable functions, and if f is some kind of a limit of the $f_n's$, then we would like to conclude that f itself is integrable, as well as the equality $\int f = \lim_{n\to\infty} \int f_n$. Such results are often employed in instances of proving that some function f is integrable and also in the construction of an integrable function.

We now examine two important convergence theorems.

Theorem 3.15. *(Lebesgue's Monotone Convergence Theorem) Let* $(A, \sum, \mu)$ *be a measure space and* $f_1,\ f_2,\ f_3,\ \cdots$ *a pointwise non-decreasing sequence of* $[0, \infty)-$ *valued* $\sum-$*measurable functions. Let* $\lim_{n\to\infty} f_n(t) := f(t)$ *for all* $t \in A$, *then* f *is* $\sum-$ *measurable and*

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu.$$

Theorem 3.16. *(Lebesgue's Dominated Convergence Theorem) Let* $\{f_n\}$ *be a sequence of complex measurable functions on a measurable space* $(A, \sum, \mu)$ *such that* $\lim_{n\to+\infty} f_n(t) = f(t)$ *exists for almost every* $t \in A$. *If there is a function* $g(t)$ *such that* $|f_n(t)| \leq g(t)$ $(n = 1,\ 2,\ 3,\ \cdots$ *for almost every* $t \in A)$, *where* $g(t)$ *is an integrable function defined on* A, *then*

$$\lim_{n\to+\infty} \int_A f_n(t) d\mu = \int_A f(t) d\mu.$$

Definition 3.27. A function $f(t)$ is said to be piecewise continuous on an interval $[a, b]$ if the interval can be partitioned by a finite number of points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ so that

i) $f(t)$ is continuous on each subinterval (t_{i-1}, t_i);

ii) $f(t)$ approaches a finite limit as the endpoints of each subinterval are approached from within the

interval.

Piecewise continuous functions express many natural relationships that occur in physics, engineering, etc, and most importantly in impulsive differential equations where the solutions are said to be piece-wise continuous.

Definition 3.28. Let $\{f_n\}$ be a sequence of functions from a topological space X to be metric space Y. $\{f_n\}$ is said to be $\varepsilon - related$ at a point$x \in X$ if for every arbitrarily chosen $\varepsilon > 0$ there is a neighborhood $U(x)$ of x such that, corresponding to each point $x' \in U(x)$, a positive number $N_\varepsilon(x, x')$ can be determined satisfying the condition $\rho\,[f_n(x),\ f_n(x')] < \varepsilon$ whenever $n > N_\varepsilon(x, x')$.

Definition 3.29. Let F be a family of continuous functions from a topological space X to a metric space Y. F is said to be quasi-equicontinuous if in every infinite subset Q of F and at any point $x \in X$ there is a sequence $\{f_n\}$ contained in Q which is $\varepsilon - related$ at x.

As it stands, the main tools necessary for the proofs of the results of the thesis have been assembled. We can now proceed to put them together for the attainment of the set goals.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

Second order differential equations in general, are most important in applications. Same also applies to neutral second order delay impulsive differential equations which have been developed to model impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. The introduction of oscillation and non-oscillation theory has further boosted the concept and particularly helped in identifying more areas of applications both within and outside differential equations.

In this chapter, we investigate the oscillatory properties and asymptotic behaviour of the solutions of linear neutral impulsive differential equations of the second order, impulsive integro-differential equations and nonlinear impulsive differential equations. Also, we obtain the necessary and sufficient conditions for oscillation of solutions of linear neutral impulsive equations and finally estimate the difference between the zeros of the solutions of same equations.

4.2 Oscillation criteria

4.2.1 Nonlinear case

Here, we deal with the oscillatory behaviour of solutions of the second order neutral impulsive differential equation of the form

$$\begin{cases} [y(t)-py(t-\tau)]''+q(t)f(y(t-\sigma(t)))=0, \quad t\notin S \\ \Delta[y(t_k)-py(t_k-\tau)]'+q_kf_k(y(t_k-\sigma(t_k)))=0, \quad \forall\, t_k\in S \end{cases} \tag{135}$$

under the following assumptions:

H4.2.1: p, τ and q_k are positive numbers, $\forall\ k\in Z$;

H4.2.2: $q,\ \sigma\in C(R_+,R_+)$, $\lim_{t\to\infty}(t-\sigma(t))=\infty$, $\sigma(t)>\tau$;

H4.2.3: $f\in C(R,R)$, f is increasing and $f(-y)=-f(y)$;

H4.2.4: $f\ (y \cdot x) \geq f\ (y) f\ (x)$ when $y \cdot x > 0,\ \ f(\infty) = \infty$;

H4.2.5: $f_k\ (y \cdot x) \geq f_k\ (y)\ f_k\ (x)$ when $y \cdot x > 0,\ \ \ f_k(\infty) = \infty, \forall\ \ k \in Z,$

H4.2.6: $\lim\limits_{x \to 0} \left[\frac{f(x)}{x},\ \frac{f_k(x)}{x}\right] = \infty$ or $\lim\limits_{x \to 0} \left[\frac{f(x)}{x},\ \frac{f_k(x)}{x}\right] = 1.$

The purpose of this section is to establish a relation between the oscillation problems of equation (135) and a corresponding ordinary delay differential equation. All investigations will be restricted to the strip $H \in (\rho, \infty)$ except defined otherwise.

The following lemmas will be used to prove the main results.

Lemma 4.1. Assume that $g \in C(R_+, R_+),\ g(t) \leq t$ and $\lim\limits_{t \to \infty} g\,(t) = \infty,\ x \in PC^2\,([T,\ \infty),\ R)$ and $x\,(t) > 0,\ \ x\,(t_k) > 0,\ \ \Delta x\,(t_k) > 0,\ \ x''\,(t) \leq 0,\ \ \Delta x'\,(t_k) \leq 0$ on $[T,\ \infty)$. Then for each $\ell \in (0, 1)$, there is a $T_\ell \geq T$ such that

$$x\,(g\,(t)) \geq \ell \frac{g\,(t)}{t} x\,(t)\,, \quad t \geq T_\ell. \tag{136}$$

Proof: It is sufficient to consider only those $t \in R_+$ for which $g(t) < t$. Then by the mean value theorem and the monotone properties of y' and for $t > g(t) \geq T$, we have

$$\begin{cases} x\,(t) - x\,(g\,(t)) \leq x'\,(x\,(t))\,(x - g\,(t)) \\ x\,(t_k) - x\,\left(g\,\left(t_k\right)\right) \leq \Delta x\,(x\,(t_k))\,(x - g\,(t_k))\,. \end{cases}$$

Hence

$$\begin{cases} \frac{x(t)}{x(g(t))} \leq 1 + \frac{x'(g(t))}{x(g\,(t))}\,(t - g\,(t))\,,\ \ t > g\,(t) \geq T,\ \ t \notin\ S \\ \frac{x(t_k)}{x(g(t_k))} \leq 1 + \frac{\Delta(y(t_k))}{x(g(t_k))}\,(t_k - g\,(t_k))\,,\ \ t_k > g\,(t_k) \geq T,\ \forall\, t_k \in\ S. \end{cases}$$

Also,

$$\begin{cases} x(g(t)) \geq x(T) + x'(g(t))(g(t) - T) \\ x(g(t_k)) \geq x(T) + x'(g(t_k))(g(t_k) - T) \end{cases}$$

so that for any $0 < \ell < 1$, there is a $T_\ell \geq T$ for which the following relations hold:

$$\begin{cases} \frac{x(g(t))}{x'(g(t))} \geq \ell g(t), \ t \geq T_\ell, \ t \notin S \\ \frac{x(g(t_k))}{\Delta x(g(t_k))} \geq \ell g(t_k), \ t_k \geq T_\ell, \ t_k \in S. \end{cases}$$

Hence

$$\begin{cases} \frac{x(t)}{x(g(t))} \leq \frac{t+(\ell-1)g(t)}{\ell g(t)} \leq \frac{t}{\ell g(t)}, \ t \geq T_\ell, \ t \notin S \\ \frac{x(t)}{x(g(t_k))} \leq \frac{t_k+(\ell-1)g(t_k)}{\ell g(t_k)} \leq \frac{t_k}{\ell g(t_k)}, \ t_k \geq T_\ell, \ t_k \notin S. \end{cases}$$

This completes the proof of Lemma 4.1.

Let us discuss equation (135) for the cases where $p \in (0,1)$ and $p > 1$ respectively. The beauty of the said discussion will best be displayed in the lemmas that follow.

Lemma 4.2. Assume that $p \in (0,1)$ and the condition (H) holds. If the equation

$$\begin{cases} z''(t) + q(t) f\left(\frac{\lambda(t-\sigma(t))}{t} z(t)\right) = 0, \ t \notin S \\ \Delta z'(t_k) + q_k f_k\left(\frac{\lambda(t_k-\sigma(t_k))}{t_k} z(t_k)\right) = 0, \ \forall \ t_k \in S \end{cases} \tag{137}$$

is oscillatory for some $0 < \lambda < 1$, then the non-oscillatory solutions of equation (135) tend to zero as $t \to \infty$.

Proof: Without loss of generality, let $y(t)$ be a finally positive solution of equation (135) and define

$$z(t) = y(t) - py(t-\tau).$$

From equation (135), we have that $z''(t) \leq 0$ for $t \geq T$ and $\Delta z'(t_k) \leq 0 \ \forall \ k :$

$t_k \geq T$. If $z'(t) < 0$ finally, then

$$\begin{cases} \lim\limits_{t\to\infty} z(t) = -\infty \\ \lim\limits_{t_k\to\infty} z(t_k) = -\infty. \end{cases} \tag{138}$$

But $z(t) < 0$ finally implies that $\lim\limits_{t\to\infty} y(t) = 0$ which contradicts equation (138). Therefore, $z'(t)\,,\ \Delta z(t_k) > 0$ for $t \geq T_1$ and $\forall\, k : t_k > T$. Here, there are two possibilities for $z(t)$:

i) $z(t) > 0$ for $t \geq T$;

ii) $z(t) < 0$ for $t \geq T_1$.

For case (i), there is a $T_\ell \geq T$ such that

$$\begin{cases} z(t-\sigma(t)) \geq \frac{\ell(t-\sigma(t))}{t} z(t),\ \ t \geq T_\ell,\ \ t \notin S \\ z(t_k-\sigma(t_k)) \geq \frac{\ell(t_k-\sigma(t))}{t_k} z(t_k),\ \ t_k \geq T_\ell,\ \ \forall\, t_k \in S \end{cases}$$

by virtue of Lemma 4.1 and for each $\ell \in (0,\, 1)$. Since $0 < z(t) < y(t)$ from equation (135), we have

$$\begin{cases} z''(t) + q(t) f\left(\frac{\ell(t-\sigma(t))}{t} z(t)\right) \leq 0,\ \ t \notin S \\ \Delta z'(t_k) + q_k f_k\left(\frac{\ell(t_k-\sigma(t_k))}{t_k} z(t_k)\right) \leq 0,\ \ \forall\ t_k \in S. \end{cases}$$

Using Theorem 3.9, we see that equation (137) has a finally positive solution. This contradicts the assumption. For case (b), as was mentioned before, this will yield $\lim\limits_{t\to\infty} y(t) = 0$. The proof of Lemma 4.2. is hereby completed.

Theorem 4.1. *In addition to the conditions of Lemma 4.2, assume further that*

$$\limsup_{t\to\infty} \left(\int_{t-\sigma(t)+\tau}^{q} (u - (t-\sigma(t) + \tau))\, q(u)\, du \right) > \begin{cases} p, & if\ \lim\limits_{x\to 0} \frac{f(x)}{x} = 1 \\ 0, & if\ \lim\limits_{x\to 0} \frac{f(x)}{x} = \infty \end{cases} \tag{139}$$

$$\limsup_{t\to\infty}\left(\sum_{t-\sigma(t)+\tau\le t_k<t}(t_k-(t-\sigma(t)+\tau))\,q_k\right)$$
$$>\begin{cases} p, & if \quad \lim_{x\to 0}\frac{f_k(x)}{x}=1 \\ 0, & if \quad \lim_{x\to 0}\frac{f_k(x)}{x}=\infty \end{cases} \qquad (140)$$

Then every solution of equation (135) *is oscillatory.*

Proof: As in the proof of Lemma 4.2, it suffices to show that $z(t)<0$ for $t\ge T$ is possible under the given assumptions. Suppose that $y(t)>0$, $z''(t)$, $\Delta z'(t_k)\le 0$, $z'(t)$, $\Delta z(t_k)>0$ and $z(t)<0$ finally for $t\ge T$ and $k:\ t_k\ge T$. Then

$$\begin{cases} z(t-\sigma(t)+\tau)>-py(t-\sigma(t)), & t\notin S \\ z(t_k-\sigma(t_k)+\tau)>-py(t_k-\sigma(t_k)), & \forall\ t_k\in S. \end{cases} \qquad (141)$$

Substituting inequality(141) into equation (135), we have

$$\begin{cases} z''(t)-\frac{q(t)}{p}f(z(t-\sigma(t)+\tau))\le 0 \\ \Delta z'(t_k)\,\frac{q_k}{p}f_k(z(t_k-\sigma(t_k)+\tau))\le 0. \end{cases} \qquad (142)$$

Integrating inequality (142) from s to t for $t>s$, we obtain

$$\begin{aligned} z'(t)-z'(s) \;&-\; \frac{1}{p}\int_s^t q(u)\,f(z(u-\sigma(u)+\tau))\,du \\ &-\frac{1}{p}\sum_{s\le t_k\le t} q_k f_k(z(t_k-\sigma(t_k)+\tau))\le 0. \end{aligned} \qquad (143)$$

Integrating inequality (143) in s from $t-\sigma(t)+\tau$ to t, we have

$$z'(t)(\sigma(t)-\tau)+\Delta z(\sigma(t_k)-\tau)-\int_{t-\sigma(t)+\tau}^{t} d\,z(s)+z(t_k-\sigma(t_k)+\tau)-z(t_k)$$

$$-\frac{1}{p}\int_{t-\sigma(t)+\tau}^{t}[u-(t-\sigma(t)+\tau)]\,q(u)\,f(z(u-\sigma(u)+\tau))\,du$$

$$-\frac{1}{p}\sum_{t-\sigma(t)+\tau\leq t_k\leq t}\left[t_k-(t-\sigma(t)+\tau)\right]q_kf_k\left(z\left(t_k-\sigma(t_k)+\tau\right)\right)\leq 0.$$

Hence for t and t_k sufficiently large,

$$z\left(t-\sigma(t)+\tau\right)-z(t)-\frac{1}{p}\int_{t-\sigma(t)+\tau}^{t}\left[u-(t-\sigma(t)+\tau)\right]q(u)\times$$
$$\times\left(z\left(u-\sigma(u)+\tau\right)\right)du\leq 0,$$

$$z\left(t_k-\sigma(t_k)+\tau\right)-z(t_k)-\frac{1}{p}\sum_{t-\sigma(t)+\tau\leq t_k\leq t}\left[t_k-(t-\sigma(t_k)+\tau)\right]\times$$
$$\times\ q_kf_k\left(z\left(t_k-\sigma(t_k)+\tau\right)\right)\leq 0.$$

Dividing inequalities (144) and (144) by $z\left(t-\sigma(t)+\tau\right)$ and $z\left(t_k-\sigma(t_k)+\tau\right)$ respectively and noting the negativity of these terms, we obtain

$$1-\frac{z(t)}{z\left(t-\sigma(t)+\tau\right)}-\frac{1}{pz\left(t-\sigma(t)+\tau\right)}\int_{t-\sigma(t)+\tau}^{t}\left[u-(t-\sigma(t)+\tau)\right]\times$$
$$\times\,q(u)\,f\left(z\left(u-\sigma(u)+\tau\right)\right)du\geq 0,$$

$$1-\frac{z(t_k)}{z\left(t_k-\sigma(t_k)+\tau\right)}-\frac{1}{pz\left(t_k-\sigma(t_k)+\tau\right)}\times$$
$$\times\sum_{t-\sigma(t)+\tau\leq t_k\leq t}\left[t_k-(t-\sigma(t_k)+\tau)\right]q_kf_k\left(z\left(t_k-\sigma(t_k)+\tau\right)\right)\geq 0.$$

We note that $z(t)<0$ and $z(t_k)<0\ \ \forall\ k:\ \ t\neq t_k$ finally implies that $\lim_{t\to\infty}z(t)=0$ and $\lim_{t_k\to\infty}z(t_k)=0$. From inequalities (144) and (144), we have

$$\begin{cases} 1>\frac{1}{p}\int_{t-\sigma(t)+\tau}^{t}\left[u-(t-\sigma(t)+\tau)\right]q(u)\ \frac{f(z(u-\sigma(u)+\tau))}{z(t-\sigma(t)+\tau)}\,du \\ 1>\frac{1}{p}\sum_{t-\sigma(t)+\tau\leq t_k\leq t}\left[t_k-(t-\sigma(t)+\tau)\right]\ q_k\ \frac{f_k(z(t_k-\sigma(t_k)+\tau))}{z(t_k-\sigma(t_k)+\tau)} \end{cases}$$

which contradicts inequalities (139) and (140) respectively. This, therefore, completes the proof of Theorem 4.1.

Lemma 4.3: Assume condition (H) holds and $p = 1$. Then the non-oscillatory solutions $y(t)$ of equation (135) are bounded provided every solution of the equation

$$\begin{cases} z''(t) + q(t)\ f(Q(t)\ z(t)) = 0 \\ \Delta z'(t_k) + q_k f_k(Q(t_k)\ z(t_k)) = 0 \end{cases} \tag{144}$$

is oscillatory, where

$$\begin{cases} Q(t) = \frac{1}{3\tau t}\ (t - \sigma(t))^2 \\ Q(t_k) = \frac{1}{3\tau t_k}\ (t_k - \sigma(t_k))^2. \end{cases}$$

Proof: Let $y(t)$ be a finally positive solution of equation (135) and $z(t) = y(t) - y(t-\tau)$. Then $z''(t) \le 0,\ \Delta z'(t_k) \le 0$ for $t \ge t_0$ and $\forall\, k : t_k \ge t_0$. If $z'(t) < 0,\ \Delta z(t_k) < 0$ for $t \ge t_0$ and $\forall\ k : t_k \ge t_0$, then we have $\lim_{t\to\infty} z(t) = -\infty,\ \lim_{t\to\infty} \Delta(t_k) = -\infty$. Thus, for all large t and $t_k : k \in Z$,

$$\begin{cases} y(t) \le y(t-\tau),\ t \notin S \\ y(t_k) \le y(t_k - \tau)\ \ \forall\, t_k \in S. \end{cases} \tag{145}$$

This implies that $y(t)$ is bounded which is a contradiction to our assumption. Therefore, $z'(t) > 0,\ \Delta z(t_k) > 0$, for $t \ge t_0$ and $\forall\ k : t_k \ge t_0$.

Assume $z(t) > 0,\ z(t_k) > 0,\ t \ge t_2 \ge t_1\ \forall\, k : t_k \ge t_2 \ge t_1$. By Lemma 4.1, for any $\ell \in (0,1)$ and $i = 0, 1, 2, \cdots$, there exists $T_i \ge t_0 = T_0$ such that

$$\begin{cases} z(t - \sigma(t) - i\,\tau) \ge \frac{\ell\,(t - \sigma(t) - i\tau)}{t}\, z(t), \\ \qquad\qquad t - \sigma(t) \ge T_i,\ t \notin S \\ z(t_k - \sigma(t_k) - i\,\tau) \ge \frac{\ell(t_k - \sigma(t_k) - i\tau)}{t_k}\, z(t_k), \\ \qquad\qquad t_k - \sigma(t_k) \ge T_i, \forall\, t_k \in S. \end{cases} \tag{146}$$

Since

$$\begin{cases} y\,(t-\sigma\,(t)) = \sum_{i=0}^{n-1} z\,(t-\sigma\,(t)\; - \; i\tau) \; + \; y\,(t-\sigma\,(t)\; - \; n\tau) \\ \qquad\qquad\qquad\qquad \geq \sum_{i=1}^{n} z\,(t-\sigma\,(t)\; - \; i\tau)\,, \quad t \notin S \\ y\,(t_k-\sigma\,(t_k)) = \; \sum_{i=1}^{n-1} z\,(t_k-\sigma\,(t_k)\; - \; i\tau) \; + \; y\,(t_k-\sigma\,(t_k)\; - \; n\tau) \\ \qquad\qquad\qquad\qquad \geq \sum_{i=1}^{n} z\,(t_k-\sigma\,(t_k)\; - \; i\tau)\,, \; \forall\, t_k \in \; S, \end{cases}$$

$\left(here \quad \sum_{i=0}^{n-1} = 0\right)$, from equation (135) we have

$$\begin{cases} z''\,(t) + q\,(t)\,f\,(\sum_{i=1}^{n} z\,(t-\sigma\,(t)\; - \; i\tau)) \leq 0, \;\; t \notin S \\ \Delta\, z'\,(t_k) + q_k f_k\,(\sum_{i=1}^{n} z\;(t_k-\sigma\,(t_k)\; - \; i\tau)) \leq 0, \;\; \forall\, t_k \in S. \end{cases}$$

Using inequality (146), we obtain

$$\begin{cases} z''\,(t) + q\,(t)\,f\left(\frac{\ell}{t}\sum_{i=0}^{n}\;(t-\sigma\,(t)\; - \; i\tau)\; z\,(t)\right) \leq 0, \;\; t \; \notin \; S \\ \Delta\, z'\,(t_k) + q_k f_k\left(\frac{\ell}{t_k}\sum_{i-0}^{n}(t_k-\sigma\,(t_k)\; - \; i\tau)\; z\,(t_k)\right) \leq 0, \;\; \forall\, t_k \in S, \end{cases}$$

that is,

$$\begin{cases} z''\,(t) + q\,(t)\,f\left(\frac{\ell}{t}\;(n+1)\left(t-\sigma\,(t)\; - \; \frac{n}{2}\tau\right)\; z\,(t)\right) \leq 0, \;\; t \notin S \\ \Delta z'\,(t_k) + q_k f_k\left(\frac{\ell}{t_k}\,(n+1)\left(t_k-\sigma\,(t_k)\; - \; \frac{n}{2}\tau\right)\; z\,(t_k)\right) \leq 0, \;\; \forall\, t_k \in S. \end{cases}$$

Since $n\tau \; \leq \; t-\sigma\,(t) - T_0 < \; (n+1)\,\tau$, we have

$$\begin{cases} z''\,(t) + q\,(t)\,f\left(\frac{\ell}{2\tau t}\;\left[(t-\sigma\,(t))^2 - T_0^2\right] z\,(t)\right) \leq 0, \;\; t \; \notin \; S \\ \Delta z'\,(t_k) + q_k f_k\left(\frac{\ell}{2\tau t_k}\left[(t_k-\sigma\,(t_k))^2 - T_0^2\right] z\,(t_k)\right) \leq 0, \;\; \forall\, t_k \in S. \end{cases}$$

Choose $T \; \geq \; T_0$ large enough, then it follows that

$$\begin{cases} z''\,(t) + q\,(t)\,f\left(\frac{1}{3\tau t}\;(t-\sigma\,(t))^2\, z\,(t)\right) \leq 0, \;\; t \; \geq \; T, \;\; t \; \notin \; S \\ \Delta z'\,(t_k) + q_k f_k\left(\frac{1}{3\tau t_k}\,(t_k-\sigma\,(t_k))^2\, z\,(t_k)\right) \leq 0, \;\; t_k \; \geq \; T, \;\; \forall\, t_k \in S. \end{cases}$$

Noting that $z\,(t)\,,\; z\,(t_k)$ and $z\,(T)$ are upper and lower solutions of equation (144) respectively, and using the known result in Theorem 3.9, we observe that there is a solution $y(t)$ of equation (144) satisfying

$$\begin{cases} z(T) \leq z(t) \leq z(t) \\ z(T) \leq x(t_k) \leq z(t_k). \end{cases}$$

This contradicts the fact that equation (144) is oscillatory.

Next we assume that $z(t) < 0,\ z(t_k) < 0$, for $t \geq t_2 \geq t_1$ and $\forall\, k: t_k \geq t_2 \geq t_1$. Then

$$\begin{cases} y(t) < y(t-\tau),\ t \geq t_2,\ t \notin S \\ y(t_k) < y(t_k - \tau),\ t_k \geq t_2,\ \forall\, t_k \in S. \end{cases}$$

This implies that $y(t)$ is bounded which completes the proof of Lemma 4.3.

Definition 4.1. Let E be a subset of R_+. Define

$$\rho_t(E) = \frac{\mu\{E \bigcap [0,t]\}}{t} \text{ and } \rho(E) = \limsup_{t\to\infty} \rho_t(E),$$

where μ is a Lebesgue measure.

Lemma 4.4. Assume $p > 1$. Then the non-oscillatory solutions $y(t)$ of equation (135) satisfy $y(t) < py(t-\tau)$ finally provided the following conditions hold:

i)

$$\begin{cases} z''(t) + q(t) f(\Omega\ (t,\ \lambda)\ z\ (t)) = 0,\ t \notin S \\ \Delta z'(t_k) + q_k f_k(\Omega\ (t_k,\ \lambda)\ z\ (t_k)) = 0,\ \forall\, t_k \in S \end{cases} \tag{147}$$

is oscillatory for all $0 < \lambda < 1$, where

$$\begin{cases} \Omega\ (t,\ \lambda) = \frac{\lambda}{t}\, p^{\frac{t-\sigma(t)}{\tau}} \\ \Omega\ (t_k,\ \lambda) = \frac{\lambda}{t_k}\, p^{\frac{t_k-\sigma\ (t_k)}{\tau}}; \end{cases}$$

ii)

$$\limsup_{\substack{t\to\infty \\ t\notin E}} \left[p_1^{-\frac{t}{\tau}} \int_0^t (t-u)q(u)f(u-\sigma(u)+\tau)du \right.$$
$$\left. +p_{1k}^{-\frac{t_k}{\tau}} \sum_{0\le t_k\le t} (t-t_k)q_k(t_k)f_k(t_k-\sigma(t_k)+\tau)\right] > 0 \tag{148}$$

holds for some $p_1 > p$ and any set E with $\rho(E) = 0$.

Proof: First we claim that if the set $E \subset R_+$ and $\rho(E) = \rho > 0$, then for any $t_0 \in R_+$ and integer n, there exists a $T \in (t_0, t_0 + \tau)$ such that the set $\{T + i\tau\}_{i=1}^{\infty}$ intersects E at least n times. If not, there exists a $t_0 \in R_+$ and an integer N, such that $\{T + i\tau\}_{i=1}^{\infty}$ intersects E at most N times for any $T \in [t_0, t_0 + \tau)$. This implies that $\mu(E) < \infty$. But $\rho(E) = \rho > 0$ means there exists $t_k \to \infty$ such that $\rho_{t_n}(E) \ge \frac{\rho}{2} > 0$. Thus,

$$\mu\left\{E \bigcap [0,\ t_n]\right\} \ge \frac{\rho}{2} t_n \to \infty \ \ as\ \ n \to \infty$$

is impossible.

Again, let $y(t)$ be a finally positive solution of the equation

$$\begin{cases} (p\,(t)\,y'\,(t))' + \sum_{i=1}^{n} q_i\,(t)\,y\,(\tau_i\,(t)) + \int_{r(a)}^{t} \ell\,(s,\ t)\,y\,(s)\,ds = 0, \\ \qquad\qquad t \in [a,\ b)\,,\ \ t \notin\ S \\ (p\,(t_k)\,\Delta y\,(t_k))' + \sum_{i=1}^{n} q_{ik} y\,(\tau_i\,(t_k)) + \sum_{\sigma(a)\le t_k \le t} \ell\,(t_k, t)\,y\,(t_k) = 0, \\ \qquad\qquad t_k \in [a,\ b)\,,\ \ \forall\, t_k \in S \end{cases} \tag{149}$$

and set

$$z\,(t) = y\,(t) - py\,(t-\tau)\,.$$

Then $z''\,(t) \ \le\ 0,\ \ \Delta z'\,(t_k) \le\ 0$ finally. Here, we observe that there are three possibilities:

$$\text{i)}\begin{cases} z'(t) > 0, & z(t) > 0 \\ \Delta z(t_k) > 0, & z(t_k) > 0 \end{cases};$$

$$\text{ii)}\begin{cases} z'(t) < 0, & z(t) < 0 \\ \Delta z(t_k) < 0, & z(t_k) < 0 \end{cases};$$

$$\text{iii)}\begin{cases} z'(t) > 0, & z(t) < 0 \\ \Delta z(t_k) > 0, & z(t_k) < 0 \end{cases}$$

finally.

i) Assume

$$\begin{cases} z'(t) > 0, & z(t) > 0, \ t \notin S \\ \Delta z(t_k) > 0, & z(t_k) > 0, \ \forall\, t_k \in S \end{cases}$$

for $t \geq t_0 \geq 0$ and $\forall\, k : t_k \geq t_o \geq 0$. Then equation (146) holds, and for any $t,\ t_k \in R_{T_0} = \{t\,; t + \sigma(\tau) \geq T_0\}$, there exists a positive integer n such that

$$\begin{cases} T_0 \leq t - \sigma(t) - n\tau < T_0 + \tau \\ T_0 \leq t_k - \sigma(t_k) - n\tau < T_0 + \tau. \end{cases}$$

Since

$$\begin{cases} y(t - \sigma(t)) = \sum_{i=0}^{n-1} p^i z(t - \sigma(t) - i\tau) + p^n y(t - \sigma(t) - n\tau) \\ \qquad\qquad\qquad \geq \sum_{i=0}^{n} p^i z(t - \sigma(t) - i\,\tau), \ t \notin S \\ y(t_k - \sigma(t_k)) = \sum_{i=0}^{n-1} p^i z(t_k - \sigma(t_k) - i\tau) + p^n y(t_k - \sigma(t_k) - n\tau) \\ \qquad\qquad\qquad \geq \sum_{i=0}^{n} p^i z(t_k - \sigma(t_k) - i\tau), \ \forall\, t_k \in S, \end{cases}$$

from equation (135) we have

$$\begin{cases} z''(t) + q(t) f\left(\sum_{i=0}^{n} p^i z(t - \sigma(t) - i\tau)\right) \leq 0, \ t \notin S \\ \Delta z'(t_k) + q_k f_k\left(\sum_{i=0}^{n} p^i z(t_k - \sigma(t_k) - i\tau)\right) \leq 0, \forall\, t_k \in S. \end{cases}$$

In view of equation (146), we have

$$\begin{cases} z''(t) + q(t) f\left(\frac{\ell}{t} \sum_{i=0}^{n} p^i (t-\sigma(t) - i\tau) z(t)\right) \le 0, \ t \notin S \\ \Delta z'(t_k) + q_k f_k \left(\frac{\ell}{t_k} \sum_{i=0}^{n} p^i (t_k - \sigma(t_k) - i\tau) z(t_k)\right) \le 0, \ \forall t_k \in S, \end{cases}$$

that is,

$$\begin{cases} z''(t) + q(t) f \left[\left(\frac{\ell}{t}(t-\sigma(t)) \frac{p^{n+1}}{p-1} - \frac{l\tau}{t} \sum_{i=1}^{n} p^i\right) z(t)\right] \\ \qquad \le 0, \ t \notin S \\ \Delta z'(t_k) + q_k f_k \left[\left(\frac{\ell}{t_k}(t_k - \sigma(t_k)) \frac{p^{n+1}}{p-1} - \frac{l\tau}{t_k} \sum_{i=1}^{n} p^i\right) z(t_k)\right] \\ \qquad \le 0, \forall t_k \in S. \end{cases} \tag{150}$$

Since

$$\sum_{i=1}^{n} i p^i = \frac{np^{n+2} - (n+1) p^{n+1} + p}{(p-2)^2},$$

we have

$$\frac{\ell}{t}(t-\sigma(t)) \frac{p^{n+1}-1}{p-1} - \frac{\ell\tau}{t} \sum_{i=1}^{n} i p^i$$

$$= \frac{\ell}{(p-1)^2 t}\left[(t-\sigma(t))\left(p^{n+2} - p^{n+1}\right) - \tau\left(np^{n+2} - (n+1) p^{n+1} + p\right)\right]$$

$$= \frac{\ell}{(p-1)^2 t}[t-\sigma(t) - n\tau) p^{n+2} - (t-\sigma(t) - (n+1)\tau) p^{n+1}$$
$$- (t-\sigma(t)+\tau) p + (t-\sigma(t)]$$

$$\ge \frac{\ell}{(p-1)^2 t}\left[T_0 p^{n+2} - T_0 P^{n+1} - (t-\sigma(t) + \tau) P + (t-\sigma(t))\right]$$

$$\ge \frac{\ell}{t} p^{n+2} \ge \frac{1}{t} p \frac{t-\sigma(t) - T_0 + \tau}{t} \ge \frac{\lambda}{t} p \frac{t-\sigma(t)}{t}, \ t \notin S \tag{151}$$

and

$$\frac{\ell}{t_k}\left(t_k-\sigma\left(t_k\right)\right)\frac{p^{n+1}-1}{p-1}-\frac{\ell\tau}{t_k}\sum_{i=1}^{n}ip^i$$

$$= \frac{\ell}{(p-1)^2 t_k}\left[\left(t_k-\sigma\left(t_k\right)\right)\left(p^{n+2}-p^{n+1}-p+1\right)-\tau\left(np^{n+2}-(n+1)p^{n+1}+p\right)\right]$$

$$= \frac{\ell}{(p-1)^2 t_k}\left[t_k-\sigma\left(t_k\right)-n\tau\right)p^{n+2}-\left(t_k-\sigma\left(t_k\right)-(n+1)\tau\right)p^{n+1}$$

$$-\left(t_k-\sigma\left(t_k\right)\ +\tau\right)p+\left(t_k-\sigma\left(t_k\right)\right]$$

$$\geq \frac{\ell}{(p-1)^2 t_k}\left[T_0P^{n+2}-T_0P^{n+1}-\left(t_k-\sigma\left(t_k\right)+\tau\right)p\ +\left(t_k-\sigma\left(t_k\right)\right)\right]$$

$$\geq \frac{1}{t_k}P^{n+2}\geq\frac{1}{t_k}P^{\frac{t_k-\sigma\left(t_k\right)-T_0+\tau}{\tau}}\geq\frac{\lambda}{t_k}p^{\frac{t_k-\sigma\left(t_k\right)}{\tau}},\ \ \forall\, t_k\notin S \qquad (152)$$

for some $\lambda \in (0,\ 1)$ if T_0 and $t,\ \ t_k$ are sufficiently large. Substituting inequalities (151) and (152) into inequality (150) we obtain

$$\begin{cases} z''(t)+q\ (t)\,f\left(\frac{\lambda}{t}\ P^{\ \frac{t-\sigma(t)}{\tau}}\ z\,(t)\right)\leq 0,\ \ t\notin S \\ \Delta\, z'\ (t_k)+q_kf_k\left(\frac{\lambda}{t_k}p^{\frac{t_k-\sigma(t_k)}{\tau}}\ z\ (t_k)\right)\leq 0,\ \ \forall\, t_k\in S. \end{cases}$$

Noting that $z(t), z(t_k)$ and $z(T_0)$ are upper and lower solutions of equation (147) respectively, and by the known result in Theorem 3.9, we observe that there is a solution $x(t)$ of equation (147) which satisfies the relation.

$$\begin{cases} z\,(T_0)\leq x\,(t)\ \leq\ z\ (t),\ \ t\notin S \\ z\ \,(T_0)\leq x\ (t_k)\ \leq\ z\ (t_k),\ \ \forall\, t_k\in S. \end{cases}$$

This contradicts the fact that equation (147) is oscillatory for all $0\ <\ \lambda\ <\ 1$.

ii) Assume

$$\begin{cases} z'(t) < 0, \ z(t) < 0, \ t \notin S \\ \Delta z(t_k) < 0, \ z(t_k) < 0, \ \forall\, t_k \in S \end{cases}$$

for $t \geq t_0 \geq 0$ and $k: t_k \geq t_0 \geq 0$. Then

$$\begin{cases} z(t) \leq -\omega t, \ t \geq t_0, \ t \notin S \\ z(t_K) \leq -\omega_k t_k, \ t_k \geq t_0, \ \forall\, t_k \in S \end{cases}$$

for some $\omega, \omega_k > 0$. We begin by saying that $z(t) \geq -p_1^{\frac{t}{\tau}}$. Here $p_1 > p$ is arbitrary, that is, if $E = \left\{t: \ z(t) < -p^{\frac{t}{\tau}}\right\}$, then, $\rho(E) = \rho$. Otherwise, $\rho(E) = \rho > 0$. As in the beginning of the proof for any n, there exists a $T_1 \in [t_1, t_0 + \tau)$ such that the set $\{T_1 + i\tau\}_{i=1}^{\infty}$ intersects E at least n times. Assume

$$M = \max_{t_0 \leq t, t_k \leq t_0 + \tau} \{y(t), \ y(t_k)\}.$$

Then if n is sufficiently large,

$$y(T_1 + n\tau) \leq p^n y(T_1) + z(T_1 + n\tau) \leq p^n M - p_1^{\frac{T_1 + n\tau}{\tau}} = p^n M - p_1^{n + \frac{T_1}{\tau}} < 0,$$

which contradicts the fact that $y(t)$ is finally positive.

It is immediately observed that condition (ii) implies that

$$\int_0^{\infty} q(u)\, f(u - \sigma(u) + \tau)\, du + \sum_{0 \leq t_k < \infty} q_k f_k(t_k - \sigma(t_k) + \tau) = \infty. \quad (153)$$

Condition (ii) also implies that

$$\begin{cases} z'(t) < -\mu, \ t \notin S \\ \Delta z(t_k) < -\mu, \ \forall\, t_k \in S \end{cases} \quad (154)$$

finally, for all $\mu > 0$. Otherwise, for the same $\mu > 0$ the condition

$$\begin{cases} z'(t) \geq -\mu, \ t \notin S \\ \Delta z(t_k) \geq -\mu, \ \forall\, t_k \in S \end{cases}$$

would have been satisfied for all $t, \ t_k \geq T_2$. On the other hand,

$$\begin{cases} y(t-\tau) \geq \frac{1}{p} z(t), \ t \notin S \\ y(t_k-\tau) \geq \frac{1}{p} z(t_k), \ \forall\, t_k \in S, \end{cases}$$

thus

$$\begin{cases} z''(t) + q(t) f\left(-\frac{1}{p} z(t-\sigma(t)+\tau)\right) \leq 0, \ t \notin S \\ \Delta z'(t_k) + q_k f_k\left(-\frac{1}{p} z(t_k-\sigma(t_k)+\tau)\right) \leq 0, \ \forall\, t_k \in S. \end{cases} \tag{155}$$

Integrating inequality (155) from T_2 to t, we obtain

$$\begin{aligned} z'(t) + \Delta z(t_k) \ &+ \int_{T_2}^{t} q(u) f\left(-\frac{1}{p} z(u-\sigma(u)+\tau)\right) du \\ &+ \sum_{T_2 \leq t_k < t} q_k f_k\left(-\frac{1}{p} z(t_k-\sigma(t_k)+\tau)\right) \leq 0, \end{aligned}$$

or

$$\begin{cases} z'(t) + \int_{T_2}^{t} q(u)\, f\left(-\frac{1}{p} z(u-\sigma(u)+\tau)\right) du \leq 0, \ t \notin S \\ \Delta z(t_k) + \sum_{T_2 \leq t_k < t} q_k f_k\left(-\frac{1}{p} z(t_k-\sigma(t_k)+\tau)\right) \leq 0, \ \forall\, t_k \in S. \end{cases}$$

Noting that

$$\begin{cases} z(t-\sigma(t)+\tau) \leq -\omega(t-\sigma(t)+\tau), \ t \notin S \\ z(t_k-\sigma(t_k)+\tau) \leq -\omega_k(t_k-\sigma(t_k)+\tau), \ \forall\, t_k \in S, \end{cases}$$

iii) Assume

$$\begin{cases} z'(t) > 0, & z(t) < 0, \quad t \notin S \\ \Delta z(t_k) > 0, & z(t_k) < 0, \quad \forall\ t_k \in S \end{cases}$$

for $t \geq t_0 \geq 0$ and $k : t_k \geq t_0 \geq 0$. Then $y(t) < py(t-\tau)$ is obvious. This completes the proof of Lemma 4.4.

Corollary 4.1. In addition to the assumptions of Lemma 4.3, further assume that σ is a positive constant and

$$\sum_{i=0}^{\infty} \left(\int_{T+i\tau}^{T+i\tau+\alpha} (u-T)\, q(u)\, du + \sum_{T+i\tau \leq t_k < T+i\tau+\alpha} (t_k - T)\, q_k \right) = \infty \qquad (157)$$

holds for any $T \in R_+$ and $0 < \alpha \leq \tau$, then all non-oscillatory solutions of equation (135) tend to zero as t and $t_k \to \infty$.

Proof: Let us assume by contradiction that there exists a finally positive solution $y(t)$ satisfying $\lim_{t\to\infty} \sup y(t) > 0$, and this can only occur when $z''(t),\ \Delta z'(t_k) \leq 0,\ z'(t),\ \Delta z(t_k) > 0$ and $z(t) < 0$, for all $t \geq t_0 \geq 0$ and $k : t_k \geq t_0 \geq 0$. Hence, $z'(t) \to 0$ and $z(t) \to 0$ as $t,\ t_k \to 0$. If $\lim_{t\to\infty} \inf y(t) > 0$, then $y(t) \geq a > 0,\ t,\ t_k \geq t_1 \geq t_0$. Integrating equation (135) twice, we obtain

$$z(t) + \int_t^{\infty} (u-t)\, q(u)\, f(a)\, du + \sum_{t \leq t < \infty} (t_k - t)\, q_k f_k(a) < 0$$

which implies

$$\lim_{t\to\infty} \sup \left[\int_t^{\infty} (u-t)\, q(u)\, du + \sum_{t \leq t_k < \infty} (t_k - t)\, q_k \right] \leq 0$$

and contradicts equation (157). Thus,

$$\lim_{t\to\infty} \sup y(t) > 0 \text{ and } \lim_{t\to\infty} \inf y(t) = 0.$$

Then, we can choose $t_2 > t_1 \geq t_0$ such that $y(t_2 - \sigma) > y(t_1 - \sigma)$. We claim that

$$\lim_{t\to\infty} \inf y(t_2 - \sigma + n\tau) > 0. \tag{158}$$

In fact,

$$y(t_j - \sigma + n\,\tau) = \sum_{i=1}^{n} z\,(t_j - \sigma + n\tau) + y(t_j - \sigma),\ \ j = 1,\ 2.$$

Since $z(t_2 - \sigma + i\tau) \geq z(t_1 - \sigma + i\tau)$ for $j = 1, 2, \cdots, n$, and $\lim_{t\to\infty} \inf y(t_1 - \sigma + n\,\tau) \geq 0$,
we have

$$\lim_{t\to\infty} \inf y(t_2 - \sigma + n\tau) \geq y(t_2 - \sigma) - y(t_1 - \sigma) > 0.$$

Now choose $t_0 \leq t_1 < t_2 < t_3$ such that for any $T \in [t_2, t_3]$,

$$y(t_1 - \sigma) < t(t_2 - \sigma) \leq y(T - \sigma).$$

From the above discussion, we observe that inequality (158) holds, that is, there exists a $\mu > 0$ such that $y(t_2 - \sigma + n\,\tau) \geq \mu$ for all n. It is now obvious that for $T \in [t_2,\ t_3]$,

$$\begin{aligned} y(T - \sigma + n\tau) &= \sum_{i=1}^{n} z(T - \sigma + i\tau) + y(T - \sigma) \\ &\geq \sum_{i=t}^{n} z(t_2 - \sigma + i\tau) + y(t_2 - \sigma) \\ &= y(t_2 - \sigma + n\tau) \geq \mu. \end{aligned}$$

From equation (135), we have

$$-z'(s) + \int_s^t q(u)\ f(y(u - \sigma))\,du + \sum_{s\leq t_k<t} q_k f_k(y(t_k - \sigma)) \ \leq\ 0,\ \ t_0 \leq s \leq t,$$

$$
\begin{aligned}
z\left(t_0\right) \quad &+ \quad \int_{t_0}^{t}\left(u-t_0\right) q\left(u\right)\, f\left(y\left(u-\sigma\right)\right) du \\
&+ \sum_{t_0 \leq t_k < t}\left(t_k-t_0\right)\, q_k f_k\left(y\left(t_k-\sigma\right)\right) \;\leq\; 0,\; \forall\, t_k \geq t_0. \qquad (159)
\end{aligned}
$$

Hence

$$
z\left(t_0\right)+\; f\left(\mu\right)\sum_{i=0}^{n}\left[\int_{t_2+i\tau}^{t_3+i\tau}\left(u-t_0\right)\, q\left(u\right)\, du+\sum_{t_2+i\tau\leq t_k<t_3+i\tau}\left(t_k-t_0\right) q_k\right] \leq\; 0,
$$

and then

$$
z\left(t_0\right)+f\left(\mu\right)\sum_{i=1}^{n}\left[\int_{t_2+i\tau}^{t_3+i\tau}\left(u-t_2\right)\, q\left(u\right)\, du+\sum_{t_2+i\tau\leq t_k<t_3+i\tau}\left(t_k-t_2\right)\, q_k\right] \leq 0,
$$

thus contradicts equation (157). This completes the proof of corollary 4.1.

Corollary 4.2. In addition to the assumptions of Lemma 4.4, assume further that σ is a positive constant,

$$
\int_{t}^{\infty}\left(u-t\right)\, q\left(u\right)\, du+\sum_{t\leq t_k<\infty}\left(t_k-t\right)\, q_k=\infty \qquad (160)
$$

and

$$
\sum_{i=0}^{\infty}\left[f\left(p^i\right)\int_{T+i\tau}^{T+i\tau+\alpha}\left(u-T\right)\, q\left(u\right)+f_k\left(p^i\right)\sum_{T+i\tau\leq t_k<T+i\tau+\alpha}\left(t_k-T\right)\, q_k\right]=\infty \qquad (161)
$$

hold for any $T \in R_+$ and $0 < \alpha \leq \tau$. Then all non-oscillatory solutions of equation (135) tend to zero as $t \to \infty$.

Proof: Approaching this by contradiction like in the proof of corollary 4.1, we observe that there exists a finally positive solution $y(t)$ satisfying the conditions

$$
\lim_{t\to\infty} \sup\, y\left(t\right) \;>\; 0 \;\text{ and }\; \lim_{t\to\infty}\; \inf\, y\left(t\right)=0.
$$

From the proof of Lemma 4.4, this can only occur when $z''(t), \Delta z'(t_k) \leq 0$, $z'(t), \Delta z(t_k) > 0$ and $z(t) < 0$, for $t \geq t_0$ and $k : t_k \geq t_0$. Choose $t_2 > t_1 \geq t_0$ such that $y(t_2 - \sigma) > y(t_1 - \sigma)$. Since

$$y(t_2 - \sigma + n\tau) = \sum_{i=1}^{n} p^{n-i} z(t_2 - \sigma + i\tau) + p^n y(t_2 - \sigma),$$

$$y(t_1 - \sigma + n\tau) = \sum_{i=1}^{n} p^{n-i} z(t_1 - \sigma + i\tau) + p^n y(t_1 - \sigma),$$

$$z(t_2 - \sigma + n\tau) \geq z(t_1 - \sigma + i\tau), \; i = 1, 2, \cdots, n,$$

and

$$y(t_1 - \sigma + n\tau) > 0, \; n = 0, 1, 2, \cdots,$$

we see that

$$y(t_2 - \sigma + n\tau) \geq p^n (y(t_2 - \sigma) - y(t_1 - \sigma)) = Ap^n.$$

Similar to the proof of Corollary 4.1, we can show that there is an interval $[t_2, t_3]$ such that

$$y(T - \sigma + nt) \geq Ap^n$$

for $T \in [t_2, t_3]$ and for all n. From inequality (159), we obtain

$$z(t_0) + f(A) \sum_{i=0}^{n} \left[\int_{t_2+i\tau}^{t_3+i\tau} (u - t_2)\, q(u)\, f(p^n)\, du + \sum_{t_2+i\tau \leq t_k < t_3+i\tau} (t_k - t_2)\, q_k\, f_k(p^n) \right] \leq 0$$

which contradicts equation (161). This completes the proof of Corollary 4.2.

Now, we are ready to state the criterion for the oscillation of the solutions of equation (135).

finally, where $q^*(t) = \min\{q(t), q(t-\tau)\}$ and $q_k^* = \min\{q(t_k), q(t_k-\tau)\}$.

Proof: Suppose $y(t) > 0$ for $t \geq t_0$. Then $x(t) > 0$ for $t \geq t_0 + \tau, x''(t)$, $\Delta x'(t_k) < 0$ for $t \geq t_1 = t_0 + \max\{\sigma, \tau\}$ and $k : t_k \geq t_1 = t_0 + \max\{\sigma, \tau\}$. Therefore, $x'(t)$, $\Delta x(t_k) > 0$ for $t \geq t_1$ and $k : t_k \geq t_1$. Then

$$z''(t) = x''(t) + px''(t-\tau) \leq -q(t)\left[y(t-\sigma) + py(t-\tau-\sigma)\right]$$

$$= -q^*(t)\, x(t-\sigma) \tag{165}$$

and

$$\Delta z'(t_k) = \Delta x'(t_k) + p\Delta x'(t_k-\tau) \leq -q_k^*\left[y(t_k-\sigma) + py(t_k-\tau-\sigma)\right]$$

$$= -q_k^* x(t_k-\sigma). \tag{166}$$

Similar to the above, we have $z(t) > 0$, $z'(t), \Delta z(t_k) > 0$ and $z''(t), \Delta z'(t_k) \leq 0$ for $t \geq t_2 \geq t_1$, $\forall\, k : t_k \geq t_2 \geq t_1$ and

$$\begin{aligned} z''(t) \leq -q*(t)\, x(t-\sigma) &\leq -\frac{q*(t)}{1+p}\left[x(t-\sigma) + px(t-\sigma-\tau)\right] \\ &= -\frac{q*(t)}{1+p}\, z(t-\sigma), \end{aligned}$$

and

$$\begin{aligned} \Delta z'(t_k) \leq -q_k^*\, x\,(t_k-\sigma) &\leq -\frac{q_k^*}{1+p}\left[x\,(t_k-\sigma) + px\,(t_k-\sigma-\tau)\right] \\ &= -\frac{q_k^*}{1+p} z\,(t_k-\sigma). \end{aligned}$$

This completes the proof of Lemma 4.5.

Theorem 4.3. *Let $p > 0$, $q_k \geq 0$ and $q \in PC(R_+, R_+)$. Assume that the second order impulsive differential equation*

$$\begin{cases} x''(t) + \lambda q(t) \frac{t-\sigma}{t} x(t) = 0, & t \notin S \\ \Delta x'(t_k) + \lambda q_k \frac{t_k-\sigma}{t_k} x(t_k) = 0, & \forall t_k \in S \end{cases} \tag{167}$$

is oscillatory for some $\lambda \in (0,1)$. Then every solution of equation (162) is oscillatory.

Proof: Let us assume by contradiction that there exists a finally positive solution $y(t)$ of equation (162) and $z(t)$ is defined by equation (163). Then $z(t)$ satisfies all conditions of Lemma 4.1. Consequently, for every $\ell \in (0, 1)$, there exists a $t_\ell \geq 0$ such that

$$\begin{cases} z(t-\sigma) \geq \ell \frac{t-\sigma}{t} z(t), & \text{for } t \geq t_\ell, \ t \notin S \\ z(t_k - \sigma) \geq \ell \frac{t_k-\sigma}{t_k} z(t_k), & \text{for } t_k \geq t_\ell, \ \forall t_k \in S, \end{cases} \tag{168}$$

which implies that

$$\begin{cases} z''(t) + \frac{\ell(t-\sigma)\, q^*(t)}{(1+p)\, t} z(t) = 0, & \text{for } t \geq t_\ell, \ t \notin S \\ \Delta z'(t_k) + \frac{\ell(t_k-\sigma)\, q_k^*}{(1+p)\, t_k} z(t_k) = 0, & \text{for } t_k \geq t_\ell, \ \forall t_k \in S. \end{cases} \tag{169}$$

By Lemma 4.5, inequality (164) is true. Combining inequalities (164) and (169), we obtain

$$\begin{cases} z''(t) + \frac{\ell(t-\sigma)\, q^*(t)}{(1+p)\, t} z(t) \leq 0, & for \ t \geq t_l, \ t \notin S \\ \Delta z'(t_k) + \frac{\ell(t_k-\sigma)\, q_k^*}{(1+p)\, t_k} z(t_k) \leq 0, & for \ t_k \geq t_l, \ \forall t_k \in S, \end{cases} \tag{170}$$

which implies that

$$\begin{cases} z''(t) + \frac{\ell(t-\sigma)\, q*(t)}{(1+p)\, t}\, z(t) &= 0, \ t \notin S \\ \Delta z'(t_k) + \frac{\ell(t_k-\sigma)\, q_k^*}{(1+p)\, t_k}\, z(t_k) &= 0, \ \forall\, t_k \in S \end{cases} \tag{171}$$

has a non-oscillatory solution. This contradicts our initial assumption and thus, completes the proof of Theorem 4.3.

From this theorem, every oscillation criterion for the second order impulsive differential equation (167) becomes an oscillation criterion for the second order neutral impulsive differential equation (162).

Corollary 4.3. Let $p > 0,\ q_k \geq 0$ and $q \in PC(R_+,\ R_+)$. Then every solution of equation (162) is oscillatory if for some $\alpha \in (0,\ 1)$,

$$\int_0^{\infty} t^{\alpha} q(t)\ dt + \sum_{0 \leq t_k < \infty} t_k^{\alpha} q_k = \infty \tag{172}$$

We now return to the linear equation with variable coefficient p as follows:

$$\begin{cases} [y(t) + p(t)\, y(t-\tau)]'' + q(t) y(t-\sigma) = 0, \ t \geq t_0, \ t \notin S \\ \Delta [y(t_k) + p_k y(t_k - \tau)]' + q_k y(t_k - \sigma) = 0, \ t_k \geq t_0, \forall\, t_k \in S. \end{cases} \tag{173}$$

Theorem 4.4. *Assume that*

i) $\tau > 0,\ \sigma > 0,\ p_k > 0;$

ii) $q \in PC(R_+,\ R_+)$ *and* $q(t) \geq q_0 > 0;$

iii) $p \in PC^1(R_+,\ R)$ *and there exist constants* p_1 *and* p_2 *such that* $p_1 \leq p(t) \leq p_2$ *and* $p(t)$ *is not finally negative. Then every solution of equation* (173) *is oscillatory.*

Proof: By contradiction, we assume that $y(t)$ is a finally positive solution of equation (173). Set

$$z(t) = y(t) + p(t)\, y(t-\tau). \tag{174}$$

Using arguments similar to those in the previous theorems, we can show that $z(t) < 0$ finally. This contradicts condition (iii) and thus completes the proof of Theorem 4.4.

4.3 Classification of non-oscillatory solutions

Consider the second order nonlinear neutral impulsive differential

$$\begin{cases} \left[y(t) - \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i)\right]'' \\ \qquad + \sum_{j=1}^{n} f_j\left(t,\, y(g_{jl}(t)),\, \cdots,\, y(g_{jl})\right)) \\ \qquad = 0,\ t \geq t_0 \in R_+,\ \ t \notin S \\ \Delta\left[y(t_k) - \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i)\right]' \\ \qquad + \sum_{j=1}^{n} f_{jk}\left(t_k, y(g_{jl}(t_k)),\, \cdots,\, y(g_{jl}(t_k))\right)) \\ \qquad = 0,\ t_k \geq t_0 \in R_+,\ \forall\ t_k \in S \end{cases} \tag{175}$$

We introduce the following conditions:

H4.3.1: $\tau_i > 0,\ p_{ik} \geq 0,\ \ p_i \in PC^1([t_0,\infty),\, R_+),\ \ i = 1,\, 2,\, \cdots,\, m$ and there exists $\delta \in (0,1]$ such that

$$\sum_{i=1}^{m} p_i(t) + \sum_{j=1}^{n} p_j \leq 1 - \delta, \quad t, \geq t_0 \in R_+;$$

H4.3.2: $g_{js} \in C([t_0,\infty),\, R),\ \lim_{t\to\infty} g_{js}(t) = \infty,\ j = 1,\ 2,\ \cdots,\ n,\ s = 1,\ 2,\ \cdots,\ \ell;$

H4.3.3: $f_j \in PC\left([t_0,\infty) \times R^{\ell}, R\right),\ \ x_1 f_j(t, x_1, \cdots, x_\ell) > 0;\, x_1 f_{jk}(t_k, x_1 \cdots, x_l)$ > 0 for $x_1 x_i > 0,\ i = 1,\ 2,\ \cdots,\ \ell,\ \ j = 1,\ 2,\ \cdots,\ n.$ Moreover,

$$\begin{cases} |f_j\ (t, y_1,\ \cdots,\ y_\ell)| \ \geq\ |f_j\ (t, x_1,\ \cdots,\ x_\ell)| \\ |f_{jk}\ (t_k, y_1\ \cdots,\ y_\ell)| \ \geq\ |f_{jk}\ (t_k, x_1,\ \cdots,\ x_\ell)| \end{cases}$$

whenever

$$|x_i| \leq |y_i| \text{ and } y_i x_i > 0, \ i = 1, 2, \cdots, \ell, \ j = 1, 2, \cdots, n;$$

H4.3.4: Set

$$x(t) = y(t) - \sum_{i=1}^{m} p_i(t)\, y(t - \tau_i). \tag{176}$$

In this section, we give the classification of non-oscillatory solutions of equation (175). But first, we establish the following lemmas which will be useful in the discussion of the main results.

Lemma 4.6. Let $y(t)$ be a finally positive (or negative) solution of equation (175). If $\lim_{t\to\infty} y(t) = 0$, then $x(t)$ is finally negative (or positive) and $\lim_{t\to\infty} x(t) = 0$. Otherwise, $x(t)$ is finally positive (or negative).

Proof: Let $y(t)$ be a finally positive solution of equation (175). From the same equation (175), $x''(t)$, $\Delta x'(t_k) > 0$ or $x'(t)$, $\Delta x(t_k) < 0$ finally. Also, $x(t) > 0$ or $x(t) < 0$ finally. If $\lim_{t\to\infty} y(t) = 0$, from equation (176), it follows that $\lim_{t\to\infty} x(t) = 0$. Since $x(t)$ is monotonic, so $\lim_{t\to\infty} x'(t) = 0$, $\lim_{t_k\to\infty} \Delta x(t_k) = 0$ which implies that $x'(t) > 0$, $\Delta x(t_k) > 0$. Therefore, $x(t) < 0$ finally. If $\lim_{t\to\infty} y(t) \neq 0$, then $\limsup_{t\to\infty} y(t) > 0$. We show that $x(t) > 0$ finally. If not, then $x(t) < 0$ finally. If $y(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $y(t_n) = \max_{t_0 \leq t < t_n} y(t)$ and $\lim_{n\to\infty} y(t_n) = \infty$. From equation (176), we obtain

$$x(t_n) = y(t_n) - \sum_{i=1}^{m} p_i(t_n)\, y(t_n - \tau_i) \geq y(t_n) \left(1 - \sum_{i=1}^{m} p_i(t_n)\right). \tag{177}$$

Thus, $\lim_{n\to\infty} x(t_n) = \infty$, which is a contradiction. If $y(t)$ is bounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = \limsup_{t\to\infty} y(t)$. Since the sequences $\{p_i(t_n)\}$and $\{y(t_n - \tau_i)\}$ are bounded, there exists convergent subsequences. Without loss of generality, we may assume that $\lim_{n\to\infty} y(t_n - \tau_i)$ and $\lim_{n\to\infty} p_i(t_n)$, $i = 1, 2 \cdots, m$, exist. Hence

$$\begin{aligned} 0 \geq \lim_{n\to\infty} x(t_n) &= \lim_{n\to\infty} \left(y(t_n) - \sum_{i=1}^{m} p_i(t_n)\, y(t_n - \tau_i) \right) \\ &\geq \lim_{t\to\infty} \sup y(t) \left(1 - \sum_{i=1}^{m} p_i(t_n) \right) > 0, \end{aligned}$$

which, again, is a contradiction. Therefore, $x(t) > 0$ finally. A similar proof can be repeated if $y(t) < 0$ finally.

Lemma 4.7. Assume that $\lim_{t\to\infty} \sum_{i=1}^{m} p_i(t) = P \in (0, 1]$, and $y(t)$ is a finally positive (or negative) solution of equation (175). If $\lim_{t\to\infty} x(t) = a \in R$, then $\lim_{t\to\infty} y(t) = \frac{a}{1-p}$. If $\lim_{t\to\infty} x(t) = \infty$ (*or* $-\infty$), then $\lim_{t\to\infty} y(t) = \infty$ (*or* $-\infty$).

Proof: Let $y(t)$ be a finally positive solution of equation (175), then $y(t) \geq x(t)$ finally. If $\lim_{t\to\infty} x(t) = \infty$, then $\lim_{t\to\infty} y(t) = \infty$. Now we consider the case that $\lim_{t\to\infty} x(t) = a \in R$. Thus, $x(t)$ is bounded which implies, by equation (177), that $y(t)$ is bounded. Therefore, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = \lim_{t\to\infty} \sup\, y(t)$. As before, without loss of generality, we may assume that $\lim_{n\to\infty} p_i(t_n)$ and $\lim_{n\to\infty} y(t_n - \tau_i)$, $i = 1, 2, \cdots, n$ exist. Hence

$$\begin{aligned} a = \lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} y(t_n) &- \sum_{i=1}^{m} \lim_{n\to\infty} p_i(t_n) \lim_{n\to\infty} y(t_n - \tau_i) \\ &\geq \lim_{t\to\infty} \sup y(t)(1-p), \end{aligned}$$

that is,

$$\frac{a}{1-p} \geq \lim_{t\to\infty} \sup y(t). \tag{178}$$

On the other hand, there exists $\{t_n'\}$ such that $\lim_{n\to\infty} y(t_n') = \lim_{t\to\infty} \inf y(t)$. Without loss of generality, we assume that $\lim_{n\to\infty} p_i(t_n')$ and $\lim_{n\to\infty} y(t_n' - \tau_i)$, $i =$

1, 2, $\cdots$, m exist. Hence

$$a = \lim_{n\to\infty} x\left(t'_n\right) = \lim_{n\to\infty} y\left(t'_n\right) \;-\; \sum_{i=1}^{m} \lim_{n\to\infty} p_i\left(t'_n\right) \lim_{n\to\infty} y\left(t'_n - \tau_i\right) \leq \lim_{t\to\infty} \inf\, y\left(t\right)\left(1-p\right)$$

or

$$\frac{a}{1-p} \leq \lim_{t\to\infty} \inf\, y\left(t\right). \tag{179}$$

Combining inequalities (178) and (179), we obtain $\lim_{t\to\infty} y\left(t\right) = \frac{a}{1-p}$. A similar argument can be repeated if $y(t) < 0$.

We are now ready to prove the following results.

Theorem 4.5. *Assume that* $\lim_{t\to\infty} \sum_{i=1}^{m} p_i\left(t\right) = p \in [0,1)$. *Let* $y(t)$ *be a non-oscillatory solution of equation* (175). *Let* Λ *denote the set of all non-oscillatory solutions of equation* (175), *and define*

$$\Lambda^{(0,0,0)} = \Big\{ y \in \Lambda :\ \lim_{t\to\infty} y\left(t\right) = 0,\ \lim_{t\to\infty} x\left(t\right) = 0,\ \lim_{t,t_k\to\infty} \left(x'\left(t\right),\ \Delta x\left(t_k\right)\right) = 0 \Big\},$$

$$\Lambda^{(b,a,0)} = \Big\{ y \in \Lambda :\ \lim_{t\to\infty} y\left(t\right) = b := \frac{a}{1-p},\ \lim_{t\to\infty} x\left(t\right) = a,\ \lim_{t,t_k\to\infty} \left(x'\left(t\right),\ \Delta x\left(t_k\right)\right) = 0 \Big\},$$

$$\Lambda^{(\infty,\infty,0)} = \Big\{ y \in \Lambda :\ \lim_{t\to\infty} y\left(t\right) = \infty,\ \lim_{t\to\infty} x\left(t\right) = \infty,\ \lim_{t,t_k\to\infty} \left(x'\left(t\right),\ \Delta x\left(t_k\right)\right) = 0 \Big\},$$

$$\Lambda^{(\infty,\infty,d)} = \Big\{ y \in \Lambda :\ \lim_{t\to\infty} y\left(t\right) = \infty,\ \lim_{t\to\infty} x\left(t\right) = \infty,\ \lim_{t,t_k\to\infty} \left(x'\left(t\right),\ \Delta x\left(t_k\right)\right) = d \neq 0 \Big\}.$$

Then

$$\Lambda = \Lambda^{(0,0,0)} \bigcup \Lambda^{(b,a,0)} \bigcup \Lambda^{(\infty,\infty,0)} \bigcup \Lambda^{(\infty,\infty,d)}.$$

Proof: Without loss of generality, let $y(t)$ be a finally positive solution of equation (175). If $\lim_{t\to\infty} y(t) = 0$, then by Lemma 4.6, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t,t_k\to\infty} (x'(t), \Delta x(t_k)) = 0$, that is, $y \in \Lambda^{(0,0,0)}$. If $\lim_{t\to\infty} y(t) \neq 0$, then by Lemma 4.6, $x(t) > 0$ finally and it therefore implies that $x'(t), \Delta x(t_k) > 0$ and $x''(t), \Delta x'(t_k) < 0$ finally. If $\lim_{t\to\infty} x(t) = a > 0$ exists, then $\lim_{t,t_k\to\infty} (x'(t), \Delta x(t_k)) = 0$. By Lemma 4.7, we have $\lim_{t\to\infty} y(t) = \frac{a}{1-p} = b$, that is, $y \in \Lambda^{(b,a,0)}$. *If* $\lim_{t\to\infty} x(t) = \infty$, then by Lemma 4.7, $\lim_{t\to\infty} y(t) = \infty$. Since $x''(t), \Delta x'(t_k) < 0$ and $x'(t), \Delta x(t_k) > 0$, we obtain $\lim_{t,t_k\to\infty} (x'(t), \Delta x(t_k)) = d$, where $d = 0$ or $d > 0$. Then either $y \in \Lambda^{(\infty,\infty,0)}$ or $y \in \Lambda^{(\infty,\infty,d)}$.

This completes the proof of Theorem 4.5.

In what follows, we shall show some existence results for each kind of non-oscillatory solution of equation (175).

Theorem 4.6. *Assume that there exist two constants $h_1 > h_2 > 0$ such that*

$$|p_i(t_2) - p_i(t_1)| \leq h_1 |t_2 - t_1|, \; |p_i(t_{2k}) - p_i(t_{1k})| \leq h_1 |t_{2k} - t_{1k}|, \quad i = 1, 2, \cdots, m,$$

$$\sum_{i=1}^{m} p_i(t) \exp(h_1\tau_i) + \exp(h_1 t) \sum_{i=1}^{m} p_{ik} \exp(-h_1(t_k - \tau_i)) > 1$$

$$\geq \sum_{i=1}^{m} p_i(t) \exp(h_2\tau_i) + \exp(h_2 t) \sum_{i=1}^{m} p_{ik} \exp(-h_2(t_k - \tau_i)) \tag{180}$$

and

$$\left(\sum_{i=1}^{m} p_i(t) \exp(h_1\tau_i) + \exp(h_1 t) \sum_{i=1}^{m} p_{ik} \exp(-h_1(t_k - \tau_i)) - 1\right) \exp(-h_1 t)$$

$$\geq \int_t^{x_n} (u - t) \sum_{j=1}^{m} f_j(u, \exp(-h_2 g_{j1}(u)), \ldots, \exp(-h_2 g_{jl}(u))) \, du$$

$$+\sum_{t\le t_k<\infty}(t_k-t)\sum_{j=1}^{n} f_{jk}\left(t_k,\exp\left(-h_2 g_{j1}\left(t_k\right)\right)\right),\ \cdots,\times$$
$$\times\exp\left(-h_2 g_{jl}\left(t_k\right)\right) \qquad (181)$$

finally. Then equation (175) *has a solution* $y \in \Lambda^{(0,0,0)}$.

Proof: Let us denote by B_p the space of all bounded piece-wise continuous functions in $PC\left([t_0,\infty)\right)$ and define the sup norm in B_p as follows:

$$\|y\| := \sup_{t\ge t_0}|y(t)|.$$

Set

$$\Omega = \begin{cases} y \in B_p : \exp\left(-h_1 t\right) \le y(t) \le \exp\left(-h_2 t\right) \\ |y(t_2)-y(t_1)| \le L\,|t_2-t_1|,\ |y(t_{2k}) - y(t_{1k})| \le L\,|t_{2k}-t_{1k}|, \end{cases}$$

for $t_1, t_2 \ge t_0$, $\forall\, k : t_{1k}, t_{2k} \ge t_0$ and for $L \ge h_1$. Then Ω is a nonempty, closed convex bounded set in B_p.

For the sake of convenience, denote

$$\begin{cases} f\left(u, y\left(g\left(u\right)\right)\right) = \sum_{j=1}^{n} f_j\left(u, y\left(g_{j1}\left(u\right)\right),\ \cdots,\ y\left(g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k, y\left(g\left(t_k\right)\right)\right) = \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{j1}\left(t_k\right)\right),\ \cdots,\ y\left(g_{jl}\left(t_k\right)\right)\right), \end{cases} \qquad (182)$$

$$\begin{cases} f\left(u, \exp\left(-h_2 g\left(u\right)\right)\right) \\ \qquad = \sum_{j=1}^{n} f_j\left(u, \exp\left(-h_2 t_{j1}\left(u\right)\right),\ \cdots,\ \exp\left(-h_2 g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k, \exp\left(-h_2 g\left(t_k\right)\right)\right) \\ \qquad = \sum_{j=1}^{n} f_{jk}\left(t_k, \exp\left(-h_2 g_{j1}\left(t_k\right)\right),\ \cdots,\ \exp\left(-h_2 g_{jl}\left(t_k\right)\right)\right). \end{cases} \qquad (183)$$

Define a mapping J on Ω as follows:

$$(Jy)(t) = \begin{cases} \sum_{i=1}^{m} p_i(t)\ y(t-\tau_i) + \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i) \\ \qquad - \int_t^{\infty} (u-t) f(u, y(g(u)))\, du \\ \qquad - \sum_{t \le t_k < \infty} (t_k - t) + f_k(t_k, y(g(t_k))), \\ \qquad t,\ t_k \ge T \\ \exp(-K(y)t) + \exp(-K(y)t_k), \quad t_0 \le t, t_k < T, \end{cases} \tag{184}$$

where

$$K(y) = -\frac{\ln(Jy)(T)}{T},$$

T is sufficiently large such that $t - \tau_i \ge t_0;\ t_k - \tau_i \ge t_0;\ g_{js}(t_k) \ge t_0;\ i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n;\ s = 1, 2, \cdots, \ell$, for $t,\ t_k \ge T$.

Now, we see that condition (181) implies that

$$\int_T^{\infty} f(u, \exp(-h_2 g(u)))\, du + \sum_{T \le t_k < \infty} f_k(t_k, \exp(-h_2 g(t_k))) < \infty,$$

while from condition H4.3.1, it follows that for a given $\alpha \in (1-\delta, 1)$,

$$\begin{cases} \left(\alpha - \sum_{i=1}^{m} p_i(t)\right) L \ge [\alpha - (1-\delta)]\, L > 0 \\ \left(\alpha - \sum_{i=1}^{m} p_{ik}\right) \ge [\alpha - (t-\delta)]\, L. \end{cases} \tag{185}$$

Therefore, T can be chosen so large that for $t,\ t_k \ge T$,

$$\begin{cases} \int_T^{\infty} f(u, \exp(-h_2 g(u)))\, du \le \left(\alpha - \sum_{i=1}^{m} p_i(t)\right) L \\ \sum_{T \le t_k < \infty} f_k(t_k) \exp(-h_2 g(t_k)) \le \left(\alpha - \sum_{i=1}^{m} p_{ik}\right) L, \end{cases} \tag{186}$$

and

$$\begin{cases} \alpha + \sum_{i=1}^{m} \exp(-h_2(t-\tau_i)) \le \frac{1}{2} \\ \left(\alpha + \sum_{i=1}^{m} \exp(-h_2(t_k - \tau_i))\right) \le \frac{1}{2}\ \frac{|t_2 - t_1|}{|t_{2k} - t_{1k}|}. \end{cases}$$

Hence from inequalities (180) and (181), it follows that

$$(Jy)\ (t)\ \leq\ \sum_{i=1}^{m} p_i\,(t)\ y\ (t-\tau_i)\ +\ \sum_{i=1}^{m} p_{ik}\,y\ (t_k-\tau_i)$$

$$\leq\ \sum_{i=1}^{m} p_i\ (t)\ \exp\ (-h_2\ (t_k-\tau_i)) + \sum_{i=1}^{m} p_{ik} \exp\ (-h_2\ (t_k-\tau_i))$$

$$\leq\ \exp\ (-h_2\,t) \left[\sum_{i=1}^{m} p_i\ (t)\ \exp\ (h_2\,\tau_i)\ +\ \exp\ (h_2 t)\ \sum_{i=1}^{m} p_{ik} \exp\ (-h_2\ (t_k-\tau_i))\right]$$

$$\leq\ \exp\ (-h_2\,t)\ \ for\ \ t,\ t_k \geq T,$$

and

$$(Jy)\ (t)\ \geq\ \sum_{i=1}^{m} p_i\,(t)\ \exp\ (-h_1\ (t-\tau_i)) + \sum_{i=1}^{m} p_{ik}\ \exp\ (-h_1\ (t_k-\tau_i))$$

$$-\int_t^{\infty} (u-t)\ f\ (u,\ \exp\ (-h_2 g\ (u)))\ du - \sum_{it\leq t_k<\infty} (t_k-t)\ f_k\ (t_k,\ \exp\ (-h_2 g\ (t_k)))$$

$$= \exp\ (-h_1 t) + \exp\ (-h_1 t) \left(\sum_{i=1}^{m} p_i\ (t)\ \exp\ (h_1 \tau_i) \right.$$

$$\left. + \exp\ (h_1 t) \sum_{i=1}^{m} p_{ik}\ \exp\ (-h_1\,(t_k-\tau_i))\right)$$

$$-\int_t^{\infty} (u-t)\ f\ (u_1\ \exp\ (-h_2 g\ (u)))\ du$$

$$-\sum_{t\leq t_k<\infty} (t_k-t)\ f_k\,(t_k,\ \exp\ (-h_2 g\ (t_k))) \geq \exp\ (-h_1 t)\ \ for\ \ t,\ t_k \geq T.$$

That is,

$$\exp\ (-h_1 t)\ \leq\ (Jy)\ (t)\ \leq\ \exp\ (-h_2 t)\,,\ \ t\ \geq T,$$

$$\exp\ (-h_1\ (t_k))\ \leq\ (Jy)\ (t_k)\ \leq\ \exp\ (-h_2 t_k)\,,\ \ t_k\ \geq T.$$

By the definition of $K(y)$ and the statement

$$\exp\ (-h_1 T) \leq (Jy)\ (T)\ \leq\ \exp\ (-h_2 T)\,,$$

It is clear that $h_2 \le K(y) \le h_1$. Hence

$$\exp(-h_1 t) \le (Jy)(t) \le \exp(-h_2 t), \quad t_0 \le t,\ t_k < T.$$

Next, we show that

$$|(Jy)(t_2) - (Jy)(t_1)| \le L\,|t_2 - t_1|, \tag{187}$$

for $t_1,\ t_2 \in [t_0, \infty)$ and $k:\ t_{1k},\ t_{2k} \in [t_0, \infty)$. Without loss of generality, we assume that $t_2 \ge t_1 \ge t_0$ and $\forall\ k:\ t_{2k} \ge t_{1k} \ge t_0$. Indeed, for $t_2 \ge t_1 \ge T$ and $\forall\ k:\ t_{2k} \ge t_{1k} \ge T$, using condition (186) and inequality (187), we have that

$$|(Jy)(t_2) - (Jy)(t_1)| = |(Jy)(t_2) + (Jy)(t_{2k}) - (Jy)(t_1) - (Jy)(t_{1k})|$$

$$\le \sum_{i=1}^{m} |p_i(t_1)\, y(t_1 - \tau_i) + p_i(t_{1k})\, y(t_{1k} - \tau_i) \\ - p_i(t_2)\, y(t_2 - \tau_i) - p_i(t_{2k})\, y(t_{2k} - \tau_i)|$$

$$+ \int_{t_1}^{\infty} (u - t_1)\, f(u_1 y(g(u)))\, du + \sum_{t_1 \le t_{1k} < \infty} (t_{1k} - t_1)\, f_k(t_{1k}, y(g(t_{1k}))) \\ - \int_{t_2}^{\infty} (u - t_2)\, f(u, y(g(u)))\, du$$

$$- \sum_{t_2 \le t_{2k} < \infty} (t_{2k} - t_2)\, f_k(t_{2k}, y(g(t_{2k})))$$

$$\le \sum_{i=1}^{m} |p_i(t_1)\, y(t_1 - \tau_i) - p_i(t_2)\, y(t_2 - \tau_i)| \\ + \sum_{i=1}^{m} |p_i(t_{1k})\, y(t_{1k} - \tau_i) - p_i(t_{2k})\, y(t_{2k} - \tau_i)|$$

$$+ \left| \int_{t_1}^{\infty} (u - t_1)\, f(u_1 y(g(u)))\, du - \int_{t_2}^{\infty} (u - t_2)\, f(u_1 y(g(u)))\, du \right|$$

$$+ \left| \sum_{t_1 \le t_{1k} < \infty} (t_{1k} - t_1)\, f_k (t_{1k}, y (g (t_{1k}))) - \sum_{t_2 \le t_{2k} < \infty} (t_{2k} - t_2)\, f_k (t_{2k}, y (g\ (t_{2k}))) \right|$$

$$\le \sum_{i=1}^{m} p_i\ (t_2)\ |y\ (t_2 - \tau_i)\ -\ y\ (t_1 - \tau_i)| + \sum_{i=1}^{m} |p_i (t_2)\ -\ p_i\ (t_1)|\ y\ (t_1 - \tau_i)$$

$$+ \left| \int_{t_1}^{t_2} (u - t_2)\ f\ (u_1 y\ (g\ (u)))\ du\ +\ \int_{t_2}^{\infty} (t_2 - t_1)\ f\ (u, y\ (g\ (u)))\, du \right|$$

$$+ \sum_{i=1}^{m} p_i\ (t_{2k})\ |y\ (t_{2k} - \tau_i) -\ y\ (t_{1k} - \tau_i)| + \sum_{i=1}^{m} |p_i\ (t_{2k})\ -\ p_i\ (t_{1k})|\ y\ (t_{1k} - \tau_i)$$

$$+ \left| \sum_{t_1 \le t_k \le t_2} (t_k - t_{1k})\ f\ (t_k, y\ (g\ (t_k)))\ +\ \sum_{t_2 \le t_k < \infty} (t_{2k} - t_{1k})\ f_k (t_k, y\ (g\ (t_k))) \right|$$

$$\le \left[\sum_{i=1}^{m} (p_i\, (t_2)\ +\ \exp\ (-h_2\, (t_1 - \tau_i)))\ L + \int_{t_1}^{\infty} f\, (u,\ \exp\ (-h_2 g\ (u)))\ du \right] \times$$
$$\times\ |t_2 - t_1|$$

$$+ \left[\sum_{i=1}^{m} (p_i\ (t_{2k}) +\ \exp\ (-h_2\, (t_{1k} - \tau_i)))\ L\ +\ \sum_{t_1 \le t_k < \infty} f_k\, (t_k,\ \exp\ (-h_2 g\ (t_k))) \right] \times$$
$$\times\ |\ t_{2k} - t_{1k}|$$

$$\le \left\{ \left[\sum_{i=1}^{m} p_i\ (t_2) + \sum_{i=1}^{m} \exp\ (-h_2\ (t_1 - \tau_i)) \right]\ +\ \left(\alpha\ -\ \sum_{i=1}^{m} p_i\ (t_2) \right) \right\}\ L\ |\ t_2 - t_1|$$

$$+ \left\{ \left[\sum_{i=1}^{m} p_i\ (t_{2k})\ +\ \sum_{i=1}^{m} \exp\ (-h_2)\ (t_{1k} - \tau_i) \right]\ +\ \left(\alpha\ -\ \sum_{i=1}^{m} p_i\ (t_{2k}) \right) \right\} \times$$
$$\times L\ |\ t_{2k} - t_{1k}|$$

$$= \left[\sum_{i=1}^{m} \exp\ (-h_2\ (t_1 - \tau_i))\ +\ \alpha \right]\ L\ |t_2 - t_1|\ \ +\ \ \left[\sum_{i=1}^{m} \exp\ (-h_2\ (t_{1k} - \tau_i))\ +\ \alpha \right] \times$$
$$\times L\ |\ t_{2k} - t_{1k}|$$

$$\le\ \frac{L}{2}\ |t_2 - t_1|\ +\ \frac{L}{2}\ \frac{|t_{2k} - t_{1k}|}{t} \cdot \frac{|t_2 - t_1|}{|t_{2k} - t_{1k}|}$$

$$\le\ L\ |t_2 - t_1|\,.$$

For $t_0 \leq t_1 \leq t_2 \leq T$ and $\forall\, k : t_0 \leq t_{1k} \leq t_{2k} \leq T$, we have

$$|(Jy)(t_2) - (Jy)(t_1)| = |(Jy)(t_2) + (Jy)(t_{2k}) - (Jy)(t_1) - (Jy)(t_{1k})|$$

$$= |\exp(-K(y)(t_2)) + \exp(-K(y)(t_{2k})) - \exp(-K(y)(t_1)) - \exp(-K(y)(t_{1k}))|$$

$$\leq |\exp(-K(y)(t_2)) - \exp(-K(y)(t_1))| + |\exp(-K(y)(t_{2k})) - \exp(-K(y)(t_{1k}))|$$

$$\leq \frac{L}{2}\,|t_2 - t_1| + \frac{L\,|t_{2k} - t_{1k}|}{2} \times \frac{|t_2 - t_1|}{|t_{2k} - t_{1k}|} = L\,|t_2 - t_1|.$$

For $t_0 < t_1 \leq T \leq t_2$ and $\forall\, k : t_0 < t_{1k} \leq T \leq t_{2k}$, we obtain

$$|(Jy)(t_2) - (Jy)(t_1)| \leq |(Jy)(t_2) - (Jy)(t_1)| + |(Jy)(t_{2k}) - (Jy)(t_{1k})|$$

$$\leq |(Jy)(t_2) - (Jy)(T)| + |(Jy)(T) - (Jy)(t_1)| + |(Jy)(t_{2k}) - (Jy)(T)| + |(Jy)(T) - (Jy)(t_{1k})| \leq \frac{L}{2}\,|t_2 - T| + \frac{L}{2}\,|T - t_1| + \frac{L\,|t_2 - t_1|}{2\,|t_{2k} - t_{1k}|}\,t_{2k} - T \left| + \frac{L\,|t_2 - t_1|}{2\,|t_{2k} - t_{1k}|} \right| |T - t_{1k}|$$

$$= \frac{L}{2}\,|t_2 - t_1| + \frac{L}{2}\,|t_2 - t_1| = L\,|t_2 - t_1|.$$

We have proved that inequality (187) holds for all $t_0 \leq t_1 \leq t_2$ and $\forall\, k : t_0 \leq t_{1k} \leq t_{2k}$. Therefore, $J\Omega \subseteq \Omega$. Hence, J is piece-wise continuous. Since $J\Omega \subseteq \Omega$, $J\Omega$ is uniformly bounded.

Set $y \in \Omega$. It immediately implies that

$$|(Jy)(t)| \leq b_0,$$

where $b_0 > 0$ and

$$|(Jy)(t_2) - (Jy)(t_1)| \leq L\,|t_2 - t_1|$$

for $t_2 \geq t_1 \geq t_0$ and $k:\ t_{2k} \geq t_0$. Without loss of generality, we set

$$b_0 = \exp(-h_2 t),\ \ t,\, t_k \geq t_0.$$

Hence, for any arbitrarily pre-assigned small positive numberε, there exists a sufficiently large $T' > t_0$ such that whenever $\exp(-h_2 t) < \frac{\varepsilon}{2}$,

$$|(Jy)(t_2) - (Jy)(t_1)| \leq \exp(-h_2 t_2) + \exp(-h_2 t_1) \leq \varepsilon \tag{188}$$

for $t,\ t_k \geq T'$, $t_2 \geq t_1 \geq T'$ and $k:\ t_{2k} \geq t_{1k} \geq T'$.

On the other hand, if we set $\lambda = \frac{\varepsilon}{L}$ and assume that $|t_2 - t_1| < \lambda$, then for all $t_0 \leq t_1 \leq t_2 \leq T'$ and $k:\ t_0 \leq t_{1k} \leq t_{2k} \leq T'$, it becomes clear that

$$|(Jy)(t_2) - (Jy)(t_1)| \leq \varepsilon \tag{189}$$

Thus, from inequalities (188) and (189), we can affirm that $J\Omega$ is quasi-equicontinuous. Therefore, $J\Omega$ is relatively compact. By virtue of Schauder-Tychonoff fixed point theorem, the mapping J has a fixed point $y^* \in J$ such that $y^* = Jy^*$. Then y^* is a positive solution of equation (175) and $y^* \in \Lambda^{(0,0,0)}$. This completes the proof of Theorem 4.6.

Theorem 4.7. *Assume that* $\lim\limits_{t\to\infty} \sum_{i=1}^{m} p_i(t) + \lim\limits_{t_k\to\infty} \sum_{i=1}^{m} p_{ik} = p \in [0, 1)$. *Then equation* (175) *has a non-oscillatory solution* $y \in \Lambda^{(b,a,0)}$ $(b,\ a \neq 0)$ *if and only* *if*

$$\int_{t_0}^{\infty} u \left| \sum_{j=1}^{n} f_j \ (u,\ b_1,\ ...,\ b_1) \right| du$$
$$+ \sum_{t_0 \leq t_k < \infty} t_k \left| \sum_{j=1}^{n} f_{jk} \left(t_k,\ b_1,\ ...,\ b_1\right) \right| < \infty \tag{190}$$

for $b_1 \neq 0$.

Proof

i) **Necessity:** Without loss of generality, let $y(t) \in \Lambda^{(b,a,0)}$ be a finally positive solution of equation (175). From Theorem 4.5, we know that $b > 0$ and $a > 0$. Using notations in equations (182) and (183), we obtain from equations (175) and (176),

$$\begin{cases} x''\ (t) \ = \ - f\left(t,\ y\ (g\ (t))\right) \\ \Delta x'\ (t_k) = f_k \left(t_k,\ y\ (g\ (t_k))\right). \end{cases}$$

Integrating it from s *to* ∞ for $s \geq t_0$, we have

$$x'\ (s) \ = \ \int_s^{\infty} f\ (u,\ y\ (g\ (u)))\ du \ + \sum_{s \leq t_k < \infty} f_k \left(t_k, y\ (g\ (t_k))\right). \tag{191}$$

Again, integrating equation(191) from T to t, where T is sufficiently large, we obtain

$$\begin{aligned} x\ (t) \ = \ x\ (T) \ &+ \ \int_T^t (u - T)\ f\ (u,\ y\ (g\ (u)))\ du \\ &+ \int_t^{\infty} (t - T)\ f\ (u,\ y\ (g\ (u)))\ du \\ &+ \sum_{T \leq t_k \leq t} (t_k - T)\ f_k\ (t_k,\ y\ (g\ (t_k))) \\ &+ \sum_{t \leq t_k < \infty} (t - T)\ f_k\ (t_k,\ y\ (g\ (t_k))). \end{aligned} \tag{192}$$

Since $\lim_{u \to \infty} y\left(g_{jh}\left(u\right)\right) \ = \ b \ > \ 0$ and $\lim_{t_k \to \infty} y\left(g_{jh}\left(t_k\right)\right) \ = \ b \ > \ 0,\ j \ = \ 1,\ 2,\ \cdots,\ n,\ \ h = 1,\ 2,\ \cdots,\ \ell$, there exists a $T \geq t_0$ such that $y\left(g_{jh}\ (u)\right) \geq \frac{b}{2}$

for $u \geq T$ and $y(g_{jh}(t_k)) \geq \frac{b}{2}$ for $k : t_k \geq T$. Hence from equation (192) we have

$$\int_T^t (u-T) \left| \sum_{j=1}^{n} f_j \left(u, \frac{b}{2}, \cdots, \frac{b}{2}\right) \right| du + \sum_{T \leq t_k \leq t} (t_k - T) \left| \sum_{j=1}^{n} f_{jk} \left(t_k, \frac{b}{2}, \cdots, \frac{b}{2}\right) \right| < x(t) - x(T)$$

which implies that condition (190) holds.

ii) **Sufficiency:** Set $b_1 > 0$ and $A > 0$ so that $A < (1-p)\, b_1$. From condition (190) there exists a sufficiently large T so that for $t,\ t_k \geq T$ we have $t - \tau_i \geq t_0,\ \ t_k - \tau_i \geq t_0,\ \ i = 1, 2, \cdots, m$, and $g_{jh}(t) \geq t_o,\ \ g_{jh}(t_k) \geq t_0,\ j = 1, 2, \cdots, n,\ \ h = 1, 2, \cdots, \ell$ and

$$\frac{\mathrm{A}}{b_1} + \sum_{i=1}^{m} (p_i(t) + p_{ik}) + \frac{1}{b_1} \int_T^{\infty} u \sum_{j=1}^{n} f_j(u, b_1, \ldots, b_1)\, du + \frac{1}{b} \sum_{T \leq t_k < \infty} t_k \sum_{j=1}^{n} f_{jk}(t_k, b_1, \ldots, b_1) \leq 1. \tag{193}$$

Let Ω be the set of all piece-wise continuous functions $y(t) \in [t_0, \infty)$ such that $0 \leq y(t) \leq b_1,\ \ t,\ t_k \geq t_0$. Define a mapping J in Ω as follows:

$$(Jy)(t) = \begin{cases} \mathrm{A} + \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i) + \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i) \\ \quad + \int_T^t u\, f(u, y(g(u)))\, du \\ \quad + \int_t^{\infty} t\, f(u, y(g(u)))\, du \\ \quad + \sum_{T \leq t_k \leq t} t_k f_k(t_k, y(g(t_k))) \\ \quad + \sum_{t \leq t_k < \infty} t\, f_k(t_k, y(g(t_k))), \\ \qquad t,\ t_k \geq T \\ (Jy)(T), \quad t_0 \leq t,\ t_k < T. \end{cases} \tag{194}$$

Set

$y_0(t) = 0,\ \ t \geq t_0;$

$$y_\ell(t) = (Ty_{\ell-1})(t), \;\; t \ge t_0, \;\; \ell = 1, 2, \cdots . \tag{195}$$

It immediately follows that $y_0(t) < y_1(t) = \mathrm{A} \le b_1, \;\; t \ge t_0$. By induction, we obtain

$$\mathrm{A} \le y_\ell(t) \le y_{\ell+1}(t) \le b_1, \;\; t \ge t_0, \;\; \ell = 1, 2, \cdots .$$

Thus, $\lim_{\ell\to\infty} y_\ell(t) \le y(t)$ exists and $\mathrm{A} \le y(t) \le b_1, \;\; t \in [t_0, \infty)$. By Lebesgue's monotone convergence theorem, we obtain from equation (195) the result

$$y(t) = \begin{cases} \mathrm{A} + \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i) + \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i) \\ \qquad + \int_T^t u\, f(u, y(g(u)))\, du \\ \qquad + \int_t^\infty t\, f(u, y(g(u)))\, du \\ \qquad + \sum_{T \le t_k \le t} t_k f_k(t_k, y(g(t_k))) \\ \qquad + \sum_{t \le t_k < \infty} t f_k(t_k, y(g(u))), \\ \qquad t, t_k \ge T \\ y(T), \quad t_0 \le t, t_k < T. \end{cases}$$

Hence, $y(t)$ is a positive solution of equation (175). Since $0 < \mathrm{A} \le y(t) < b_1$, from Theorem 4.5, $y \in \Lambda^{(b,a,0)}$. This completes the proof of Theorem 4.7.

Using reasoning analogous to that given in the proof of Theorem 4.7 above, we can verify the following results.

Theorem 4.8. *Assume that* $\lim_{t\to\infty} \sum_{i=1}^{m} p_i(t) + \lim_{t_k\to\infty} \sum_{i=1}^{m} p_{ik} = p \in [0, 1)$. *Then equation* (175) *has a non-oscillatory solution* $y \in \Lambda^{(\infty,\infty,d)}$, $(d \ne 0)$ *if and only if*

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{n} f_j(u, d_1 g_{j1}(u), \ldots, d_1 g_{j\ell}(u)) \right| du + \sum_{t_0 \le t_k < \infty} \left| \sum_{j=1}^{n} f_{jk}(t_k, d_1 g_{ji}(t_k), \ldots, d_1 g_{j\ell}(t_k)) \right| < \infty, \tag{196}$$

$$q_k = \frac{1}{4}\left(t_k^{-3/2} - \tfrac{1}{2}\left(t_k - 1\right)^{-3/2}\right) t_k^{-1/6}.$$

For large t and t_k, $q(t) \sim Mt^{-5/3}$ and $q_k \sim Mt_k^{-5/3}$. It is obvious that inequalities (197) and (198) are satisfied. From Theorem 4.9, equation (200) has a solution $y \in \Lambda^{(\infty,\infty,0)}$. In fact, $y(t) = \sqrt{t}$
is such a solution of equation (200).

Remark 4.1. The above arguments can be applied to the equation

$$\begin{cases} \left[y(t) - \sum_{i=1}^{m} p_i(t)\, y(t - \tau_i)\right]'' \\ \qquad\qquad = \sum_{j=1}^{n} f_j\left(t, y\left(g_{j1}(t)\right), \cdots, y\left(g_{j\ell}(t)\right)\right), \\ \qquad\qquad\qquad\qquad t \geq t_0, \ \ t \notin S \\ \Delta\left[y(t_k) - \sum_{i=1}^{m} p_{ik}\, y(t_k - \tau_i)\right]' \\ \qquad\qquad = \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{j1}(t_k)\right), \cdots, y\left(g_{j\ell}(t_k)\right)\right), \\ \qquad\qquad\qquad\qquad t_k \geq t_0, \ \forall\, t_k \in S. \end{cases} \tag{201}$$

For instance, under the assumptions of Theorem 4.5, we have

$$\Lambda = \Lambda^{(0,0,0)} \bigcup \Lambda^{(b,a,0)} \bigcup \Lambda^{(\infty,\infty,\alpha)} \bigcup \Lambda^{(\infty,\infty,\infty)}.$$

Therefore Theorems 4.7 and 4.8 hold for equation (201). Furthermore, equation (201) has a non-oscillatory solution $y(t) \in \Lambda^{(\infty,\infty,\infty)}$ if

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{n} f_j\left(t, d_1 g_{j1}(t), \ldots, d_1 g_{j\ell}(t)\right) \right| dt \\ + \sum_{t_0 \leq t_k < \infty} \left| \sum_{j=1}^{n} f_{jk}\left(t_k, d_1 g_{j1}(t_k), \ldots, d_1 g_{j\ell}(t_k)\right) \right| < \infty \tag{202}$$

for some $d_1 \neq 0$.

At this point we now present another result for the second order linear equation

$$\begin{cases} [y(t) - py(t-\tau)]'' + q(t)y(g(t)) = 0, \ t \geq t_0, \ t \notin S \\ \Delta [y(t_k) - py(t_k-\tau)]' + q_k y(g(t_k)) = 0, \ t_k \geq t_0, \ \forall t_k \in S, \end{cases} \tag{203}$$

where the condition $p \in [0,1)$ is not required.

Theorem 4.10. *Assume that*

i) $p, \tau > 0,\ q_k \geq 0,\ q \in PC([t_0,\infty), R_+),\ g \in C([t_0,\infty), R),\ g(t+\tau) < t,$ $t \geq t_0$ *and* $\lim_{t\to\infty} g(t) = \infty$;

ii) there exists a constant $\alpha > 0$ *such that for sufficiently large* t,

$$\begin{aligned} &\tfrac{1}{p}e^{-\alpha\tau} + \tfrac{1}{p}\textstyle\int_{t+\tau}^{\infty}(s-t-\tau)\,q(s)\exp[\alpha(t-g(s))]\,ds \\ &\quad + \tfrac{1}{p}\textstyle\sum_{t+\tau\leq t_k\leq\infty}(t_k-t-\tau)\,q_k\exp[\alpha(t-g(t_k))] \leq 1. \end{aligned} \tag{204}$$

Then equation (203) *has a positive solution* $y(t)$ *that converges to zero as* t *tends to infinity.*

Proof: If the equality in equation (204) holds finally, then we can verify that $y(t) = e^{-\alpha t}$ is the expected solution. Otherwise, we assume that there exists $T \geq t_0$ such that $t+\tau \geq 0,\ \ g(t+\tau) \geq t_0$ for $t \geq T$, and

$$\begin{aligned} \mu = \frac{1}{p}e^{-\alpha\tau} &+ \frac{1}{p}\int_{T+t}^{\infty}(s-T-\tau)\,q(s)\exp[\alpha(T-g(s))]\,ds \\ &+\frac{1}{p}\sum_{T+\tau\leq t_k<\infty}(t_k-T-\tau)\,q_k\exp[\alpha(T-g(t_k))] < 1 \end{aligned} \tag{205}$$

and inequality (204) holds for $t \geq T$.

Let B_p denote the Banach space of all piece-wise continuous bounded functions defined on $[t_0,\infty)$ endowed with a sup norm. Let Ω be the subset of B_p defined by

$$\Omega = \{x \in B_p :\ 0 \leq x(t) \leq 1 \ \ for \ \ t \geq t_0\}.$$

Define a map $J: \ \Omega \to B_p$ as follows:

$$(Jx)\ (t) = (J_1 x)\ (t) + \ (J_2 x)\ (t)\,,$$

where

$$(J_1 x)\ (t) = \begin{cases} \frac{1}{p} e^{-\alpha\tau} x\ (t+\tau)\,, \ \ t \geq T \\ (J_1 x)\ (T) + \ \exp\left[\varepsilon\ (T-t)\right] - 1, \ \ t_0 \leq t \leq T \end{cases}$$

and

$$(J_2\, x)\ (t) = \begin{cases} \frac{1}{p}\int_{t+\tau}^{\infty} (s-t-\tau)\ q\,(s) \exp\left[\alpha\,(t - g\,(s))\right] x\,(g\,(s))\, ds \\ \qquad + \frac{1}{p}\sum_{t+\tau \leq t_k < \infty} (t_k - t - \tau)\ q_k \\ \qquad \exp\left[\alpha\,(t - g\,(t_k))\right] x\,(g\,(t_k))\,, \\ \qquad t \geq T \\ (J_2 x)\ (T)\,, \ \ t_0 \leq t \leq T, \end{cases}$$

where

$$\varepsilon = \frac{\ln\ (2-\mu)}{(T - t_0)}.$$

We can show that the map J satisfies all the assumptions of Krasnoselskii's fixed point Theorem, and so J has a fixed point x in Ω. Clearly, $x\,(t) > 0$ for $t \geq t_0$. Consequently, it is easy to verify that

$$y\,(t) = x\,(t)\ e^{-\alpha t}$$

is a solution of equation (203). This completes the proof of Theorem 4.10.

Corollary 4.4. Assume that $0 < p < 1, \ \ \tau > 0$ and there exist constants $q*, \ \sigma > \tau$ such that $0 \leq q\ (t) \leq \ q*, \ \ g\ (t) \geq t - \sigma$ finally. If the "majorant" that is, inequality (204) holds. Then, by Theorem 4.9, equation (203) has a positive solution.

4.4 Unstable type equations

4.4.1 Equations with constant coefficient p

Consider the second order linear neutral impulsive differential equation of the form

$$\begin{cases} [y(t) - py(t-\tau)]'' = q(t)\, y(g(t)), \ t \geq t_0, \ t \notin S \\ \Delta [y(t_k) - py(t_k - \tau)]' = q_k y(g(t_k)), \ t_k \geq t_0, \ \forall\, t_k \in S, \end{cases} \tag{208}$$

where $p \in R$, $q_k \geq 0$, $q \in PC([t_0, \infty), R_+)$; $g \in C([t_0, \infty), R)$; $\lim_{t \to \infty} g(t) = \infty$; $\tau > 0$.

In general, equation (208) always has an unbounded non-oscillatory solution. Therefore our task now is to find conditions for which all bounded solutions of equation (208) are oscillatory.

Theorem 4.11. *Assume that*

i) $0 < p < 1$, $\tau > 0$ *are constants;*

ii) $g(t) \leq 1$ *and* g *is non-decreasing for* $t \geq t_0$*;*

iii) the inequality

$$\lim_{t \to \infty} \sup \left[\int_{g(t)}^{t} (s - g(t))\, q(s)\, ds + \sum_{g(t) \leq t_k < \infty} (t_k - g(t))\, q_k \right] > 1 \tag{209}$$

holds.

Then every bounded solution $y(t)$ *of equation* (208) *is oscillatory.*

Proof: Assume by contradiction that $y(t)$ is a finally positive bounded solution of equation (208). Define

$$z(t) = y(t) - py(t-\tau). \tag{210}$$

We have $z''(t) > 0$ for $t \geq T \geq t_0$, $\Delta z'(t_k) > 0$ for $k : t_k \geq T \geq t_0$. If $z'(t), \Delta z(t_k) > 0$ for $t \geq T' > T$ and $k : t_k \geq T' > T$, then $\lim_{t\to\infty} z(t) = \infty$, which contradicts the boundedness of $y(t)$. Therefore, $z'(t), \Delta z(t_k) \leq 0$ for $t \geq T$ and $k : t_k \geq T$.

Here, we observe that there exists two possibilities for $z(t)$:

i) $z(t) > 0$ for $t \geq T$;

ii) $z(t) < 0$ for $t \geq T'' \geq T$.

In case (i), we integrate equation (208) from s to t and obtain

$$z'(t) - z'(s) = \int_s^t q(s)\, y(g(u))\, du + \sum_{s\leq t_k\leq t} q_k y(g(t_k)). \tag{211}$$

Again, integrating equation (211) in s from $g(t)$ to t, we obtain

$$\begin{aligned} z'(t)(t-g(t)) - z(t) + z(g(t)) &= \int_{g(t)}^t \int_s^t q(u)\, y(g(u))\, duds \\ &\quad + \sum_{g(t)\leq t_k\leq t} \sum_{s\leq t_k\leq t} q_k y(g(t_k)) \end{aligned}$$

$$= \int_{g(t)}^t (s-g(t))\, q(s)\, y(g(s))\, ds + \sum_{g(t)\leq t_k\leq t} (t_k - g(t))\, q_k y(g(t_k))$$

$$> \int_{g(t)}^t (s-g(t))\, q(s)\, z(g(s))\, ds + \sum_{g(t)\leq t_k\leq t} (t_k - g(t))\, q_k\, z(g(t_k))$$

$$\geq z(g(t)) \left[\int_{g(t)}^t (s-g(t))\, q(s)\, ds + \sum_{g(t)\leq t_k t} (t_k - g(t))\, q_k\right].$$

Hence for $t \geq T$,

$$z(t) + z(g(t)) \left(\int_{g(t)}^t (s-g(t))\, q(s)\, ds + \sum_{g(t)\leq t_k\leq t} (t_k - g(t))\, q_k - 1\right) \leq 0,$$

which contradicts the positivity of $z(t)$ and condition (209).

In case (ii), we have

$$y(t) < py(t-\tau) < p^2 y(t-2\tau) < \cdots < p^n y(t-n\tau)$$

for $t \geq T_2 + n\tau$, which implies that $\lim_{t\to\infty} y(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$. This is a contradiction and therefore completes the proof of Theorem 4.11.

Remark 4.2. Theorem 4.11 is also true for $p = 0$.

Theorem 4.12. *Assume that*

i) $p < 0$, $q_k > 0$ *and* $q(t) > 0$, *for all* $k \in Z$ *and* $t \geq t_0$*;*

ii) $g(t) = t - \sigma$, *where* σ*is a constant,* $\sigma > \tau$*;*

iii) There exists $\alpha > 0$ *such that*

$$\lim_{t,t_k\to\infty} \sup \left\{ \frac{q(t)}{q(t-\tau)}, \frac{q_k}{q(t_k-\tau)} \right\} = \alpha \tag{212}$$

and

$$\lim_{t\to\infty} \sup \left[\int_{t-(\sigma-\tau)}^{t} (s-(t-(\sigma-\tau)))\, q(s)\, ds + \sum_{t-(\sigma-\tau)\leq t \leq t} (t_k-(t-(\sigma-\tau))) \right] > 1-\alpha p. \tag{213}$$

Then every bounded solution of equation (208) *is oscillatory.*

Proof: Let us assume, by contradiction, that $y(t)$ is a bounded, finally positive solution of equation (208) and that $z(t)$ is defined by equation (210). As shown before, $z''(t)$, $\Delta z'(t_k) > 0$, $z'(t)$, $\Delta z(t_k) < 0$ and $z(t) > 0$ finally, where $z(t)$ *is defined by equation (210).*

From condition (212) and inequality (213), there exists a constant $M > 1$ such that

$$\lim_{t,t_k\to\infty} \sup \left[\int_{t-(\sigma-\tau)}^{t} (s-(t-(\sigma-\tau)))\, q(s)\, ds + \sum_{t-(\sigma-\tau)\le t_k\le t} (t_k-(t-(\sigma-\tau)))\right] > 1 - M\alpha p, \tag{214}$$

$$\frac{q(t)}{q(t-\tau)},\ \frac{q_k}{q(t_k-\tau)} < M\alpha,\ \ t, t_k \ge t_1, \tag{215}$$

where t_1 is a sufficiently large number. We rewrite equation (208) in the form

$$\begin{cases} z''(t) - p\,\frac{q(t)}{q(t-\tau)}\, z''(t-\tau) = q(t)\, z(t-\sigma), \ \ t \notin S \\ \Delta z'(t_k) - p\,\frac{q_k}{q(t_k-\tau)}\, \Delta z'(t_k-\tau) = q_k z(t_k-\sigma), \ \forall\, t_k \in S. \end{cases} \tag{216}$$

Substituting inequalities (215) into equation (216), we obtain

$$\begin{cases} z''(t) - M\alpha p\, z''(t-\tau) \ge q(t)\, z(t-\sigma), \ \ t \ge t_1, \ \ t \notin S \\ \Delta z'(t_k) - M\alpha p\, \Delta z'(t_k-\tau) \ge q_k z(t_k-\sigma), \ \ t_k \ge t_1, \ \ \forall\, t_k \in S. \end{cases} \tag{217}$$

Set

$$\omega(t) = z(t) - M\alpha p z(t-\tau). \tag{218}$$

Then

$$\begin{cases} \omega''(t) \ge q(t)\, z(t-\sigma) > 0, \ \ t \ge t_1, \ \ t \notin S \\ \Delta\omega'(t_k) \ge q_k z(t_k-\sigma) > 0, \ \ t_k \ge t_1, \ \ \forall\, t_k \in S. \end{cases} \tag{219}$$

By the boundedness of the function $y(t)$, it is seen that $\omega(t) > 0, \quad \omega'(t), \Delta\omega(t_k) \le 0$ for $t,\ t_k \ge t_2 \ge t_1$. Since $z(t)$ is decreasing,

$$\omega(t) = z(t) - M\alpha p z(t-\tau) \le (1 - M\alpha p) z(t-\tau), \quad t \ge t_2. \tag{220}$$

Combining inequalities (219) and (220), we obtain

$$\begin{cases} \omega''(t) \geq \frac{1}{1-M\alpha p} q(t)\, \omega(t-(\sigma-\tau)), \ t \notin S \\ \Delta\omega'(t_k) \geq \frac{1}{1-M\alpha p} q_k \omega(t_k-(\sigma-\tau)), \ \forall\, t_k \in S. \end{cases} \tag{221}$$

Integrating inequality (221) from s to t for $t \geq s \geq t_2$ we have

$$\begin{aligned} \omega'(t) - \omega'(s) \ &\geq \ \frac{1}{1-M\alpha p} \int_s^t q(u)\, \omega(u-(\sigma-\tau))\, du \\ &+ \frac{1}{1-M\alpha p} \sum_s q_k \omega(t_k-(\sigma-\tau)). \end{aligned} \tag{222}$$

We integrate inequality (222) in s from $t-(\sigma-\tau)$ to t and obtain

$$\omega'(t)(\sigma-\tau) - \omega(t) + \omega(t-(\sigma-\tau))$$

$$\begin{aligned} &\geq \ \frac{1}{1-M\alpha p} \left[\int_{t-(\sigma-s)}^t \int_s^t q(u)\, \omega\, (u-(\sigma-\tau))\, duds \right. \\ &\qquad \left. + \sum_{t-(\sigma-\tau)\leq t_k \leq t} \ \sum_{s \leq t \leq t} q_k \omega\, (t_k-(\sigma-\tau)) \right] \\ &= \ \frac{1}{1-M\alpha p} \int_{t-(\sigma-\tau)}^t (u-(t-(\sigma-\tau)))\, q\, (u)\, \omega\, (u-(\sigma-\tau))\, du \\ &+ \frac{1}{1-M\alpha p} \sum_{t-(\sigma-\tau)\leq t_k \leq t} (t_k-(t-(\sigma-\tau)))\, q_k \omega(t_k-(\sigma-\tau)) \\ &\geq \ \frac{\omega(t-(\sigma-\tau))}{1-M\alpha p} \left[\int_{t-(\sigma-\tau)}^t (u-(t-(\sigma-\tau)))\, q(u)\, du \right. \\ &\qquad \left. + \sum_{t-(\sigma-\tau)\leq t_k \leq t} (t_k-(t-(\sigma-\tau)))\, q_k \right], \ \ t, t_k \geq t_2. \end{aligned}$$

Thus,

$$\begin{aligned} \omega(t) \ + \ \omega(t-(\sigma-\tau)) &\left\{ \frac{1}{1-M\alpha p} \left[\int_{t-(\alpha-\tau)}^t (u-(t-(\alpha-\tau)))q(u)\, du \right. \right. \\ &\quad \left. \left. + \sum_{t-(\alpha-\tau)\leq t_k \leq t} (t_k-(t-(\alpha-\tau)))q_k \right] - 1 \right\} \leq 0. \end{aligned}$$

This contradicts condition (214) and therefore completes the proof of Theorem 4.12.

Theorem 4.13. *Assume that*

i) $p = 1, \ \ q_k > 0$ *and* $\tau > 0$;

ii) $g\ (t) \leq t$ *and* g *is non-decreasing for* $t \geq t_0$*;*

iii) either

$$\int_{t_0}^{\infty} t\, q\,(t)\ dt + \sum_{t_0 \leq t_k < \infty} t_k q_k = \infty \tag{223}$$

or

$$\lim_{t, t_k \to \infty} \left[t \int_t^{\infty} q\ (s)\ ds + t_k \sum_{t \leq t_k < \infty} q_k \right] = \infty. \tag{224}$$

Then every bounded solution of equation (208) *is oscillatory.*

Proof: Let us assume, by contradiction, that $y(t)$ is a bounded finally positive solution of equation (208) and $z(t)$ is defined by equation (210). There are two possibilities for $z(t)$ here:

a) $z''\,(t)\,, \Delta\, z'\,(t_k) \geq \ 0, \quad z'\,(t)\,, \Delta\, z\,(t_k) \leq 0, \quad z\,(t) < 0$ for $t, t_k \geq t_1 \geq t_0$;

b) $z''\,(t)\,, \Delta t'\,(t_k) \geq 0, \quad z'\,(t)\,, \Delta z\,(t_k) \leq 0, \quad z\,(t) > 0$ for $t, t_k \geq t_1 \geq t_0$.

In case (a), there exists a finite number $\alpha > 0$ such that

$$\lim_{t \to \infty} z\,(t) = -\alpha.$$

Thus, there exists $t_2 \geq t_1$ such that $-\alpha < z\,(t) < -\frac{\alpha}{2}, \quad t \geq t_2$, that is,

$$-\alpha < y\,(t) - y\,(t - \tau) < -\frac{\alpha}{2}, \quad t \geq t_2.$$

Hence $y\,(t - \tau) > \frac{\alpha}{2}, \ \ t \geq t_2$. Then there exists $t_3 \geq t_2$ such that $y\,(g\,(t)) > \frac{\alpha}{2}, \ \ t \geq$

t_3. From equation (208), we have

$$\begin{cases} z''(t) \geq \frac{\alpha}{2} q(t), \ \ t \geq t_3, \ \ t \notin S \\ \Delta z'(t_k) \geq \frac{\alpha}{2} q_k, \ \ t_k \geq t_3, \ \forall\, t_k \in S. \end{cases} \tag{225}$$

In case (b), we have

$$y(t) > y(t-\tau), \ \ t \geq t_1.$$

Then there exists $L > 0$ such that $y(t) \geq L, \ \ t \geq t_1$. Hence

$$\begin{cases} z''(t) \geq L\, q(t), \ \ t \geq t_3, \ \ t \notin S \\ \Delta z'(t_k) \geq L\, q_k, \ \ t_k \geq t_3, \ \ t_k \in S. \end{cases} \tag{226}$$

Therefore, in both cases, we are led to the same inequality (226). Integrating inequality (226) from t to T for $T > t, \ \ t_k \geq t_3$, we have

$$z'(T) - z'(t) \geq L \left[\int_t^T q(s)\, ds + \sum_{\tau \leq t_k \leq T} q_k \right], \ \ \ t_3 \geq t, \ t_k < T,$$

which implies that

$$\int_{t_0}^{\infty} q(s)\, ds + \sum_{t_0 \leq t_k < \infty} q_k < \infty,$$

and so

$$-\, z'(t) \geq L \left[\int_t^{\infty} q(s) + \sum_{t \leq t_k < \infty} q_k \right]. \tag{227}$$

Integrating inequality (227) from t to T for $T > t$, we obtain

$$z(t) \geq z(T) + L \left[\int_t^T \int_s^{\infty} q(u)\, duds + \sum_{t \leq t_k < \infty} q_k \right]$$

$$= z(T) + L\left[\int_t^T (u-T)\,q(u)\,du \;+\; (T-t)\int_T^\infty q(u)\,du \right.$$
$$\left. + \sum_{t\le t_k T} (t_k - T)\,q_k + \sum_{T\le t_k<\infty} q_k\right],\quad t_1 t_k \ge t_2,$$

which leads to a contradiction to the boundedness of $z(t)$ in either of the cases in equation (223) or (224). This completes the proof of Theorem 4.13.

Example 4.3. Consider the equation

$$\begin{cases} [y(t) - y(t-2\pi)]'' = \frac{2\pi}{t-\pi} y(t-\pi) \\ [\Delta y(t_k) - y(t_k - 2\pi)]' = \frac{2\pi}{t_k-\pi} y(t_k - \pi). \end{cases} \tag{228}$$

It is easy to see that all the assumptions of Theorem 4.13 are satisfied. Therefore, every bounded solution of equation (228) is oscillatory.

Equation (228) may have unbounded oscillatory solutions. For example, equation (228) has a solution $y(t) = t\sin t$.

Theorem 4.14. *Assume that* $p > 1$. *Then equation* (208) *has a bounded positive solution if and only if*

$$\int_{t_0}^\infty t\,q(t)\,dt + \sum_{t_0\le t_k<\infty} t_k < \infty. \tag{229}$$

Proof

i) **Necessity:** Let $y(t)$ be a bounded positive solution of equation (208) and $z(t)$ is defined by equation (210). Then equation (208) becomes

$$\begin{cases} z''(t) = q(t)\,y(g(t)),\quad t\notin S \\ \Delta z'(t_k) = q_k\,y(g(t_k)),\quad \forall\, t_k \in S. \end{cases}$$

At this point, there exists two possibilities for $z(t)$:

a) $z''(t), \Delta z'(t_k) \ge 0;\quad z'(t), \Delta z(t_k) \le 0;\quad z(t) < 0,\ \ t, t_k \ge t_1 \ge t_0;$

b) $z''(t), \Delta z'(t_k) \ge 0;\quad z'(t), \Delta z(t_k) \le 0;\quad z(t) > 0,\ \ t, t_k \ge t_1 t_0.$

As in the proof of Theorem 4.13, in either case we are led to the inequality

$$\begin{cases} z''(t) \geq L\,q(t), & t \geq t_2, \ t \notin S \\ \Delta z'(t_k) \geq L\,q_k, & t_k \geq t_2, \ \forall\, t_k \in S, \end{cases} \tag{230}$$

where $L > 0$ is a constant and t_2 is a sufficiently large number. Integrating inequality (230) twice, we obtain

$$\begin{aligned} z(t) - z(T) &\geq L\left[\int_t^T \int_s^T q(u)\,duds + \sum_{t \leq t_k \leq T} \sum_{s \leq t \leq T} q_k\right] \\ &= L\left[\int_t^T (u-t)\,q(u)\,du + \sum_{t \leq t_k \leq T} (t_k - t)\,q_k\right], \end{aligned}$$

for $T > t, t_k \geq t_2$. Letting $T \to \infty$, it is obvious that inequality (229) is satisfied.

ii) **Sufficiency:** The sufficiency part of Theorem 4.14 is derived from the following more general result.

Theorem 4.15. *Assume that* $p > 1$, $q_k > 0$ *and* $q \in PC([T_0, \infty), R)$ *such that*

$$\int_{t_0}^{\infty} s\,|q(s)|\,ds + \sum_{t_0 \leq t_k < \infty} t_k\,|q_k| < \infty. \tag{231}$$

Then equation (208) *has a bounded positive solution.*

Proof: Let $T \geq t_0$ be sufficiently large so that $t + \tau \geq t_0$, $g(t+\tau) \geq t_0$ for $t \geq T$, and

$$\int_{t+\tau}^{\infty} s\,|q(s)|\,ds + \sum_{t+\tau \leq t_k < \infty} t_k\,|q_k| \leq \frac{p-1}{4}, \quad t, t_k \geq T. \tag{232}$$

Consider the Banach space B_p of all piece-wise continuous bounded functions defined on $[t_0, \infty)$with the sup norm. Set

$$\Omega = \left\{ y \in B_p : \frac{p}{2} \leq y(t) \leq 2p, \ t \geq t_0 \right\}.$$

Clearly, Ω is a bounded closed convex subset of B_p. Define a mapping $J : \Omega \to B_p$

as follows

$$\begin{cases} p-1+\frac{1}{p}y\left(t+\tau\right)-\frac{1}{p}\int_{t+\tau}^{\infty}\left(s-t-\tau\right)\,q\left(s\right)y\left(g\left(s\right)\right)ds \\ \qquad\qquad -\frac{1}{p}\sum_{t+\tau\leq t_k<\infty}\left(t_k-t-\tau\right)\,q_k y\left(g\left(t_k\right)\right),\;\; t\geq\tau \\ \left(Jy\right)\left(T\right),\quad t_0\leq t\leq T. \end{cases} \tag{233}$$

For any $y\in\Omega$ and from inequality (232) we obtain

$$\begin{aligned} \left(J\right)\left(t\right) &\leq p+1\frac{1}{p}\int_{t+\tau}^{\infty}\left(s-t-\tau\right)\left|q\left(s\right)\right|\left|y\left(g\left(s\right)\right)\right|ds \\ &\quad +\frac{1}{p}\sum_{t+\tau\leq t_k<\infty}\left(t_k-t-\tau\right)\left|q_k\right|\left|y\left(g\left(t_k\right)\right)\right|\leq 2p, \end{aligned}$$

for $t,t_k\geq T$, and

$$\begin{aligned} \left(Jy\right)\left(t\right) &\geq p-\frac{1}{2}-\frac{1}{p}\int_{t+\tau}^{\infty}\left(s-t-\tau\right)\left|q\left(s\right)\right|\left|y\left(g\left(s\right)\right)\right|ds \\ &\quad -\frac{1}{p}\sum_{t+\tau\leq t_k<\infty}\left(t_k-t-\tau\right)\left|q_k\right|\left|y\left(g\left(t_k\right)\right)\right|\geq\frac{p}{2}, \end{aligned}$$

for $t,t_k\geq T$. Therefore, $T\Omega\subseteq\Omega$.

We shall show that J is a contraction mapping on Ω. In fact, for any $y_1,\ y_2\in\Omega$,

$$\left|\left(Jy_1\right)\left(t\right)-\left(Jy_2\right)\left(t\right)\right|\leq\frac{1}{p}\left|y_1\left(t+\tau\right)-y_2\left(t+\tau\right)\right|$$

$$+\frac{1}{p}\int_{t+\tau}^{\infty}\left(s-t-\tau\right)\left|q\left(s\right)\right|\left|y_1\left(g\left(s\right)\right)-y_2\left(g\left(s\right)\right)\right|ds$$

$$+\frac{1}{p}\sum_{t+\tau\leq t_k<\infty}\left(t_k-t-\tau\right)\left|q_k\right|\left|y_1\left(g\left(t_k\right)\right)-y_2\left(g\left(t_k\right)\right)\right|$$

$$\leq\frac{1}{p}\left\|y_1-y_2\right\|+\frac{1}{p}\left\|y_1-y_2\right\|\left[\int_{t+\tau}^{\infty}\left(s-t-\tau\right)\left|q\left(s\right)\right|ds+\sum_{t+\tau\leq t_k<\infty}\left(t_k-t-\tau\right)\left|q_k\right|\right]$$

$$\leq\left|y_1-y_2\right|\left|\frac{1}{p}\left(1+\frac{p-1}{4}\right)=\frac{1}{4}\left(1+\frac{3}{p}\right)\right|\left\|y_1-y_2\right\|,\quad t,t_k\geq T,$$

which implies that

$$\begin{aligned}\|J\,y_1 - J\,y_2\| &= \sup_{t,t_k\geq t_0} |(Jy_1)\,(t) - (Jy_2)\,(t)| \\ &= \sup_{t,t_k\geq T} |(T\,y_1)\,(t) - (T\,y_2)\,(t)| \leq \frac{1}{4}\left(1+\frac{3}{p}\right)\|y_1 - y_2\|.\end{aligned}$$

Since $\frac{1}{4}\left(1+\frac{3}{p}\right) < 1$, it follows that J is a contraction mapping. Hence there exists a fixed point $y \in \Omega$. Then

$$\begin{cases} [y\,(t+\tau) - y\,(t)]'' = q\,(t+\tau)\,y\,(t+\tau)), & t \geq T, \ t \notin S \\ \Delta\,[y\,(t_k+\tau) - y\,(t_k)]' = q\,(t_k+\tau)\,y\,(t_k+\tau)), & t_k \geq T, \ \forall\, t_k \in S, \end{cases}$$

That is, $y(t)$ is a bounded positive solution of equation (208). This completes the proof of Theorem 4.15.

Remark 4.3. Using a reasoning analogous to that given in the proof above, we can show that Theorem 4.15 is also true for $p \in (0,1)$.

The following result is about the existence of asymptotically decaying positive solution of equation (208).

Theorem 4.16. *Assume that* $0 < p < 1$ *and that there exists a constant* $\alpha > 0$ *such that*

$$\begin{aligned} pe^{\alpha\tau} &+ \int_t^\infty (s-t)\,q\,(s)\,\exp\left[\alpha\,(t - g\,(s))\right]ds \\ &+ \sum_{t\leq t_k<\infty} (t_k - t)\,q_k \exp\left[\alpha\,(t - g\,(t_k))\right] \leq 1 \end{aligned} \tag{234}$$

finally. Then equation (208) *has a positive solution* $y(t)$ *satisfying* $y\,(t) \to 0$ *as* $t \to \infty$.

Proof: Notice that if the equality in condition (234) holds finally, then equation (208) has a positive solution

$$y\,(t) = e^{-\alpha t}.$$

In the rest of the proof, we may assume that there exists a number $T > t_0$ such that

$$t-\tau \geq t_0, \;\; g(t) \geq t_0 \;\text{ for }\; t \geq T$$

and

$$\beta = pe^{\alpha t} + \int_T^{\infty} (s-T)\, q(s) \exp\left[\alpha\left(T - g(s)\right)\right] ds + \sum_{T \leq t_k < \infty} (t_k - T)\, q_k \exp\left[\alpha\left(T - g(t_k)\right)\right] < 1 \tag{235}$$

and condition (234) holds for $t \geq T$.

Let B_p denote the Banach space of all piece-wise continuous bounded functions defined on $[t_0, \infty)$ and endowed with the sup norm. Let Ω be the subset of B_p defined by

$$\Omega = \{x \in B_p : 0 \leq x(t) \leq 1, \quad t \geq t_0\}.$$

Define a map $J : \Omega \to B_p$ as follows:

$$(Jx)(t) = (J_1 x)(t) + (J_2 x)(t),$$

where $\varepsilon = In\frac{(2-\beta)}{(T-t_0)}$ and

$$(J_2x)(t) = \begin{cases} \int_t^\infty (s-t)\, q(s) \exp[\alpha(t-g(s))]\, x(g(s))\, ds \\ \qquad + \sum_{t\le t_k<\infty} (t_k - t)\; q_k \times \\ \qquad \times\ \exp[\alpha(t-g(t_k))]\, x(g(t_k)), \\ \qquad t \ge T \\ (J_2x)(T), \quad t_0 \le t \le T. \end{cases} \tag{237}$$

Notice that the integral in (237) is defined whenever $x \in \Omega$. Obviously, the set Ω is closed, bounded and convex in B_p.

We shall show that for every pair $y, x \in \Omega$

$$J_1y + J_2x \in \Omega. \tag{238}$$

In fact, for any $y, x \in \Omega$, we have

$$\begin{aligned}(J_1y)(t) + (J_2x)(t) = pe^{\alpha\tau}y(t-\tau) + \int_t^\infty (s-t)\, q(s) \times \\ \times \exp[\alpha(t-g(s))]\; x(g(s))\, ds \\ + \sum_{t\le t_k<\infty} (t_kt)\, q_k \exp[\alpha(t-g(t_k))]\; x(g(t_k))\end{aligned}$$

$$\begin{aligned}\le pe^{\alpha\tau} + \int_\tau^\infty (s-t)\, q(s)\, \exp[\alpha\; (t-g(s))]\, ds \\ + \sum_{t\le t_k\infty} (t_k - t)\, q_k \exp[\alpha\; (t-g(t_k))] \le 1, \\ t \ge T\end{aligned}$$

and

$$
\begin{aligned}
(J_1 y)(t) + (J_2 x)(t) &= (J_1 y)(T) + (J_2 x)(T) + \exp\left[\varepsilon\,(T-t)\right] - 1 \\
&= \beta + \exp\left[\varepsilon\,(T-t)\right] - 1 \\
&\le \beta + \exp\left[\varepsilon\,(T-t_0)\right] - 1 \\
&= 1, \\
&t_0 \le t \le T
\end{aligned}
$$

Obviously, $(J_1 y)(t) + (J_2 x)(t) \ge 0$ for $t \ge t_0$. Thus, condition (238) is true. From inequality (235), we know that $pe^{\alpha\tau} < 1$, which implies that J_1 is a contraction mapping.

We shall now show that J_2 is completely continuous. In fact, from inequality (234), there exists a positive constant M such that

$$
\int_t^\infty q(s)\exp\left[\alpha\,(t-g(s))\right]ds + \sum_{t\le t_k<\infty} q_k \exp\left[\alpha\,(t-g(t_k))\right] \le M,
$$

for $t \ge T$. Thus, we have

$$
\begin{aligned}
\left|\frac{d}{dt}(J_2 x)(t)\right| &= \Bigg|\int_t^\infty q(s)\exp\left[\alpha(t-g(s))\right] x(g(s))\,ds \\
&\quad +\alpha \int_t^\infty (s-t)\,q(s)\exp\left[\alpha(t-g(s))\right] x(g(s))\,ds \\
&\quad + \sum_{t\le t_k<\infty} q_k \exp\left[\alpha(t-g(t_k))\right] x(g(t_k)) \\
&\quad +\alpha \sum_{t\le t_k<\infty} (t_k-t)\,q_k \exp\left[\alpha(t-g(t_k))\right] x(g(t_k))\Bigg| \\
&\le \Bigg|\int_t^\infty q(s)\exp\left[\alpha(t-g(s))\right] x(g(s))\,ds \\
&\quad + \sum_{t\le t_k<\infty} q_k \exp\left[\alpha\,(t-g(t_k))\right] x(g(t_k)) \\
&\quad +\alpha\Bigg[\int_t^\infty (s-t)\,q(s)\exp\left[\alpha\,(t-g(s))\right] x(g(s))\,ds \\
&\quad + \sum_{t\le t_k<\infty} (g_k-t)\,q_k \exp\left[\alpha\,(t-g(t_k))\right] x(g(t_k))\Bigg]\Bigg| \\
&\le M+\alpha,\ \ t>T
\end{aligned}
$$

and

$$\frac{d}{dt}\left(J_2 x\right)(t) = 0, \quad t_0 \leq t < T.$$

This shows the quasi-equicontinuity of the family $J_2\,\Omega$. On the other hand, it is immediately obvious that J_2 is piece-wise continuous and the family of $J_2\,\Omega$ is uniformly bounded. Therefore, J_2 is completely compact.

By Krasnoselskii's fixed point theorem, J has a fixed point $x \in \Omega$.

That is,

$$x\,(t) = \begin{cases} pe^{\alpha\tau} x\,(t-\tau) + \int_t^\infty (s-t)\,q\,(s) \\ \qquad \exp\left[\alpha\,\left(t - g\,(s)\right)\right]\,y\,(g\,(s))\,ds \\ \qquad + \sum_{t \leq t_k < \infty} (t_k - t)\,q_k \\ \qquad \exp\left[\alpha\,\left(t - g\,(t_k)\right)\right]\,y\,(g\,(t_k))\,, \\ \qquad t \geq T \\ x\,(T) + \exp\left[\varepsilon\,\left(T - t\right)\right] - 1, \;\; t_0 \leq t \leq T. \end{cases} \tag{239}$$

Since $x\,(t) \geq \exp\left[\varepsilon\,(T-t)\right] - 1 > 0$ for $t_0 \leq t < T$, it follows that $x(t) > 0$ for $t \geq t_0$. Set

$$y\,(t) = x\,(t)\,e^{-\alpha\tau}.$$

Then equation (239) becomes

$$\begin{aligned} y\,(t) = p\,y\,(t-\tau) \;\; &+ \;\; \int_t^\infty (s-t)\,q\,(s)\;y\,(g\,(s))\,ds \\ &+ \sum_{t \leq t_k < \infty} (t_k - t)\;q_k y\,(g\,(t_k))\,, \;\; t \geq T. \end{aligned} \tag{240}$$

Thus, $y(t)$ is a positive solution of equation (208) and $y\,(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 4.16.

Remark 4.4. The conclusion of Theorem 4.16 still holds for the case $p = 0$ if $g(t) < t$.

Corollary 4.5. Assume that $0 < p < 1$, and there exist constants $q^* > 0$ and $\sigma > 0$ such that

$$0 \le q(t) \le q*, \;\; g(t) \ge t - \sigma. \tag{241}$$

If the "majorant" equation

$$\begin{cases} [y(t) - p\,y(t-\tau)]'' = q * y(t-\sigma),\ t \ge t_0,\ \ t \notin S \\ \Delta\,[y(t_k) - p\,y(t_k - \tau)]' = q_k^* y(t_k - \sigma),\ t_k \ge t_0,\ \forall\, t_k \in S \end{cases} \tag{242}$$

has a bounded positive solution, then equation (208) also has a positive solution $y(t)$ that converges to zero as t tends to infinity.

Proof: The corresponding characteristic equation of equation (144) has the form

$$F(\lambda) \equiv \lambda^2 \left(1 - pe^{-\lambda t}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_1}\right) - q^* e^{-\lambda\sigma}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_2} = 0 \tag{243}$$

or

$$F(\lambda) \equiv \lambda^2 \left(-1 + pe^{-\lambda\tau}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_2}\right) + q^* e^{-\lambda\sigma}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_2} = 0.$$

Let $\alpha = -\lambda > 0$, then

$$-1 + pe^{\alpha\tau}\left(1 - \frac{q_k^*}{q^*}\alpha\right)^{n_1} + q^* e^{\alpha\sigma}\left(1 - \frac{q_k^*}{q^*}\alpha\right)^{n_2} = 0.$$

Considering the case where $n_1 = n_2 = 0$, we obtain

$$pe^{\alpha\tau} + \frac{q^*}{\alpha^2}e^{\alpha\sigma} = 1. \tag{244}$$

Equation (242) has a bounded positive solution if and only if its characteristic equation (243) has a real root $\alpha \in (0, \infty)$. This immediately means that equation (244) holds. Combining conditions (241) and equation (244), we realize that for sufficiently large t,

$$pe^{\alpha\tau} + \int_t^{\infty} (s-t)\; q(s) \exp\left[\alpha\,(t-\sigma)\right] ds + \sum_{t \le t_k < \infty} (t_k - t)\; q_k \exp\left[\alpha\,(t-\sigma)\right]$$

$$\le\; pe^{\alpha\tau} + \frac{1}{\alpha^2} q^* e^{\alpha\sigma} = 1.$$

By Theorem 4.16, equation (208) has a positive solution $y(t)$ which converges to zero as $t \to \infty$. This completes the proof of Corollary 4.5.

Example 4.4. Consider the equation

$$\begin{cases} \left[y(t) - \frac{1}{2e} y(t-2)\right]'' = \left(\frac{1}{8e} - \frac{1}{t}\right) y(t-2), & t \notin S \\ \Delta \left[y(t_k) - \frac{1}{2e} y(t-2)\right]' = \frac{1}{8e} y(t_k - 2), & \forall\; t_k \in S. \end{cases} \tag{245}$$

In our notation, $p = \frac{1}{2e}, \quad q* = \frac{1}{8e}, \quad \tau = 2 \;\; and \;\; \sigma = 2$. The "*majorant*" equation is

$$\begin{cases} \left[y(t) - \frac{1}{2e} y(t-2)\right]'' = \frac{1}{8e} y(t-2), & t \notin S \\ \Delta \left[y(t_k) - \frac{1}{2e} y(t-2)\right]' = \frac{1}{8e} y(t_k - 2), & \forall\, t_k \in S \end{cases} \tag{246}$$

and the characteristic equation (243) becomes

$$\lambda^2 \left(1 - \frac{1}{2e} e^{-2\lambda} (1+\lambda)^{n_1}\right) + \frac{1}{8e} e^{-2\lambda} (1+\lambda)^{n_2} = 0$$

or

$$\lambda^2 \left(-1 + \frac{1}{2e} e^{-2\lambda} (1+\lambda)^{n_1}\right) + \frac{1}{8e} e^{-2\lambda} (1+\lambda)^{n_2} = 0. \tag{247}$$

Setting $n_1 = n_2 \;\; and \;\; \alpha = -\lambda$, equation (244) becomes

$$\frac{1}{2e} e^{2\alpha} + \frac{1}{8e\alpha^2} e^{2\alpha} = 1. \tag{248}$$

It is obvious that $\alpha = -\frac{1}{2}$. Consequently, $\lambda = \frac{1}{2}$ is a real root of equation (247), and hence, equation (246) has a bounded positive solution. By Corollary 4.5, equation (245) has a positive solution $y(t)$ that converges to zero as $t \to \infty$.

4.4.2 Equations with variable coefficient p

We now consider the second order neutral impulsive differential equation

$$\begin{cases} [y(t) - p(t)\,y(t-\tau)]'' = q(t)\,y(t-\sigma), & t \geq t_0, \ t \notin S \\ \Delta [y(t_k) - p_k y(t_k - \tau)]' = q_k y(t_k - \sigma), & t_k \geq t_0, \ \forall\, t_k \in S, \end{cases} \tag{249}$$

where $\tau,\ \sigma \in (0, \infty)$; $q_k \in R$; $p, q \in PC([t_0, \infty), R)$.

Theorem 4.17. *Assume that*

i) $0 \leq p(t) \leq 1, \quad t \geq t_0$;

ii) $0 < h_1 \leq q(t) \leq h_2, \quad t \geq t_0$;

iii) for any $\lambda > 0$,

$$\begin{cases} \liminf\limits_{t\to\infty} \left\{ p(t-\sigma)\frac{q(t)}{q(t-\tau)} e^{\lambda\tau} + \frac{1}{\lambda^2} q(t)\, e^{\lambda\sigma} \right\} > 1 \\ \liminf\limits_{t_k\to\infty} \left\{ p(t_k - \sigma)\frac{2_k}{q(t-\tau)} e^{\lambda\tau} + \frac{1}{n^2} q_k e^{\lambda\sigma} \right\} > 1. \end{cases} \tag{250}$$

Then every bounded solution of equation (249) *is oscillatory.*

Proof: Let $y(t)$ be a bounded, finally positive solution of equation (249). Assume further that $y(t-\tau),\ y(t-\sigma) > 0 \ for \ t \geq t_1 \geq t_0$. Set

$$z(t) = y(t) - p(t)y(t-\tau). \tag{251}$$

It is not difficult to show that $z''(t), \Delta z'(t_k) > 0$; $z'(t), \Delta z(t_k) < 0$; $z(t) > 0$ for $t \geq t_1$, where t_1 is large enough and

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0.$$

Then from condition (ii)

$$\begin{cases} z''(t) \geq h_1 y(t-\sigma), & t \geq t_1, \ t \notin S \\ \Delta z'(t_k) \geq h_1 y(t_k - \sigma), & t_k \geq t_1, \ \forall\, t_k \in S \end{cases} \tag{252}$$

and

$$\begin{cases} z''(t) \geq h_2 y(t-\sigma), & t \geq t_1, \ t \notin S \\ \Delta z'(t_k) \geq h_2 y(t_k - \sigma), & t_k \geq t_1, \ \forall\, t_k \in S. \end{cases} \tag{253}$$

Define a set Λ as follows:

$$\Lambda = \left\{ \lambda > 0 : \begin{array}{l} z''(t) > \lambda^2 z(t) \\ \Delta z'(t_k) > \lambda^2 z(t_k) \end{array} \right. \tag{254}$$

finally. It is clearly seen that $\sqrt{h_1} \in \Lambda$, that is, Λ is nonempty. We shall show that Λ is bounded above. In fact, condition (252) implies that

$$\begin{cases} z''(t) \geq h_1 z(t-\sigma), & t \geq t_1 + \sigma, \ t \notin S \\ \Delta z'(t_k) \geq h_1 z(t_k - \sigma), & t_k \geq t_1 + \sigma, \ \forall\, t_k \in S. \end{cases} \tag{255}$$

Integrating inequality(255) from t to $t + \frac{\sigma}{2}$, we obtain

$$z'\left(t + \frac{\sigma}{2}\right) - z'(t) \geq h_1 \left[\int_t^{t+\sigma_2} z(s-\sigma)\, ds + \sum_{t \leq t_k \leq t + \sigma/2} z(t_k - \sigma) \right]$$

$$> \frac{\sigma}{2} h_1 \left[z\left(t - \frac{\sigma}{2}\right) + z\left(t_k - \frac{\sigma}{2}\right) \right] \ > \frac{\sigma}{2} h_1 z\left(t - \frac{\sigma}{2}\right), \quad t,\ t_k \geq t_1 + \sigma,$$

and then

$$z(t) - z\left(t + \frac{\sigma}{4}\right) > h_1 \frac{\sigma}{2} \left[\int_t^{t+\sigma/4} z\left(s - \frac{\sigma}{2}\right) ds + \sum_{t \leq t_k \leq t + \sigma/4} z\left(t_k - \frac{\sigma}{2}\right) \right]$$

$$> h_1 \left(\frac{\sigma^2}{8}\right) \left[z\left(t - \frac{\sigma}{4}\right) + z\left(t_k - \frac{\sigma}{4}\right) \right] > h_1 \left(\frac{\sigma^2}{8}\right) z\left(t - \frac{\sigma}{4}\right).$$

This implies that

$$z(t) > \alpha z\left(t - \frac{\sigma}{4}\right), \quad t \geq t_1 + \sigma, \tag{256}$$

where

$$\alpha = h_1\left(\frac{\sigma^2}{8}\right).$$

Applying condition (256) four times, we discover that

$$z\,(t) > \alpha\, z\,(t - \sigma)\,, \;\; t \geq t_1 + 2\sigma. \tag{257}$$

In view of the boundness of $y(t)$, it is not difficult to see that

$$\lim_{t\to\infty} \inf y\,(t) = 0.$$

Choose a sequence $\{s_n\}$ such that $s_n \geq t_1 + 2\sigma, \quad n = 1,\ 2,\ \cdots, \quad \lim_{n\to\infty} s_n = \infty$, and

$$y\,(s_n - \sigma) = \min\,\{y\,(s) : t_1 \leq s \leq s_n - \sigma\}\,,\;\; n = 1,\ 2,\ \cdots.$$

Integrating inequality (252) twice, we have

$$z\,(t-\sigma) > h_1\left[\int_{-t-\sigma}^{t}\int_{s}^{t} y\,(u-\sigma)\,duds + \sum_{t-\sigma\leq t_k<t}\,\sum_{s\leq t_k\leq t} y\,(t_k - \sigma)\right], \quad t, t_k \geq t_1 + \sigma$$

and

$$z\,(s_n-\sigma) > h_1\left[\int_{s_n-\sigma}^{s_n}\int_{s}^{s_n} y\,(u-\sigma)\,duds + \sum_{s_n-\sigma\leq t_k\leq s_n}\,\sum_{s\leq t_k\leq s_n} y\,(t_k - \sigma)\right]$$

$$\geq \frac{\sigma^2}{2} h_1 \left[y\left(s_n - \sigma\right) + y\left(s_k - \sigma\right) \right], \; n = 1, 2, \cdots.$$

$$\geq \frac{\sigma^2}{2} h_1 \, y\left(s_n - \sigma\right), \; n = 1, 2, \cdots,$$

that is,

$$y\left(s_n - \sigma\right) < \beta \, z\left(s_n - \sigma\right), \; n = 1, 2, \cdots,$$

where

$$\beta = \frac{2}{h_1 \sigma^2}.$$

Then from inequalities (253) and (257), we obtain

$$z''\left(s_n\right) \leq h_2 y\left(s_n - \sigma\right) < \beta \, h_2 z\left(s_n - \sigma\right) < \alpha^{-4} \beta \, h_2 \, z\left(s_n\right), \quad n = 1, 2, \cdots$$

which implies that $\sqrt{\alpha^{-4}\beta \, h_2} \in \Lambda$, that is, Λ is bounded above.

Set $\lambda_0 = \sup \Lambda$. Then $\lambda_0 \in \left(0, \sqrt{\alpha^{-4}\beta \, h_2}\right)$. For any $\alpha \in (0, 1)$ we discover that for sufficiently large t,

$$\begin{cases} z''(t) \geq \left(\alpha \, \lambda_0\right)^2 \, z(t), \; t \notin S \\ \Delta z'\left(t_k\right) \geq \left(\alpha \, \lambda_0\right)^2 \, z\left(t_k\right), \; \forall \, t_k \in S. \end{cases} \tag{258}$$

Set

$$\bar{z}(t) = z'(t) + \alpha \, h_0 z(t).$$

Then

$$\bar{z}'(t) - \alpha \, \lambda_0 \, z(t) = z''(t) - \left(\alpha \, \lambda_0\right)^2 \, z(t) \geq 0$$

finally. It implies that $\bar{z}(t) \, e^{-\alpha \lambda_0 t}$ is non-decreasing. Since $z(t) \to 0$, $z'(t) \to 0$ as

$t \to \infty$, so $\bar{z}(t) \to 0$ as $t \to \infty$. Thus, $\bar{z}(t) < 0$, that is,

$$z'(t) + \alpha\,\lambda_0 z(t) \le 0$$

finally. Set $\omega(t) = z(t)\, e^{\alpha\,\lambda_0 t}$. Then

$$\begin{cases} \omega'(t) \;=\; [z'(t) + \alpha\,\lambda_0 z(t)]\, e^{\alpha\,\lambda_0 t} \le 0 \\ \Delta\omega(t_k) = [\Delta z(t_k) + \alpha\,\lambda_0 z(t_k)]\, e^{\alpha\lambda_0 t_k} \le 0. \end{cases}$$

We can rewrite equation (249) in the form

$$\begin{cases} z''(t) = p(t-\sigma)\, \frac{q(t)}{q(t-\tau)} z''(t-\tau) + q(t)\; z(t-\sigma) \\ \Delta z'(t_k) = p(t_k - \sigma)\, \frac{q_k}{q(t_k-\tau)}\, \Delta z'(t_k - \tau) + q_k z(t_k - \sigma). \end{cases} \tag{259}$$

Then by condition (258), we have

$$z''(t) \ge (\alpha\,\lambda_0)^2\, p(t-\sigma)\;\frac{q(t)}{q(t-\tau)}\, z(t-\tau) + q(t)\; z(t-\sigma)$$

$$= (\alpha\,\lambda_0)^2\, p(t-\sigma)\;\frac{q(t)}{q(t-\tau)}\,\omega(t-\tau)\, e^{-\alpha\lambda_0(t-\tau)} + q(t)\;\omega(t-\sigma)\, e^{-\alpha\lambda_0}\,(t-\sigma)$$

$$\ge \left[(\alpha\,\lambda_0)^2\, p(t-\sigma)\;\frac{q(t)}{q(t-\tau)} e^{\alpha\lambda_0\tau} + q(t)\, e^{\alpha\lambda_0\sigma}\right]\; z(t),$$

$$\Delta z'(t_k) \ge (\alpha\,\lambda_0)^2\, p(t_k-\sigma)\;\frac{q_k}{q(t_k-\tau)}\, z(t_k-\tau) + q_k z(t_k-\sigma)$$

$$= (\alpha\,\lambda_0)^2\, p(t_k-\sigma)\;\frac{q_k}{q(t_k-\tau)}\,\omega(t_k-\tau)\, e^{-\alpha\lambda_0(t_k-\tau)} + q_k\omega(t_k-\sigma)\, e^{-\alpha\lambda_0(t_k-\sigma)}$$

$$\ge \left[(\alpha\,\lambda_0)^2\, p(t_k-\sigma)\;\frac{q_k}{q(t_k-\tau)} e^{\alpha\lambda_0\tau} + q(t)\, e^{\alpha\lambda_0\sigma}\right]\; z(t_k),$$

which implies that

$$\begin{cases} \inf\limits_{t \ge T} \left\{(\alpha\,\lambda_0)^2\, P(t-\sigma)\;\frac{q(t)}{q(t-\tau)} e^{\alpha\,\lambda_0\tau} + q(t)\, e^{\alpha\lambda_0\sigma}\right\} \le \lambda_0^2 \\ \inf\limits_{t_k \ge T} \left\{(\alpha\,\lambda_0)^2\, p(t_k-\sigma)\;\frac{q_k}{q(t_k-\tau)} e^{\alpha\lambda_0\tau} + q_k e^{\alpha\lambda_0\sigma}\right\} \ge \lambda_0^2. \end{cases}$$

Letting $\alpha \to 1$, we have

$$\begin{cases} \inf\limits_{t\geq T}\left\{\lambda_0^2 p\left(t-\sigma\right)\frac{q(t)}{q(t-\tau)}e^{\lambda_0\tau}+q\left(t\right)e^{\lambda_0\sigma}\right\}\leq\lambda_0^2 \\ \inf\limits_{t_k\geq T}\left\{\lambda_0^2 p\left(t_k-\sigma\right)\frac{q_k}{q(t_k-\tau)}e^{\lambda_0\tau}+q_k e^{\lambda_0\sigma}\right\}\leq\lambda_0^2 \end{cases}$$

which contradicts condition (250). This completes the proof of Theorem 4.17.

Remark 4.5. Noting that $e^y \geq 1, \quad e^y \geq \frac{e^2}{4}y^2$, for $y \geq 0$, condition (250) can be replaced by condition

$$\lim_{t\to\infty}\inf\left\{p\left(t-\sigma\right)\frac{q\left(t\right)}{q\left(t-\tau\right)}+\frac{e^2}{4}\sigma^2 q\left(t\right)\right\}>1$$

$$\lim_{t_k\to\infty}\inf\left\{p\left(t_k-\sigma\right)\frac{q_k}{q\left(t_k-\tau\right)}+\frac{e^2}{4}\sigma^2 q_k\right\}>1.$$

Remark 4.6. In the case where $p(t)\equiv p, \quad q(t)\equiv q$ are constants, condition (250) is also a necessary condition for the bounded oscillation of equation (249).

Theorem 4.18. *Assume that* $p\left(t\right)\leq 0, \quad q\left(t\right)>0, \quad \sigma>\tau$,

$$\begin{cases} \lim\limits_{t\to\infty}\sup\left\{-p\left(t-\sigma\right)\frac{q(t)}{q(t-\tau)}\right\}=\alpha\in\left(0,\,\infty\right) \\ \lim\limits_{t_k\to\infty}\sup\left\{-p\left(t_k-\sigma\right)\frac{q_k}{q(t_k-\tau)}\right\}=\alpha\in\left(0,\,\infty\right) \end{cases} \tag{260}$$

and

$$\lim_{t\to\infty}\sup\left[\int_{t-(\sigma-\tau)}^{t}\left(s-t+\left(\sigma-\tau\right)\right)q\left(s\right)ds + \sum_{t-(\sigma-\tau)\leq t_k\leq t}\left(t_k-t+\left(\sigma-\tau\right)\right)q_k\right]>1-\alpha. \tag{261}$$

Then every bounded solution of equation (249) *is oscillatory.*

Proof: Let $y(t)$ be a bounded, finally positive solution of equation (249) with $y\left(t-\tau\right)\geq 0, \;$ for $\; t\geq t_1$. Then $z''\left(t\right), \Delta z'\left(t_k\right)\geq 0, \quad z'\left(t\right), \Delta z\left(t_k\right)<0, \quad z\left(t\right)>$

$0,\ t \geq t_1$. From condition (261), there exists a constant $h > 1$ such that

$$\limsup_{t\to\infty} \left[\int_{t-(\sigma-\tau)}^{t} (s - t + (\sigma - \tau))\, q(s)\, ds \right.$$
$$\left. + \sum_{t-(\sigma-\tau)\leq t_k \leq t} (t_k - t + (\sigma - \tau))\, q_k \right] > 1 - h_\alpha \quad (262)$$

and there is a $t_2 > t_1$ such that

$$\begin{cases} -p(t-\tau)\, \frac{q(t)}{q(t-\tau)} \leq h\,\alpha, \ t \geq t_2, \ t \notin S \\ -p(t_k-\tau)\, \frac{q_k}{q(t_k-\tau)} \leq h\,\alpha, \ t_k \geq t_2, \ \forall\, t_k \in S. \end{cases} \quad (263)$$

From equation (259), we have

$$\begin{cases} z''(t) - h\,\alpha\, z''(t-\sigma) \geq q(t)\, z(t-\sigma), \ t \geq t_2, \ t \notin S \\ \Delta z'(t_k) - h\,\alpha\, \Delta z'(t_k - \sigma) \geq q_k z(t_k - \sigma), \ t_k \geq t_2, \ \forall\, t_k \in S. \end{cases}$$

We set

$$\omega(t) = z(t) - h\,\alpha\, z(t-\tau).$$

Then

$$\begin{cases} \omega''(t) \geq q(t)\, z(t-\sigma), \ t \geq t_2, \ t \notin S \\ \Delta\omega'(t_k) \geq q_k z(t_k - \sigma), \ t_k \geq t_2, \ \forall\, t_k \in S, \end{cases} \quad (264)$$

and it is immediately obvious that $\omega(t) > 0, \ \omega'(t),\ \Delta\omega(t_k) \leq 0, \quad t,\ t_k \geq t_2$. By the monotone property of $z(t)$, we have

$$\omega(t) = z(t) - h\,\alpha\, z(t-\tau) \leq (1 - h\,\alpha)\, z(t-\tau), \ t \geq t_2$$

or

$$z(t) \geq 1\, \frac{1}{1 - h\,\alpha}\, \omega(t+\tau), \ t \geq t_2.$$

Substituting this into inequality (264), we obtain

$$\begin{cases} \omega''(t) \geq \frac{1}{1-h\alpha} q(t)\, \omega(t-(\sigma-\tau)), \ \ t \geq t_2+\sigma, \ \ t \notin S \\ \Delta\omega'(t_k) \geq \frac{1}{1-h\alpha} q_k \omega(t_k-(\sigma-\tau)), \ \ t_k \geq t_2+\sigma, \ \ \forall\, t_k \in S. \end{cases} \tag{265}$$

Integrating inequality (265) from s to t for $t \geq s$, we have

$$\omega'(t) - \omega'(s) \geq \frac{1}{1-h\alpha}\left[\int_s^t q(u)\, \omega(u-(\sigma-\tau))\, du + \sum_{s \leq t_k \leq t} q_k \omega(t_k-(\sigma-\tau))\right], \ \ s \geq t_2.$$

Integrating the resulting inequality again in s from $t-(\sigma-\tau)$ to t, we have

$$\omega'(t)(\sigma-\tau) - \omega(t) + \omega(t-(\sigma-\tau))$$

$$\geq \frac{1}{1-h\alpha}\left[\int_{t-(\sigma-\tau)}^t (u-t(\sigma-\tau))\, q(u)\, \omega(u-(\sigma-\tau))\, du + \sum_{t-(\sigma-\tau) \leq t_k \leq t} (t_k - t + (\sigma-\tau))\, q_k \omega(t_k-(\sigma-\tau))\right]$$

$$\geq \frac{\omega(t-(\sigma-\tau))}{1-h\alpha}\left[\int_{t-(\sigma-\tau)}^t (u-t+(\sigma-\tau))\, q(u)\, du + \sum_{t-(\sigma-\tau) \leq t_k \leq t} (t_k - t + (\sigma-\tau))\, q_k\right], \ \ t, t_k \geq t_2.$$

Thus,

$$\omega(t) + \omega(t-(\sigma-\tau))\left(-1 + \frac{1}{1-h\alpha}\left[\int_{t-(\sigma-\tau)}^t (u-t+(\sigma-\tau))\, q(u)\, du + \sum_{t-(\sigma-\tau) \leq t_k \leq t} (t_k - t + (\sigma-\tau))\, q_k\right]\right) \leq 0$$

which implies that

$$\int_{t-(\sigma-\tau)}^{t} (u = t + (\sigma - \tau))\, q(u)\, du + \sum_{t-(\sigma-\tau)\le t_k \le t} (t_k - t + (\sigma - \tau))\, q_k \le 1 - h\,\alpha, \quad t \ge t_2.$$

We reach a contradiction and thus, complete the proof of Theorem 4.18.

4.5 Forced oscillation

Consider the second order neutral impulsive differential equation with a forcing term

$$\begin{cases} [y(t) + p\,y(t-\tau)]'' + f(t, y(t-\sigma)) = Q(t), \\ \qquad\qquad t \ge t_0, \ t \notin S \\ \Delta [y(t_k) + p\,y(t_k - \tau)]' + f_k(t_k, y(t_k - \sigma)) = Q(t_k), \\ \qquad\qquad t_k \ge t_0, \ \forall\, t_k \in S. \end{cases} \tag{266}$$

We introduce the following conditions:

H4.5.1: $p, \ \tau > 0$ and $\sigma \ge 0$;

H4.5.2: $f, f_k \in C([t_0, \infty) \times R, R), \ y \cdot f(t, y), \ y \cdot f_k(t_k, y) > 0, \ y \ne 0$;

H4.5.3: there exists a function $u(t) \in C^2([t_0, \infty), R)$ such that

$$\begin{cases} Q(t) = u''(t) \\ Q(t_k) = \Delta u'(t_k) \end{cases}$$

and u changes sign on $[T, \infty)$ for any $T \ge t_0$.

Set

$$u_*(t) = \min\left\{\frac{\alpha\,(t-\tau)}{2}, \ \frac{2\,(t)}{2p}\right\},$$

$$u_*^+(t) = \max\{u_*(t), 0\}, \quad u_*^-(t) = \max\{-u_*(t), 0\}.$$

Lemma 4.8. Assume $x \in PC([t_0,\infty), R)$, $\beta \in PC([t_0,\infty), R_+)$ and $x(t) + px(t-\tau) \geq \beta(t) \geq 0$, $t \geq t_0$, where $p,\ \tau > 0$. Then for each $t^* \geq t_0 + \tau$, there exists a set $A = \{t : t^* \leq t \leq t^* + 2\tau,\ x(t-\tau) \geq \beta_*(t)\}$ with the measure $mes(A) \geq \tau$, where

$$\beta_*(t) = \min\left\{\frac{\beta(t-\tau)}{2},\ \frac{\beta(t)}{2p}\right\}.$$

Proof: For any fixed $t^* \geq t_0 + \tau$, we define a set

$$B = \left\{t : t \in [t*, t*+\tau],\ x(t) > \frac{\beta(t)}{2}\right\}.$$

If B is empty $(B = \Phi)$, then $px(t-\tau) \geq \frac{\beta(t)}{2}$, for $t \in [t*, t*+\tau]$, that is, $A = [t*, t*+\tau]$. Now we consider the case that $B = \Phi$, then $mes\ (B) = \alpha \in (0,\ \tau)$. Let $\bar{B}$ denote the closure of B. In view of the piece-wise continuity of x, we have $x(t) \geq \frac{\beta(t)}{2}$, $t \in \bar{B}$. Define a set $\bar{B} + \tau = \left\{t, t-\tau \in \bar{B}\right\}$. Then, $x(t-\tau) \geq \frac{\beta(t-\tau)}{2}$ for $t \in \left(\bar{B} + \tau\right)$.
Set

$$A = \{[t*, t*+\tau] \backslash B\}\ U\ \left(\bar{B} +\ \tau\right).$$

Then $mes\ (A) = \tau$ and $x(t-\tau) \geq \beta_*(\tau)$ on A. This completes the proof of Lemma 4.8.

Theorem 4.19. *Assume that conditions H4.5.1 - H4.5.3 hold. Further assume that f and f_k are non-decreasing in x and*

$$\begin{aligned} &\textstyle\int_E f\left(t, u_*^+(t+\tau-\sigma)\right) dt + \sum_{E \leq t_k} f_k\left(t_k, -u_*^+(t_k+\tau-\sigma)\right) = \infty, \\ &\textstyle\int_E f\left(t, -u_*^-(t+\tau-\sigma)\right) dt + \sum_{E \leq t_k} f_k\left(t_k, -u_*^-(t+\tau-\sigma)\right) = -\infty \end{aligned} \qquad (267)$$

for every closed set E whose intersection with every segment of the form $[t-\tau,\ t+\tau]$, $t \geq t_0 + \tau$, has a measure not smaller then τ. Then every solution of equation (266) is oscillatory.

Proof: Without loss of generality, let us assume by contradiction, that $y(t)$ is a finally positive solution of equation (266) for $t \geq t_0$. Set

$$z(t) = y(t) + py(t-\tau).$$

Then $(z(t) + u(t))'' < 0 \;\; for \;\; t \geq t_0 + \tau$. It is easy to show that $(z(t) - u(t))' > 0$ finally, which implies that

$$\int_{t_0}^{\infty} f(t, y(t-\sigma))\, dt + \sum_{t_0 \leq t_k < \infty} f_k(t_k, y(t_k - \sigma)) < \infty. \tag{268}$$

On the other hand, it is easy to show that $z(t) - u(t) > 0$ finally. Then we have

$$z(t) = y(t) + py(t-\tau) \geq u^{+}(t), \;\; t \geq t_0 + \tau.$$

By Lemma 4.8, for every $t^* \geq t_0 + 2\tau$, there exists a set $A = \{t : t^* \leq t \leq t^* + 2\tau,\; y(t-\tau) \geq u_*^{+}(t)\}$ with $mes\,(A) \geq \tau$. Let us consider the set $A - (\tau - \sigma) = \{t : t + (\tau - \sigma) \in A\}$. It is obvious that $mes\,(A - (\tau - \sigma)) \geq \tau$ and $y(t-\sigma) \geq u_*^{+}(t + (\tau - \sigma)), \;\; t \in (A - (\tau - \sigma))$. From condition (268), we have

$$\infty > \int_E f(t, y(t-\sigma))\; dt + \sum_{E \leq t_k} f_k(t_k, y(t_k - \sigma))$$

$$\geq \int_E f(t), u_*^{+}(t + (\tau - \sigma))\; dt + \sum_{E \leq t_k} f_k\left(t_k, u_*^{+}(t_k + (\tau - \sigma))\right)$$

which contradicts assumption (267). This completes the proof of Theorem 4.19.

The following result is for equations of the more general form

$$\begin{cases} [y(t)+p(t)y(t-\tau)]''+f(t,y(g(t)),y'(\sigma(t)))=Q(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\notin S \\ \Delta[y(t_k)+p_k y(t_k-\tau)]'+f_k(t_k,y(g(t_k)),\ \Delta y(\sigma(t_k)))=Q(t_k), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, t_k\in S \end{cases} \tag{269}$$

Theorem 4.20. *Assume that*

i) $p\in PC([t_0,\infty),R_+)$ *and* $p_k\geq 0,\ \ t_k\geq t_0;$

ii) $g,\sigma\in C([t_0,\infty),R)$ *and* $Q\in C([t_0,\infty),R),\ \ t\geq t_0;.$

iii) g *is non-decreasing and* $\lim\limits_{t\to\infty} g(t)=\infty.$

iv) $f\in C([t_0,\infty)xR^2,R)$ *and* $f(t,u,\nu)\,u>0,\ \ u\neq 0;$

v) for any $T\geq t_0;$

$$\begin{cases} \lim\limits_{t\to\infty}\inf\left[\int_T^t Q(s)\,ds+\sum_{T\leq t_k<t}Q_k\right]=-\infty, \\ \qquad\qquad\qquad \lim\limits_{t\to\infty}\sup\left[\int_T^t Q(s)\,ds+\sum_{T\leq t_k<t}Q_k\right]=\infty, \\ \lim\limits_{t\to\infty}\inf\left[\int_T^t\int_T^t Q(u)\,duds+\sum_{T\leq t_k<t}\sum_{T\leq t_k<s}Q_k\right]=-\infty, \\ \qquad\qquad\qquad \lim\limits_{t\to\infty}\sup\left[\int_F^t\int_T^s Q(u)\,duds \right.\\ \qquad\qquad\qquad\qquad \left.+\sum_{T\leq t_0<t}\sum_{T\leq t_k<s}Q_k\right]=\infty. \end{cases} \tag{270}$$

Then every solution of equation (269) *is oscillatory.*

Proof: Without loss of generality, let us assume by contradiction that $y(t)$ is a finally positive solution of equation (269). Set $z(t)=y(t)+p(t)y(t-\tau)$. Then $z(t)>0,\ \ t\geq T\geq t_0$. From equation (269), we discover that $z''(t)<Q(t)$. Thus,

$$z'(t)-z'(T)<\int_T^t Q(s)\,ds+\sum_{T\leq t_k<t}Q_k. \tag{271}$$

Theorem 4.21. *Assume that*

i) $p > 0,\ \ \sigma > \tau > 0,\ \ q_k \geq 0,\ \ \alpha,\ \beta \in (0,\ 1],\ \ q(t) \geq 0,\ \ t \geq t_0;$

ii) either

$$\limsup_{t\to\infty} \left[\int_{t-(\omega-\tau)}^{t} [u - (\sigma - \tau)]\, q(u)\, du + \sum_{t-(\sigma-\tau)\leq t_k < t} (t_k - (\sigma - \tau))\, q_k \right] > 0, \quad \beta < \alpha \tag{274}$$

or

$$\limsup_{t\to\infty} \left[\int_{t-(\sigma-\tau)}^{t} [u - (t - (\sigma - \tau))]\, q(u)\, du + \sum_{t-(\sigma-\tau)\leq t_k < t} [t_k - (t - (\sigma - \tau))]\, q_k \right] > p, \quad \beta = \alpha, \tag{275}$$

where $p \in (0,\ 1)$ *for* $\alpha = 1,\ \ p \in (0,\ \infty)$ *for* $\alpha \in (0,\ 1);$

iii) every solution of the second order impulsive differential equation

$$\begin{cases} z''(t) + \lambda q(t) \left(\frac{t-\sigma}{t}\right)^{\beta} z^{\beta}(t) = 0 \\ \Delta z'(t_k) + \lambda q_k \left(\frac{t_k-\sigma}{t_k}\right)^{\beta} z^{\beta}(t_k) = 0 \end{cases} \tag{276}$$

is oscillatory, where $0 < \lambda < 1$ *is a constant. Then every solution of equation* (273) *is oscillatory.*

Proof: Without loss of generality, let us assume by contradiction that $y(t)$ is a finally positive solution of equation (273) and define

$$z(t) = y(t) - p y^{\alpha}(t - \tau). \tag{277}$$

From equation (273), we know that $z''(t) \leq 0$. If $z'(t) < 0$ finally, then $\lim_{t\to\infty} z(t)\ \xi_n = \infty$. Thus, $\lim_{t\to\infty} y(t) = \infty$ and there exists a sequence $\{\xi_n\}$ such that $\lim_{n\to\infty} \xi_n = \infty$ and $y(\xi_n) = \max_{t_0\leq t\leq t_n} y(t) \to \infty$
$as\ \ n \to \infty$. Hence,

$$z(\xi_n) = y(\xi_n) - p\,y^{\alpha}(\xi_n - \tau) \geq y(\xi_n) - p\,y^{\alpha}(\xi_n)$$

$$= y(\xi_n)\left[1 - p\,y^{\alpha-1}(\xi_n)\right] \to \infty, \quad n \to \infty,$$

is a contradiction. Therefore, $z'(t) > 0$. If $z(t) < 0$, then $z(t) > -p\,y^{\alpha}(t-\tau)$. Then

$$y(t-\tau) > \left(-\frac{z(t)}{p}\right)^{\frac{1}{\alpha}}. \tag{278}$$

Substituting condition (278) into equation (273), we have

$$\begin{cases} z''(t) - q(t)\left(\frac{z(t-(\sigma-\tau))}{p}\right)^{\beta/\alpha} \leq 0, \ \ t \notin S \\ \Delta z'(t_k - q_k)\left(\frac{z(t_k-(\sigma-\tau))}{p}\right)^{\beta/\alpha} \leq 0, \ \ \forall\, t_k \in S. \end{cases} \tag{279}$$

As in the proof of Theorem 3.10, inequality (279) cannot have a finally negative solution under the given assumptions. This contradiction shows that $z(t) > 0$. By equation (133) of the proof of Theorem 3.13, for each $h \in (0,\ 1)$, there is a $t_h \geq t_0$ such that

$$z(t-\sigma) \geq h\frac{t-\sigma}{t}z(t), \ \text{ for } \ t \geq t_h. \tag{280}$$

Substituting inequality (280) into equation (273), we have

$$\begin{cases} z''(t) + h^{\beta}\left(\frac{t-\sigma}{t}\right)^{\beta} q_k z^{\beta}(t) \leq 0, \ \ t \notin S \\ \Delta z'(t_k) + h^{\beta}\left(\frac{t_k-\sigma}{t_k}\right)^{\beta} q_k z^{\beta}(t_k) \leq 0, \ \forall\, t_k \in S \end{cases} \tag{281}$$

which implies that equation (276) has a non-oscillatory solution, contradicting the assumptions of condition (iii). This completes the proof of Theorem 4.21.

Now we consider the unstable type of equation (273). For the sake of convenience, we write $Q(t) \equiv -q(t) \geq 0,\ \ t \geq t_0$.

Theorem 4.22. *Assume that*

i) $p,\ \tau,\ \sigma > 0,\ \ Q_k \geq 0,$ *and* $\beta \in (0,\ 1],\ \ Q(t) \geq 0,\ \ t \geq t_0$*;*

ii) the inequality

$$\limsup_{t\to\infty} \left[\int_{t-\sigma}^{t} (s-(t-\sigma))\, Q(s)\, ds + \sum_{t-\sigma\leq t_k<t} (t_k-(t-\sigma))\, Q_k\right] > 1. \qquad (282)$$

holds.

Then every bounded solution of equation (273) *is oscillatory.*

Proof: Let us assume, by contradiction, that $y(t)$ is a finally positive solution of equation (273). Then, $z''(t) \geq 0$. By the boundedness of z, we have $z'(t) < 0$ finally. If $z(t) > 0$ finally, integrating equation (273) twice, we have

$$\begin{aligned} z'(t)\gamma - z(t) + z(t-\sigma) &= \int_{t-\sigma}^{t}\int_{s}^{t} Q(u)\, y^{\beta}(u-\sigma)\, duds \\ &\quad + \sum_{t-\sigma\leq t_k<t}\ \sum_{s\leq t_k<t} Q_k y^{\beta}(t_k-\sigma) \end{aligned}$$

$$= \int_{t-\sigma}^{t} [u-(t-\sigma)]\, Q(u)\, y^{\beta}(u=\sigma)\, du + \sum_{t-\sigma\leq t_k<t} [t_k-(t-\sigma)]\, Q_k y^{\beta}(t_k-\sigma)$$

$$\geq z^{\beta}(t-\sigma)\left[\int_{t-\sigma}^{t} [u-(t-\sigma)]\, Q(u)\, du + \sum_{t-\sigma\leq t_k<t} [t_k-(t-\sigma)]\, Q_k\right]. \quad (283)$$

It is immediately seen that $\lim_{t\to\infty} z(t) = 0$. Hence there exists a $T \geq t_0$ such that

$z(t-\sigma) < 1$, for $t \geq T$. Thus, statement (283) leads to

$$z(t)+z(t-\sigma)\left[\int_{t-\sigma}^{t}[u-(t-\sigma)]\,Q(u)\,du+\sum_{t-\sigma\leq t_k<t}[t_k-(t-\sigma)]\,Q_k-1\right]\leq 0$$

which contradicts condition (282).

If $z(t) < 0$, then $z(t) \leq -d < 0$ for some $d > 0$. Hence $-p\,y^{\alpha}(t-\tau) \leq -d$, or $y^{\alpha}(t-\tau) \geq \frac{d}{p} > 0$. From equation (273), we get

$$\begin{cases} z''(t) \geq \left(\frac{d}{p}\right)^{\beta/\alpha} Q(t), & t \notin S \\ \Delta z'(t_k) \geq \left(\frac{d}{p}\right)^{\beta/\alpha} Q_k, & \forall\, t_k \in S. \end{cases} \tag{284}$$

We note from condition (282) that

$$\int_{T}^{\infty} t\,Q(t)\,dt + \sum_{T\leq t_k<\infty} t_k Q_k = \infty. \tag{285}$$

Hence inequality (284) implies that $\lim_{t\to\infty} z(t) = \infty$, which is a contradiction. This completes the proof of Theorem 4.22.

Theorem 4.23. *Assume that*

i) $p, \tau > 0, \quad \sigma \geq 0, \quad \alpha \geq 1, \quad \beta > 0, \quad Q_k \geq 0, \quad Q(t) \geq 0, \quad t \geq t_0;$

ii) There exists a constant $\lambda > 0$ *such that*

$$\alpha\,p\,\exp\{\lambda\,\alpha\,\tau + \lambda\,t\,(1-\alpha)\} \leq L < 1 \tag{286}$$

and

$$\begin{aligned} p\,\exp\{\lambda\,\alpha\,\tau+\lambda\,t\,(1-\varepsilon)\} \;+\; &\int_{t}^{\infty}(s-t)\,Q(t)\exp\{\lambda\,(t-\beta\,(s-\sigma))\}\,ds \\ &+\sum_{t\leq t_k<\infty}(t_k-t)\,Q_k\,\times \\ &\times\,\exp\{\lambda\,(t-\beta\,(t_k-\sigma))\}\leq 1 \end{aligned} \tag{287}$$

hold finally. Then equation (273) has a positive solution $y(t)$ *which converges to*

We introduce the following conditions:

H4.7.1: $\tau \in C([t_0, \infty), R)$, τ is a non-decreasing function in R_+, $\tau(t) \geq t$ for $t \in R_+$ and $\lim_{t \to t\infty} \tau(t) = +\infty$;

H4.7.2: $r \in PC^1([t_0, \infty), R_+)$ and $r(t) > 0, \quad r\left(t_k^+\right) > 0$, for $t, t_k \in R_+$;

H4.7.3: $q \in PC([t_0, \infty), R_+)$ and $q_k \geq 0, \ \ k \in N$;

H4.7.4: $\int_0^\infty \frac{dt}{r(t)} = \infty$.

Theorem 4.24. *Assume that*

$$\int_{t_0}^{\infty} q(t)\, dt + \sum_{t_0 \leq t_k < \infty} q_k = \infty. \tag{289}$$

Then every solution of equation (288) *is oscillatory.*

Proof: Let us assume, by contradiction, that $y(t)$ is a finally positive solution of equation (288). It is easy to show that $r(t)\, y'(t) > 0$ for $t \geq T \geq t_0$. Then

$$\int_{T}^{\infty} q(t)\, y(\tau(t))\, dt + \sum_{T \leq t_k < \infty} q_k y(\tau(t_k)) < \infty \tag{290}$$

which contradicts equation (289). This completes the proof of Theorem 4.24.

In what follows, we want to derive some oscillation criteria for equation (288) when

$$\int_{t_0}^{\infty} q(t)\, dt + \sum_{t_0 \leq t_k < \infty} q_k < \infty. \tag{291}$$

Lemma 4.8. Let $y(t) > 0, \ \ t \geq t_1$, be a solution of equation (288). Set

$$\omega(t) = \frac{r(t)\, y'(t)}{y(t)}. \tag{292}$$

Integrating equation (295) from t to T for $T \geq t \geq t_1$, we have

$$\omega(T) - \omega(t) + \int_t^T \frac{\omega^2(s)}{r(s)} ds + \sum_{t \leq t_k < T} \frac{\omega^2(t_k)}{r(t_k)} + \int_t^T q(s) \times$$
$$\times \exp\left(\int_s^{\tau(s)} \frac{\omega(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)}\right) ds$$

$$+ \sum_{t \leq t_k < T} q_k \exp\left(\int_s^{\tau(s)} \frac{\omega(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)}\right) = 0. \tag{296}$$

Because $r(t)\,y'(t) > 0$, so $\omega(t) > 0$. We shall show that $\lim_{t\to\infty} \omega(t) = 0$. In fact, if $\lim_{t\to\infty} r(t)\,y'(t) = c > 0$, then there exists a $t_2 \geq t_1$ such that for $t \geq t_2$,

$$y(t) \geq \left[y(t_2) + \int_{t_2}^t \frac{c}{2r(s)} ds + \sum_{t_2 \leq t_k < \tau} \frac{c}{2r(t_k)}\right] \to \infty, \quad t \to \infty,$$

and hence, $\lim_{t\to\infty} \omega(t) = 0$. If $\lim_{t\to\infty} r(t)\,y'(t) = 0$, then $\lim_{t\to\infty} \omega(t) = 0$ also. Letting $T \to \infty$, in equation (296), we obtain condition (294). This completes the proof of Lemma 4.8.

Lemma 4.9. Equation (288) has a non-oscillatory solution if and only if there exists a positive differential function $\phi(t)$ such that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \leq -q(t) \exp\left(\int_t^{\tau(t)} \frac{\omega(s)}{r(s)} ds + \sum_{t \leq t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)}\right), \quad t \geq t_2. \tag{297}$$

Proof: The necessity follows from Lemma 4.8. Now we assume that inequality (297) holds. Then, $\phi'(t) < 0$ and hence $\lim_{t\to\infty} \phi(t) = -\infty$, a contradiction. Therefore, $\lim_{t\to\infty} \phi(t) = 0$. Integrating inequality (297) from t to ∞, we obtain

$$\int_t^\infty \frac{\phi^2(s)}{r(s)}ds + \sum_{t\le t_k<\infty} \frac{\phi^2(t_k)}{r(t_k)} + \int_t^\infty q(s)\exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)}du + \sum_{s\le t_k<\tau(s)} \frac{\phi(t_k)}{r(t_k)}\right)$$
$$+ \sum_{t\le t_k<\infty} q_k \exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)}ds + \sum_{s\le t_k<\tau(s)} \frac{\phi(t_k)}{r(t_k)}\right) \le \phi(t), \quad t\ge t_2 \tag{298}$$

which implies that

$$\int_t^\infty \frac{\phi^2(s)}{r(s)}ds + \sum_{t\le t_k<\infty} \frac{\phi^2(t_k)}{r(t_k)} < \infty$$

and

$$\int_t^\infty q(s)\exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)}du + \sum_{s\le t_k<\tau(s)} \frac{\phi(t_k)}{r(t_k)}\right) ds$$
$$+ \sum_{t\le t_k<\infty} q_k \exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)}du + \sum_{s\le t_k<\tau(s)} \frac{\phi(t_k)}{r(t_k)}\right) < \infty.$$

For all functions $x(t)$ satisfying $0 \le x(t) \le \phi(t), \ t \ge t_2$, define a mapping J by

$$(Jx)(t) = \int_t^\infty \frac{x^2(s)}{r(s)}ds + \sum_{t\le t_k<\infty} \frac{x^2(t_k)}{r(t_k)} + \int_t^\infty q(s) \times$$
$$\times \exp\left(\int_s^{\tau(s)} \frac{x(u)}{r(u)}du + \sum_{s\le t_k<\tau(s)} \frac{x(t_k)}{r(t_k)}\right) ds$$
$$+ \sum_{t\le t_k<\infty} q_k \exp\left(\int_s^{\tau(s)} \frac{x(u)}{r(u)}du + \sum_{s\le t_k<\tau(s)} \frac{x(t_k)}{r(t_k)}\right), \quad t\ge t_2.$$

If is easy to see that $0 \le x_1(t) \le x_2(t), \ t \ge t_2$, implies $(Jx_1)(t) \le (Jx_2)(t), \ t \ge t_2$.

Define $y_0(t) \equiv 0$ and $y_n(t) = (Jy_{n-1})(t), \ n = 1, 2, \cdots$. Then $y_{n-1}(t) \le y_n(t) \le \phi(t), \ n = 1, 2, \cdots$, and $\lim_{n\to\infty} y_n(t) = \omega(t) \le \phi(t)$. By the Lebesgue

Dominated Convergence theorem, we have

$$\omega(t) = \int_t^{\infty} \frac{\omega^2(s)}{r(s)} ds + \sum_{t \le t_k < \infty} \frac{\omega^2(t_k)}{r(t_k)} + \int_t^{\infty} q(s) \times \\ \times \exp\left(\int_s^{\tau(s)} \frac{\omega(u)}{r(u)} du + \sum_{s \le t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)} \right) ds \\ + \sum_{t \le t_k < \infty} q_k \exp\left(\int_s^{\tau(s)} \frac{\omega(u)}{r(u)} du + \sum_{s \le t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)} \right), \quad t \ge t_2.$$

Set

$$y(t) = \exp\left(\int_{t_2}^{t} \frac{\omega(u)}{r(u)} du + \sum_{t_2 < t_k < t} \frac{\omega(t_k)}{r(t_k)} \right), \quad t \ge t_2.$$

Then

$$\omega(t) = \frac{r(t)\, y'(t)}{y(t)}$$

and

$$\begin{cases} (r(t)y'(t))' + q(t)y(\tau(t)) = 0, & t \ge t_2, \ t \notin S \\ \Delta(r(t_k)y'(t_k)) + q_k y(\tau(t_k)) = 0, & t_k \ge t_2, \ \forall\, t_k \in S, \end{cases}$$

That is, $y(t)$ is a non-oscillatory solution of equation (288). This completes the proof of Lemma 4.9.

Theorem 4.25. *If equation* (288) *has a non-oscillatory solution, then the second order linear impulsive differential equation*

$$\begin{cases} (r(t)y'(t))' + q(t)y(t) = 0, & t \notin S \\ \Delta(r(t_k)y'(t_k)) + q_k y(t_k) = 0, & \forall\, t_k \in S \end{cases} \tag{299}$$

is non-oscillatory. Conversely, if equation (299) *is oscillatory, then every solution of equation* (288) *is oscillatory.*

Proof: Assume that equation (288) has a non-oscillatory solution. By Lemma 4.9, there exists a positive differential function $\phi(t)$ such that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \leq -q(t)\exp\left(\int_t^{\tau(t)} \frac{\phi(u)}{r(u)}du + \sum_{t\leq t_k<\tau(t)} \frac{\phi(t_k)}{r(t_k)}\right), \quad t \geq t_2, \quad (300)$$

which implies that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \leq -q(t). \quad (301)$$

Taking advantage of Lemma 4.9 for the case in which $h(t) \equiv t$, equation (299) is non-oscillatory.

Consequently, the second part of the theorem is immediately obtained. This completes the proof of Theorem 4.25.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Although this research project focuses on the oscillations of neutral impulsive differential equations, we never lost sight of the fundamental problem of the theory of oscillations in our discussion. We recall that these problems include, but are not limited to, proving the existence of, and, where possible, the actual determination of oscillatory motions that are solutions of a given impulsive differential equation, and the study of the behaviour of the other solutions in relation to the given oscillations. To effectively handle this, we made a distinction between the theory of linear oscillations and that of nonlinear oscillations/non-oscillation.

In the theory of nonlinear oscillations, we considered the general nonlinear neutral delay impulsive differential equation of the form:

$$\begin{cases} [y(t) - py(t-\tau)]'' + q(t)f(y(t-\sigma(t)) = 0, & t \notin S \\ \Delta[y(t_k) - py(t_k-\tau)]' + q_k f_k(y(t_k - \sigma(t_k)) = 0, & \forall\ t_k \in S \end{cases} \tag{302}$$

under the following assumptions:

H5.1.1: p, τ and q_k are positive numbers, $\forall\ k \in Z$;

H5.1.2: $q,\ \sigma \in C(R_+, R_+),\ \lim_{t\to\infty}(t - \sigma(t)) = \infty, \quad \sigma(t) > \tau$;

H5.1.3: $f \in C(R, R)$, f is increasing and $f(-y) = -f(y)$;

H5.1.4: $f(y \cdot x) \geq f(y) f(x)$ when $y \cdot x > 0$, $f(\infty) = \infty$;

H5.1.5: $f_k(y \cdot x) \geq f_k(y)\ f_k(x)$ when $y \cdot x > 0$, $\quad f_k(\infty) = \infty, \forall\ k \in Z,$

H5.1.6: $\lim_{x\to 0}\left[\frac{f(x)}{x},\ \frac{f_k}{x}\right] = \infty$ or $\lim_{x\to 0}\left[\frac{f(x)}{x},\ \frac{f_k}{x}\right] = 1.$

We also considered the generalized form of the second order nonlinear neutral impulsive differential equation

$$\begin{cases} \left[y(t) - \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i)\right]'' \\ \qquad\qquad + \sum_{j=1}^{n} f_j\left(t,\, y\left(g_{j1}(t)\right),\, \cdots,\, y\left(g_{jl}\right)\right) \\ \qquad\qquad = 0,\ t \geq t_0 \in R_+,\ \ t \notin S \\ \Delta\left[y(t_k) - \sum_{i=1}^{m} p_{ik} y(t_k-\tau_i)\right]' \\ \qquad\qquad + \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{j1}(t_k)\right),\, \cdots,\, y\left(g_{jl}(t_k)\right)\right) \\ \qquad\qquad = 0,\ t_k \geq t_0 \in R_+,\ \forall\ t_k \in S \end{cases} \tag{303}$$

subject to the following conditions:

H5.1.7: $\tau_i > 0,\ p_{ik} \geq 0,\ \ p_i \in PC^1\left([t_0,\infty),\, R_+\right),\ \ i = 1,\, 2,\, \cdots,\, m$ and there exists $\delta \in (0,1]$ such that

$$\sum_{i=1}^{m} p_i(t) + \sum_{j=1}^{n} p_j \leq\ 1-\delta,\ \ t,\ \geq\ t_0 \in R_+;$$

H5.1.8: $g_{js}\ \in\ C\left([t_0,\infty),\, R\right),\ \lim_{t\to\infty}\ g_{js}(t)\ =\ \infty,\ j\ =\ 1,\ 2,\ \cdots,\ n,\ s\ = 1,\ 2,\ \cdots,\ \ell;$

H5.1.9: $f_j \in PC\left([t_0,\infty)\times R^{\ell}, R\right),\ \ x_1 f_j(t, x_1, \cdots, x_\ell) > 0; x_1 f_{jk}(t_k, x_1 \cdots, x_l) > 0$ for $x_1 x_i > 0,\ i = 1,\, 2,\, \cdots,\, \ell,\ \ j = 1,\, 2,\, \cdots,\, n$. Moreover,

$$\begin{cases} \left|f_j\ (t, y_1,\ \cdots,\ y_\ell)\right|\ \geq\ \left|f_j\ (t, x_1,\ \cdots,\ x_\ell)\right| \\ \left|f_{jk}\ (t_k, y_1\ \cdots,\ y_\ell)\right|\ \geq\ \left|f_{jk}\ (t_k, x_1,\ \cdots,\ x_\ell)\right| \end{cases}$$

whenever

$$|x_i| \leq |y_i|\ \text{ and }\ y_i x_i > 0,\ \ i = 1,\, 2,\, \cdots,\, \ell,\ \ j = 1,\, 2,\, \cdots,\, n;$$

H5.1.10: Set

$$x(t) = y(t)\ -\ \sum_{i=1}^{m} p_i(t)\ y\ (t-\tau_i)\,.$$

As a major achievement, we were able to establish conditions for the oscillation of all solutions of equation (302). By way of these conditions, the oscillation problem for neutral impulsive differential equation(302) was reduced to the same problem for the corresponding delay impulsive differential equations and, as the case was, to the corresponding impulsive ordinary differential equation. For the second order nonlinear neutral differential equation (303), we were able to introduce the classification of its non-oscillatory solutions and to establish various existence results of non-oscillatory solutions of different types.

In conclusion, we have observed that every solution of equation (302) oscillates if and only if the solutions of the corresponding delay impulsive differential equations are oscillatory.

Finally we discussed and developed certain theorems that helped us to:

i) Arrive at the conclusion that the solutions of nonlinear impulsive differential equations are either all oscillatory or all non-oscillatory;

ii) Establish some existence results for each kind of non-oscillatory solution of equation (303);

iii) Find the relation between oscillation/non-oscillation and other qualitative properties such as boundedness and convergence of solutions to zero;

iv) Obtain conditions for the oscillation of all solutions of nonlinear equations with a forcing term of the form

$$\begin{cases} [y(t) + py(t-\tau)]'' + f(t, y(t-\sigma)) = Q(t), \\ \qquad\qquad\qquad\qquad t \geq t_0, \ t \notin S \\ \Delta [y(t_k) + py(t_k-\tau)]' + f_k(t_k, y(t_k-\sigma)) = Q(t_k), \\ \qquad\qquad\qquad\qquad t_k \geq t_0, \ \forall\, t_k \in S. \end{cases} \tag{304}$$

subject to the following conditions:

H5.1.11: $p, \ \tau > 0$ and $\sigma \geq 0$;

H5.1.12: $f, f_k \in C([t_0, \infty) \times R, R), \ y \cdot f(t, y), \ y \cdot f_k(t_k, y) > 0, \ y \neq 0$;

at the conclusions that every oscillation criterion for the second order impulsive differential equation (167) became an oscillation criterion for the second order neutral impulsive differential equation (306), and also that,
for the linear impulsive differential equation (306), solutions are either all oscillatory or all non-oscillatory.

As a mark of achievement, we were able to

i) Establish the criteria for the existence of oscillation or non-oscillation of all solutions.

ii) Find the relation between oscillation and boundedness of all solutions.

iii) Obtain conditions such that an impulsive differential equation has an oscillatory or non-oscillatory solution with some asymptotic property.

iv) Establish conditions for oscillation of all bounded solutions of unstable type second order linear neutral impulsive differential equation of the form

$$\begin{cases} [y(t) - py(t-\tau)]'' = q(t)\, y(g(t)), \ t \geq t_0, \ t \notin S \\ \Delta [y(t_k) - py(t_k - \tau)]' = q_k y(g(t_k)), \ t_k \geq t_0, \ \forall\, t_k \in S, \end{cases} \tag{307}$$

where $p \in R, \ q_k \geq 0, \ q \in PC([t_0, \infty), R_+), \ g \in C([t_0, \infty), R); \ \lim_{t\to\infty} g(t) = \infty, \ \tau > 0$ for the cases p constant and p variable.

v) Establish conditions for existence of bounded positive solutions of equation (307).

vi) Establish conditions for the existence of asymptotically decaying positive solutions of linear equation (307).

vii) Obtain conditions for oscillation of all solutions of impulsive differential equations with advanced argument of the form

$$\begin{cases} (r(t)\, y'(t))' + q(t)\, y(\tau(t)) = 0, \ t \notin S \\ \Delta (r(t_k)\, y'(t_k)) + q_k y(\tau(t_k)) = 0, \ \forall\, t_k \in S, \end{cases}$$

where $\Delta(r(t_k)y'(t_k)) = r\left(t_k^+\right)y'\left(t_k^+\right) - r(t_k)y'(t_k)$ subject to the following conditions:

H5.1.14: $\tau \in C([t_0, \infty), R)$, τ is a non-decreasing function in R_+, $\tau(t) \geq t$ for $t \in R_+$ and $\lim_{t \to t\infty} \tau(t) = +\infty$.

H5.1.15: $r \in PC^1([t_0, \infty), R_+)$ and $r(t) > 0$, $\quad r\left(t_k^+\right) > 0$, for $t, t_k \in R_+$.

H5.1.16: $q \in PC([t_0, \infty), R_+)$ and $q_k \geq 0$, $\; k \in N$.

H5.1.17: $\int_0^\infty \frac{dt}{r(t)} = \infty$.

5.2 Suggestions for future work

Oscillation theory, though very old, is one of the most dynamic areas that has attracted investigations on the qualitative properties of differential equations. It appears that its source is inexhaustible and more often than not, continues to attract considerable interest by researchers. Simultaneously, interesting results have been obtained and this can be observed in the study of oscillatory properties of differential equations with deviating arguments.

In the last decade, an intensive investigation into the oscillatory properties of various classes of impulsive differential equations has earnestly commenced. The oscillation theory for the solution of neutral impulsive differential equations is one of the direct consequences of this great effort. However, there still remains a lot in this direction for future consideration. These include the following:

i) the study of the oscillatory nature of solutions of equation (173) when the coefficient $q(t)$ oscillates;

ii) extension of some of the results in chapter 4 to equations where the coefficient $p(t)$ is in ranges different from those described therein;

iii) the extension of results in section 4.2 to equations with positive and negative p's and/or equations with positive and negative q's;

iv) consideration of the neutral impulsive differential equation

$$\begin{cases} [y(t) + p_0\, y(t-\tau)]'' + q(t)\, y(t-\sigma) = 0, & t \notin S \\ \Delta\, [y(t_k) + p_0\, y(t_k-\tau)]' + q_k y(t_k-\sigma) = 0, & \forall\ t_k \in S, \end{cases}$$

where $\tau \in (0,\infty)$, $\sigma \in [0,\infty)$, $q \in PC([t_0,\infty), R_+)$, $q_k > 0$ and

$$\int_{t_0}^{\infty} q(s)ds + \sum_{t_0 \le t_k} q_k = +\infty.$$

The verification of whether or not every non-oscillatory solution of this equation tends to zero as $t \to +\infty$ constitutes an interesting problem;

v) consideration of the delay neutral delay impulsive differential equation

$$\begin{cases} [y(t) + p_0\, y(t-\tau)]'' + q(t)\, y(t-\sigma) = 0, & t \notin S \\ \Delta\, [y(t_k) + p_0\, y(t_k-\tau)]' + q_k y(t_k-\sigma) = 0, & \forall\ t_k \in S, \end{cases}$$

where $p \in R\backslash\{0\}$, $\tau \in (0,\infty)$, $\sigma \in [0,\infty)$, $q \in PC([0,\infty), R_+)$, $q_k > 0$ together with the given restrictions:

a) $-1 \le p < 0$. The condition

$$\int_{0}^{\infty} q(t)dt + \sum_{t_0 \le t_k} q_k = +\infty$$

should not be assumed;

b) $p > 0$ and $\sigma \le \tau$.

The study of each of the above cases and also, finding the sufficient conditions for the oscillation of all solutions under the indicated restrictions on the function $q(t)$ and the delays τ and σ provide yet another interesting investigation problem.

The developments in the field of differential equations over the last thirty years, particularly in the area of impulsive differential equations, have helped to further strengthen the understanding of the potentials of the mathematical sciences in general. There are still vast areas in the new body of knowledge yet untapped. It is believed that keeping abreast with the developments in the new area is the only way to prevent ourselves from being thrown out overboard. This work is aimed at assisting intending researchers in coping with these new developments and what is more, providing a new sense of direction to the weary ones. We sincerely hope that these goals have been achieved.

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