A PURSUIT DIFFERENTIAL GAME PROBLEM WITH MULTIPLE PLAYERS ON A CLOSED CONVEX SET WITH MIXED CONSTRAINTS

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DECLARATION

I hereby declare that this work is the product of my research efforts undertaken under the supervision of Dr. Abbas Ja'afaru Badakaya and has not been presented anywhere for the award of a degree or certificate. All sources have been duly acknowledged.

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CERTIFICATION

This is to certify that the research work for this dissertation and the subsequent write-up (Idris Ahmed SPS/13/MMT/00008) were carried out under my supervision.

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APPROVAL

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ABSTRACT

A fixed duration pursuit Differential Game of countably many pursuers and many evaders on a nonempty closed convex subset of $\mathbb{R}^n$ has been studied. Control function of the pursuers and evaders are subjected to Integral and Geometric constraints, respectively. The motions of all the players are simple and do not move out side the closed convex set. Sufficient condition under which pursuit can be completed is obtained. Strategies for the pursuers which ensure completion of the game in a finite time has been constructed. Further, illustrative examples are given.

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CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Differential Game is an extension of (sequential) game theory to the continuous-time case, which is categorized as a group of problems that are related to modelling and analysis of conflict problems in the context of a dynamical system. Differential Games can also be considered as an extension of Optimal Control problem, which is considered as Differential Game involving one player, to a game involving two players .

Differential Game has been an area of great interest to many applied mathematicians due to its application in solving real life problems; in knowledge areas such as economics, engineering, missile guidance, Behavioral Biology. Its evolution is due to inter-field research and was driven by the need for solving conflict problems during world war. Among the first games analyzed were Lion and Man Problem and the Homicidal Chauffeur game.

A recent development of Differential Game in Economics is the Stochastic Differential Game of Capitalism which was developed for the purpose of analyzing the role of uncertainty in a deterministic game.

The application of Differential Game in solving conflicts such as war (air combat) or competition between countries can be observed in the launching and guidance of a missile or a guided bomb to its intended target which could be a region in the opposing country or as a counter to an aircraft (a drone) from getting to its destination. The accuracy of the missile striking its target is a critical factor for its effectiveness and the accuracy requires knowledge of optimization theory, game theory and dynamical system. As mentioned earlier, ideas in these fields of knowledge gave birth to Differential Game.

Engineering designs such as video games and programmable robots also require concepts of Differential Game especially in the aspect of programming. A computer game which uses a controlling device (input device) is designed such that for any user to play and achieve the target of the game, they must attempt to optimize their input (which requires a lot of strategies) depending on the nature of the game. The player that responds to the users input is endowed with the ability of making several moves such that it can move any direction in the game as instructed by the user.

Artificial intelligence in modern countries uses programmed robots in carrying out dangerous military assignment and jobs that are hazardous to people such as to defuse bombs, mining and exploration of shipwrecks. These robots are built, programmed and controlled for optimal result in their executions and these require expertise in fields like Differential Games.

A typical Differential Game consists of two players, a pursuer and an evader, with conflicting goals. The motion of the pursuer and the evader is modelled by systems of differential equations. Each player attempts to control the state variables of the system into a particular target set by the use of a measurable function called Control, so as to achieve their goals. The system respond to the input of both players.

Pursuit and Evasion problems are examples of Differential Game problem, in which the pursuer tries to capture the evader in some sense, while the evader tries to prevent this capture.

A pursuit-evasion presents a mathematical abstraction of many practical problems. These include, surveillance using mobile robot where a swarm of a robots act as a pursuer try to capture the evader, or a guided missile chasing an aircraft.

An archetypal example of a pursuit-evasion game is known as the Homicidal chauffer game. In this game, the driver of a car attempts to knock down a pedestrian, who of course, does not wish to be flattened. The car can move faster than the pedestrian, but the pedestrian can maneuver himself better than the car. The question usually asked is "what is the best strategy for the pursuer (the car) and the evader (the pedestrian) to adopt in order for each to achieve their conflicting goals"? There are many versions too. For example, the lion and man problem is also a pursuit-evasion problem that was studied in the early nineteenth century which involves a lion and a man in a closed and bounded arena, both assumed to have equal maximum speed. What tactics should the lion employ to be sure of his meal? Is it possible for the man to evade capture? Can both of them achieve their goals? If the region is unbounded, can the man survive? This problem caught the attention of many researchers and after rigorous research, the answers to these questions have been applied to real life situations.

1.2 STATEMENT OF THE PROBLEM

In this research work, we consider a simple motion pursuit Differential Game problem of a fixed duration of many pursuers P_i, $\quad i=1,2,3,...,m$, and many evaders E_j, $\quad j=1,2,...,k$, in a closed convex subset of $\mathbb{R}^n$ with motions of players described by

$$\begin{array}{llll} P_i: & \dot{x}_i(t) & = & \varphi(t)u_i(t), \quad x_i(0)=x_{i0} \quad i=1,2,3,...m, \\ E_j: & \dot{y}_j(t) & = & \varphi(t)v_j(t), \quad y_j(0)=y_{j0}, \quad j=1,2,...,k, \end{array} \tag{1.2.1}$$

where the control functions $u_i(\cdot)$ and $v_j(\cdot)$ of the pursuers and evaders are such that

$$\begin{gathered} \int_0^\infty |u_i(t)|^2 dt \leq \rho_i{}^2 \\ |v_j(t)| \leq \sigma_j, \end{gathered} \tag{1.2.2}$$

where $x_i(t), u_i(t), y_j(t), v_j(t) \in \mathbb{R}^n$, u_i is control parameter of the pursuers x_i, $i=1,2,...,m,$, v_j is that of the evader y_j, $j=1,2,...,k$, ρ_i, σ_j, are given positive numbers, and $\varphi=\varphi(t)$ is a scalar measurable function that satisfies the following conditions:

$$a(\tau)=\left(\int_0^\tau \varphi^2(t)dt\right)^{\frac{1}{2}} < \infty, \quad \tau>0, \quad \lim_{\tau\to\infty} a(\tau)=\infty. \tag{1.2.3}$$

We are motivated by the works in [14] and [16] where the control functions of the players are subjected to Integral and Mixed constraints, respectively.
The research question is as follows:
"What are the sufficient conditions that can guarantee completion of pursuit at a finite time, in the game (1.2.1) and (1.2.2)"?

1.3 AIM AND OBJECTIVES

The aim of this research work is to find a sufficient condition for completion of pursuit Differential Game Problem with Multiple Players on a Closed Convex Set with Mixed Constraints. The objectives are
i. To construct the strategies of pursuer that guarantee capture at a finite time.
ii.To find a sufficient conditions which ensure the completion of the game at a finite time.

1.4 SCOPE AND LIMITATIONS

This research work will focus on a Differential Game described by (1.2.1) and (1.2.2), and that

i. We consider only pursuit problem where the players are contained in a closed convex set of $\mathbb{R}^n$.

ii. The game is of fixed duration denoted by θ.

1.5 DEFINITION OF SOME BASIC TERMS

In this section we give some basic definitions that can be very useful for understanding the concept of this write-up.

Definition 1.5.1 (State variable) *State variable of a player is one of the set of variables used to describe the state or position of that player at any given time in the space under consideration.*

Definition 1.5.2 (Constraint) *Constraint is a factor that restricts the development or transformation of a player's potential from achieving its goal.*

Definition 1.5.3 (Control Function) *A player's "control" (or steering device) is a function through which a player uses to make input in order to achieve his goal.*

Definition 1.5.4 (Admissible Strategy) *Admissible Strategy of a player is the strategy that satisfies the constraint imposed on the control function of the player.*

Definition 1.5.5 (Optimal Strategy) *Optimal Strategy is the best strategy (or control) used by a player to win a game.*

Definition 1.5.6 (Payoff Function) *Payoff Function is a formula which describes the performance of a player playing with their best strategy at any given time in the game.*

Definition 1.5.7 (Attainability Domain) *Attainability Domain is a region in the space where a player can reach at any given but finite time and cannot go beyond it.*

Definition 1.5.8 (Sigma Algebra) *Let X be a nonempty set. A collection Σ of subset of X is called a σ-algebra (sigma-algebra) in X if the following conditions hold :*
i. $\emptyset \in \Sigma$;
ii. If $A \in \Sigma$, then its compliment, A^c, is also in Σ ;
iii. If $\{A_n\}$ is a sequence of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, i.e., Σ is closed under countable unions.

Definition 1.5.9 (Measurable Space) *Measurable Space is a set considered together with the sigma algebra on the set.*

Definition 1.5.10 (Measurable Function) *Let* (X, Σ) *be a measurable space and* $E \in \Sigma$*. A function* $f : E \mapsto \mathbb{R}^*$ *is said to be measurable if for each* $\alpha \in \mathbb{R}$*, the set* $\{x \in E : f(x) < \alpha\}$ *belongs to* Σ*, where* $\mathbb{R}^*$ *is the set of extended real numbers.*

Definition 1.5.11 (Convex Set) *A set* U *in a vector space* E *is called convex if* $\forall x, y \in U$ *and* $\alpha \in [0, 1]$.

$$\alpha x + (1 - \alpha) y \in U.$$

Definition 1.5.12 (Hilbert Space) *A normed vector space* X *is said to be complete if every Cauchy sequence in* X *is convergent with limit in* X*. A complete inner product space is called a Hilbert space.*

Definition 1.5.13 (Closed Set) *A Subset* S *of a metric space* (X, d) *is closed if it is the complement of an open set.*

Definition 1.5.14 (The closest point property) *Let* H *be a Hilbert space and* S *be a closed convex subset of* H*. Then* $\forall x \in H$*,* $\exists$ *a unique* $y \in S$ *such that*

$$||x - y|| = \inf_{z \in S} ||x - z||.$$

Definition 1.5.15 (Absolutely Continuous Functions) *A function* $f : [a, b] \to \mathbb{R}$ *is said to be absolutely continuous on a* $[a, b]$ *if given* $\varepsilon > 0$*, there exist some* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,$$

whenever $\{[x_i, y_i] : i = 1, ..., n\}$ *is a finite collection of mutually disjoint subintervals of* $[a, b]$ *with* $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

CHAPTER TWO

REVIEW OF SOME LITERATURE

This chapter contains review of some works that related to our research problem.

2.1 LITERATURE REVIEW

Differential Game, due to its importance, has been an area of great interest to many applied mathematicians and its birth was as a result of inter-field research activities in game theory and optimal control. Thus a lot of publications have been devoted to this field and fundamental results were published in books (e.g. see [5, 6, 18, 23, 26]).

Pursuit-Evasion Differential Game is a Differential Game involving at least two players and it is of great importance due to its numerous applications. It's application to solving real life problems motivated a lot of researchers to study such type of Differential Game problem (see, for examples [1, 11, 12, 15, 20]). In most of these researches conditions for completion of pursuit, or conditions for evasion were mainly obtain.

Linear Differential Games with integral constraints on controls were examined in many works, for example, [2, 10, 13, 14, 16, 28].

2.2 DIFFERENTIAL GAME PROBLEM WITH DIFFERENT TYPES OF CONSTRAINTS ON CONTROL FUNCTIONS OF THE PLAYERS

Differential Game problem with different types of constraints on the control functions of the players were studied in many works, and fundamental results were published some of which include:

2.2.1 Differential Game Problem with Integral Constraints on the Control Functions of the Players

Satimov et al. [28] studied a linear pursuit Differential Game of many pursuers and one evader with integral constraints on controls of players in $\mathbb{R}^n$, described by the following equations:

$$\dot{z}_i = C_i z_i + u_i - v, \quad z_i(t_0) = z_i^0, i = 1, ..., m, \tag{2.2.1}$$

where u_i is the control parameter of the ith pursuer and v is that of the evader. The eigenvalues of the matrices C_i are assumed to be real numbers. It is proved that if the total resource of control of the pursuers is greater than that of the evader, then under certain conditions pursuit can be completed.

Ibragimov [9] examined a pursuit differential game of m pursuers and k evaders described by (2.2.1). Different from the previous work, here the eigenvalues of the matrices C_{ij} are not necessarily real, and, moreover, the number of evaders can be any. It is proved that if the total resource of control of the pursuers is greater than that of the evader, then under certain conditions pursuit can be completed. Furthermore the game was considered in $\mathbb{R}^n$, without any state constraint.

Ibragimov [13] studied a Differential Game problem of one pursuer and one evader with integral constraints, described by the following equation:

$$\begin{aligned} P &: \dot{x} = \varphi(t)u, \quad x(0) = x_0, \\ E &: \dot{y} = \varphi(t)v, \quad y(0) = y_0. \end{aligned} \tag{2.2.2}$$

In this work Game occurs on a closed convex subset S of $\mathbb{R}^n$. Evasion and pursuit problems were investigated. In the latter case, formula for optimal pursuit time was found.

Leong and Ibragimov [22] studied simple motion pursuit Differential Game of m pursuers and one evader on a closed convex subset of Hilbert space l_2. Control functions of players were subjected to integral constraints. The total resource of the pursuers is assumed to be greater than that of the evader. Strategies of the pursuers that ensure completion of pursuit from any initial position where constructed.

Ibragimov and Satimov [14] studied a multiplayer pursuit Differential Game problem on a closed convex set. Motion of the players are described by (2.2.2), and $\varphi(t)$ is a scalar measurable function that satisfies the following conditions:

$$a(\tau) = \left(\int_0^\tau \varphi^2(t)dt \right)^{\frac{1}{2}} < \infty, \quad \tau > 0, \quad \lim_{\tau \to \infty} a(\tau) = \infty. \tag{2.2.3}$$

It is proved that if the total resource of control of the pursuers is greater than that of the evaders, then pursuit can be completed.

2.2.2 Differential Game Problem with Geometric Constraints on the Control Functions of the Players

Ivanov [20] considered generalized Lion and Man problem in the case of Geometric constraints, described by the following equations:

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad |u_i| \leq 1, \quad i = 0, 1, ...m, \tag{2.2.4}$$

where $x_i(t) \in \mathbb{R}^n$, u_i, $i = 0, 1, ..., m$, are control parameters of the pursuers and u_0 is control parameter of the evader. During the game, all players may not leave a given compact subset N of $\mathbb{R}^n$. It was shown that if the number of pursuers m does not exceed the dimension of the space n, then evasion is possible; otherwise pursuit can be completed.

Ibragimov [12] studied Differential Game problem described by (2.2.3) where $\varphi(t) = 1$ with many pursuers and one evader in the space l_2 with geometric constraints imposed on the control functions of the players. Optimal strategies of the players were constructed and value of the game was found.

Ivanov and Ladyaev [19] obtained sufficient conditions for finding optimal pursuit time in the space $\mathbb{R}^2$ with $\varphi(t) = 1$ and control functions were subjected to geometric constraints. Optimal strategies of the players and value of the game were found.

2.2.3 Differential Game Problem with Mixed Constraints on the Control Functions of the Players

Ibragimov et al.[16] studied a linear Differential Game of optimal approach of many pursuers and one evader. In this case, the players motions were described by a linear system of differential equations of the same type. They obtained an estimate for the payoff functional of the game and explicitly described the strategies of the players. They also showed the existence of the game value in some specific cases and constructed players optimal strategies with an illustrative example.

Vagin and Petrov [30] obtained necessary and sufficient condition for the capture of evader in a game of simple pursuit of one evader by a group of pursuers, subject to the condition that the evader uses the same control as the pursuers and all players remain within a polyhedral region.

In the space $\mathbb{R}^n$, a Differential Game in which m dynamical objects pursuing a single object was investigated in [17] where all the players performed a simple motion in a fixed time duration described by (2.2.3) with $\varphi(t) = 1$ and mixed constraints on the players control functions. Under some certain conditions stated in the paper, they constructed an optimal strategy that guarantees capture for the pursuers and also obtained the value of the game.

In the present work motivated by the above developments we consider a Differential Game of finite number of pursuers and evaders with mixed constraints, game occurs in a nonempty closed convex set in the space $\mathbb{R}^n$. We obtain sufficient conditions of completion of the game at a finite time. The table summarize the related literature.

Table 2.1: Brief summary of the published results.

Authors	Number of pursuers	Number of evaders	The set where the pursuit occurs	Constraint on controls
Satimov et al. (1983)[28]	m	1	$\mathbb{R}^n$	Integral
Ibragimov (2004) [9]	m	k	$\mathbb{R}^n$	Integral
Inanov (2004) [20]	m	1	Closed convex subset **N** of $\mathbb{R}^n$	Geometric
Ibragimov (2002) [13]	1	1	Closed convex subset **N** of $\mathbb{R}^n$	Integral
Leong and Ibragimov (2008) [22]	m	1	Closed convex subset **N** of l_2	Integral
Ibragimov and Satimov (2012) [14]	m	k	Closed convex subset **N** of $\mathbb{R}^n$	Integral
present work	m	k	Closed convex subset **N** of $\mathbb{R}^n$	Mixed

2.3 A MULTIPLAYER PURSUIT DIFFERENTIAL GAME PROBLEM ON A CLOSED CONVEX SET WITH INTEGRAL CONSTRAINTS. [14]

2.3.1 Statement of the Problem

Consider a Differential Game of finite number of inertial players in the space $\mathbb{R}^n$ with players motions described by

$$\begin{aligned} P_i : \dot{x}_i(t) &= \varphi(t)u_i(t), \quad x_i(0) = x_{i0} \quad i = 1, ..., m, \\ E_j : \dot{y}_j(t) &= \varphi(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, ..., k, \end{aligned} \tag{2.3.1}$$

where $x_i(t), u_i(t), y_i(t), v_i(t) \in \mathbb{R}^n$, u_i is control parameter of the pursuer x_i, $i = 1, 2, ...m,$, v_j is that of the evader y_j, $j = 1, 2, ...k$, and $\varphi \equiv \varphi(t)$ is a scalar measurable function that satisfies the following conditions:

$$a(\tau) = \left(\int_0^\tau \varphi^2(t)dt \right)^{\frac{1}{2}} < \infty, \quad \tau > 0, \quad \lim_{\tau\to\infty} a(\tau) = \infty. \tag{2.3.2}$$

Definition 2.3.1 *A measurable function $u_i \equiv u_i(t) = (u_{i1}(t),...,u_{in}(t))$, $t \geq 0$, is called an admissible control of the pursuer x_i if*

$$\int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \tag{2.3.3}$$

where ρ_i, $i = 1,...,m$, are given positive numbers. We denote the set of all admissible controls of the pursuer x_i by $S(\rho_i)$.

Definition 2.3.2 *A measurable function $v_j \equiv v_j(t) = \big(v_{j1}(t),...,v_{jn}(t)\big)$, $t \geq 0$, is called an admissible control of the evader y_j if*

$$\int_0^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \tag{2.3.4}$$

where σ_j, $j = 1,...,m$, are given positive numbers. We denoted the set of all admissible controls of the evader y_j by $S(\sigma_j)$.

Definition 2.3.3 *A Borel measurable function $U_i \equiv U_i(x_i, y_j, ...y_k, v1, ..., v_k), U_i : \mathbb{R}^{2k+1} \to \mathbb{R}^n$, is called a strategy of the pursuer x_i if for any control of the evader $v_j(t)$, $t \geq 0$, the initial value problem*

$$\begin{aligned} \dot{x_i} = \varphi(t)U_i(x_i, y_j, ...y_k, v1(t), ..., v_k(t)) \quad x_i(0) = x_{i0}, \\ \dot{y_j} = \varphi(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, ...k, \end{aligned} \tag{2.3.5}$$

has a unique solution $(x_i(t), y_1(t), ..., y_k(t))$ and the inequality

$$\int_0^\infty |U_i(x_i(s), y_j(s), ...y_k(s), v1(s), ..., v_k(s))|^2 ds \leq \rho_i^2 \tag{2.3.6}$$

holds.

Definition 2.3.4 *Pursuit can be completed from the initial positions $\{x_{10}, ..., x_{m0}, y_{10}, ..., y_{k0}\}$ for the time T in the game (3.2.1)-(3.2.4), if there exist strategies U_i, $i = 1, ...m$, of the pursuers such that for any controls $v_1(\cdot), ..., v_k(\cdot)$ of the evaders and numbers $j = 1, 2, ...k$, the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, ..., m\}$ at some time $t_j \in [0, T]$.*

Given nonempty convex subset N of $\mathbb{R}^n$, according to the rule of the game all players must not leave the set N, that is $x_{i0}, x_i(t), y_{j0}, y_j(t) \in N$, $t \geq 0$, $i = 1, ..., m$, $j = 1, ..., k$. This information describes a Differential Game of many players with integral constraints

on control functions of players.
Problem: The Problem in this game is to find sufficient(s) condition of completing pursuit in the game (3.2.1)-(3.2.4).

2.3.2 Main Result by Ibragimov and Satimov [14].

Since the control parameters of the players can take the values of opposite signs, without loss of generality, we may assume that $\varphi(t) \geq 0$. Moreover, without loss of generality, we may assume that φ is not identically zero on an open interval, otherwise, if for instance $\varphi(t) = 0$, $t \in (t_1, t_2)$, then instead of φ we consider φ such that

$$\varphi_1(t) = \begin{cases} \varphi(t), & 0 \leq t \leq t_1, \\ \varphi(t + t_2 - t_1), & t > t_1. \end{cases} \tag{2.3.7}$$

Theorem 2.3.1 *If*

$$\rho_1^2 + \rho_2^2 + \ldots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_k^2, \tag{2.3.8}$$

then pursuit can be completed for a finite time T in the game (2.2.1)-(2.2.4) *from any initial position.*

Proof
(1) An auxiliary Differential Game: To prove this theorem, we first study an auxiliary differential game of one pursuer x and one evader y, which is described by the following equations:

$$\begin{aligned} P &: \dot{x} = \varphi(t)u, \quad x(0) = x_0, \\ E &: \dot{y} = \varphi(t)v, \quad y(0) = y_0, \end{aligned} \tag{2.3.9}$$

where $x(t), y(t) \in \mathbb{R}^n$, u is the control parameter of the pursuer x, and v is that of the evader y. Assume that $u = u(\cdot) \in S(\rho)$, $v = v(\cdot) \in S(\sigma)$. Pursuit is completed if $x(t') = y(t')$ at some $t' \geq 0$. Here the players move in $\mathbb{R}^n$ without any state constrain.
Set

$$u(t) = \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t), \quad t \geq 0, \tag{2.3.10}$$

where θ is an arbitrary fixed number satisfying

$$a^2(\theta) \geq \frac{|y_0 - x_0|^2}{(\rho - \sigma)^2}. \tag{2.3.11}$$

Let

$$\rho^2(t) = \rho^2 - \int_0^t |u(s)|^2 ds, \quad \sigma^2(t) = \sigma^2 - \int_0^t |v(s)|^2 ds, \quad t \geq 0,$$
$$K(\theta, x_0, y_0) = \frac{1}{a(\theta)}|y_0 - x_0|^2 + 2\sigma|y_0 - x_0|. \tag{2.3.12}$$

The following lemma was proved.

Lemma 2.3.1 *Let the pursuer use the strategy* (2.2.10).
(i) If $\rho > \sigma$, then pursuit can be completed in the game (2.2.9) *for the time θ, and moreover,*

$$\rho^2(\theta) \geq \rho^2 - \sigma^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{2.3.13}$$

(ii) If $\rho \leq \sigma$, then either $x(\theta) = y(\theta)$ or

$$\sigma^2(\theta) \leq \sigma^2 - \rho^2 + \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{2.3.14}$$

Proof of the lemma.
Let $\rho > \sigma$. In this case, we show that the control (2.2.10) is admissible and ensures the equality $x(\theta) = y(\theta)$.
Indeed,

$$\begin{aligned} x(\theta) &= x_0 + \int_0^\theta \varphi(t)u(t)dt, \\ &= x_0 + \int_0^\theta \varphi(t)\left(\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t)\right)dt, \\ &= x_0 + \frac{(y_0 - x_0)}{a^2(\theta)}\int_0^\theta \varphi^2(t)dt + \int_0^\theta \varphi(t)v(t)dt, \\ &= x_0 + \frac{(y_0 - x_0)}{a^2(\theta)}.a^2(\theta) + \int_0^\theta \varphi(t)v(t)dt, \\ &= x_0 + y_0 - x_0 + \int_0^\theta \varphi(t)v(t)dt, \\ &= y_0 + \int_0^\theta \varphi(t)v(t)dt = y(\theta). \end{aligned} \tag{2.3.15}$$

To show the admissibility of the strategy (2.2.10), Cauchy-Schwartz inequality is used as follows:

$$|y_0 - x_0| \int_0^\theta \varphi(t)|v(t)|dt \leq |y_0 - x_0|a(\theta)\sigma, \tag{2.3.16}$$

then by this inequality we have,

$$\begin{aligned}
\int_0^\theta |u(t)|^2 dt &= \int_0^\theta \left| \left(\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right) \right|^2 dt, \\
&= \int_0^\theta \left(\frac{\varphi^2(t)}{a^4(\theta)}|y_0 - x_0|^2 + 2\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0)v(t) + |v(t)|^2 \right) dt, \\
&= \int_0^\theta \frac{\varphi^2(t)}{a^4(\theta)}|y_0 - x_0|^2 dt + 2\int_0^\theta \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0)v(t)dt + \int_0^\theta |v(t)|^2 dt, \\
&\leq \frac{|y_0 - x_0|}{a^4(\theta)} \int_0^\theta \varphi^2(t)dt + 2\frac{|y_0 - x_0|}{a^2(\theta)} \int_0^\theta \varphi(t)|v(t)|dt + \int_0^\theta |v(t)|^2 dt, \\
&\leq \frac{|y_0 - x_0|^2}{a^4(\theta)}.a^2(\theta) + \frac{2}{a^2(\theta)}|y_0 - x_0|a(\theta)\sigma + \int_0^\theta |v(t)|^2 dt, \\
&= \frac{|y_0 - x_0|^2}{a^2(\theta)} + \frac{2}{a^2(\theta)}|y_0 - x_0|\sigma + \int_0^\theta |v(t)|^2 dt, \\
&= \frac{1}{a(\theta)} \left(\frac{|y_0 - x_0|^2}{a(\theta)} + 2\sigma|y_0 - x_0| \right) + \int_0^\theta |v(t)|^2 dt, \\
&= \frac{K(\theta, x_0, y_0)}{a(\theta)} + \int_0^\theta |v(t)|^2 dt.
\end{aligned} \tag{2.3.17}$$

According to (3.2.11) and $x_0 \neq y_0$, we have

$$\frac{1}{a^2(\theta)} \leq \frac{(\rho - \sigma)^2}{|y_0 - x_0|^2}.$$

Therefore,

$$\begin{aligned}
\frac{K(\theta, x_0, y_0)}{a(\theta)} + \int_0^\theta |v(t)|^2 dt &\leq \frac{1}{a^2(\theta)}|y_0 - x_0|^2 + \frac{2}{a(\theta)}|y_0 - x_0|\sigma + \int_0^\theta |v(t)|^2 dt, \\
&\leq \frac{(\rho - \sigma)^2}{|y_0 - x_0|^2}.|y_0 - x_0|^2 + 2\frac{(\rho - \sigma)}{|y_0 - x_0|}.|y_0 - x_0|\sigma + \sigma^2, \\
&= \rho^2 - 2\rho\sigma + \sigma^2 + 2\rho\sigma - 2\sigma^2 + \sigma^2 = \rho^2.
\end{aligned} \tag{2.3.18}$$

Thus,

$$\int_0^\theta |u(t)|^2 dt \leq \rho^2.$$

The admissibility of the control (2.2.10) is proved. In particular, using (2.2.17) we obtain

$$\begin{aligned}\int_0^\theta |u(t)|^2 dt &\leq \frac{K(\theta, x_0, y_0)}{a(\theta)} + \int_0^\theta |v(t)|^2 dt, \\ &\leq \frac{K(\theta, x_0, y_0)}{a(\theta)} + \sigma^2.\end{aligned} \tag{2.3.19}$$

Since

$$\rho^2(\theta) = \rho^2 - \int_0^\theta |u(t)|^2 dt,$$

from (2.2.19) we have,

$$\begin{aligned}\rho^2 - \rho^2(\theta) &= \int_0^\theta |u(t)|^2 dt, \\ &\leq \frac{K(\theta, x_0, y_0)}{a(\theta)} + \sigma^2,\end{aligned}$$

which gives

$$\rho^2(\theta) \geq \rho^2 - \sigma^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{2.3.20}$$

Thus, equation (2.2.13) holds.

We now turn to part (*ii*) of the Lemma 2.2.6. Let $\rho \leq \sigma$ and assume the pursuer uses the strategy (2.2.10) on the interval $[0, \theta]$. If for a control of the evader $v \equiv v(t)$, $t \in [0, \theta]$, the following inequality is satisfied

$$\int_0^\theta |u(t)|^2 dt = \int_0^\theta \left| \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right|^2 dt \leq \rho^2, \tag{2.3.21}$$

then clearly, the control (2.2.10) is admissible and similar to (2.2.15) we obtain that $x(\theta) = y(\theta)$. Hence, if $x(\theta) \neq y(\theta)$, then for the control (2.2.10) we must have

$$\int_0^\theta \left| \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right|^2 dt > \rho^2. \tag{2.3.22}$$

Hence, by inequality (2.2.22) and equation (2.2.17) we obtain

$$\rho^2 < \int_0^\theta |u(t)|^2 dt \leq \frac{K(\theta, x_0, y_0)}{a(\theta)} + \int_0^\theta |v(t)|^2 dt,$$

i.e;

$$\int_0^\theta |v(t)|^2 dt > \rho^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{2.3.23}$$

But from the notation,

$$\sigma^2(\theta) = \sigma^2 - \int_0^\theta |v(t)|^2 dt.$$

Then by inequality (2.2.23),

$$\rho^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)} < \sigma^2 - \sigma^2(\theta).$$

This gives

$$\sigma^2(\theta) < \sigma^2 - \rho^2 + \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{2.3.24}$$

Therefore equation (2.2.14) holds. This completes the proof of the lemma.

The last inequality can be interpreted as follows: Though $x(\theta) \neq y(\theta)$, the pursuer can force to expend the evaders energy more than $\rho^2 - \left(\frac{K(\theta, x_0, y_0)}{a(\theta)}\right)$. At the same time, using the strategy (2.2.10), the pursuer will spend all his energy by a time τ,

$$\int_0^\theta |u(t)|^2 dt = \rho^2, \quad 0 < \tau < \theta. \tag{2.3.25}$$

Then, of course the pursuer cannot move anymore and automatically $u(t) \equiv 0, t \geq \tau$.

(2) Fictitious pursuers (FPs): We now prove the theorem. For this purpose, we introduce fictitious pursuers $z_1, ..., z_m$ by the equations

$$\dot{z}_i = \varphi(t) w_i, \quad z_i(0) = x_{i0}, \quad w_i \in S(\rho_i), \quad i = 1, ..., m, \tag{2.3.26}$$

where w_i is control parameter of the pursuer z_i. FPs may go out of the set N. They move without any state constraint in $\mathbb{R}^n$. The aim of the FPs is to complete the pursuit as early as possible.
Set

$$w_m(t) = \frac{\varphi(t)}{a^2(\theta_1)}(y_{k0} - x_{m0}) + v_k(t), \quad 0 \leq t \leq \theta_1, \tag{2.3.27}$$

$$w_i(t) \equiv 0, \quad i = 1, ..., m-1, \quad 0 \leq t \leq \theta_1, \tag{2.3.28}$$

where θ_1 is any number satisfying inequalities

$$a^2(\theta_1) \geq \frac{|y_{k0} - x_{m0}|^2}{(\rho_m - \sigma_k)^2}, \tag{2.3.29}$$

$$\rho_1^2 + \rho_2^2 + ... + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + ... + \sigma_k^2 + \frac{K(\theta_1, x_{m0}, y_{k0})}{a(\theta_1)}. \tag{2.3.30}$$

Since by assumption $a(t) \to \infty$ as $t \to \infty$, it follows from (2.2.8) that such θ_1 exists. The equations (2.2.27) and (2.2.28) means that all FPs. $z_1,...,z_{m-1}$ do not move on the time interval $[0, \theta_1]$, and only one FP z_m moves according to (2.2.27). We will now show that if (2.2.8) holds and the FPs use the strategies (2.2.27) and (2.2.28). Then the pursuit problem with the pursuers $z_1,...,z_m$ and evaders $y_1,...,y_k$ is reduced to a pursuit problem with the pursuers $z_1,...,z_p$ and evaders $y_1,...,y_q$, for which $\rho_1^2(\theta_1)+...+\rho_p^2(\theta_1) > \sigma_1^2(\theta_1)+...+\sigma_q^2(\theta_1)$ and $p+q < m+k$. Hence, by the time θ_1, the number of players is reduced to $p+q$.
Indeed, we consider two possible cases: (i) $\rho_m \le \sigma_k$; (ii) $\rho_m > \sigma_k$. In the former case, that is $\rho_m \le \sigma_k$, if the equality $z_m(\theta_1) = y_k(\theta_1)$ holds, then we consider the pursuit problem with pursuers $z_1,...,z_{m-1}$ and evaders $y_1,...,y_{k-1}$ under the condition

$$\rho_1^2+\rho_2^2+...+\rho_{m-1}^2 > \sigma_1^2+\sigma_2^2+...+\sigma_{k-1}^2 \quad (p=m-1, \quad q=k-1). \tag{2.3.31}$$

In case of $z_m(\theta_1) \ne y_k(\theta_1)$, according to (2.2.14), we obtain that

$$\sigma_k^2(\theta_1) \le \sigma_k^2-\rho_m^2+\frac{K(\theta_1,x_{m0},y_{k0})}{a(\theta_1)}. \tag{2.3.32}$$

Then in view of (2.2.30) we obtain

$$\rho_1^2+\rho_2^2+...+\rho_{m-1}^2 > \sigma_1^2+\sigma_2^2+...+\sigma_k^2(\theta_1) \quad (p=m-1, \quad q=k), \tag{2.3.33}$$

and at the time θ_1, we consider the pursuit problem with the pursuers $z_1,...,z_{m-1}$ and evaders $y_1,...,y_k$. We now turn to to the case (ii), that is, $\rho_m > \sigma_k$. In this case the pursuer z_m certainly ensurers the equality $z_m(\theta_1) = y_k(\theta_1)$ and according to Lemma 2.2.1,

$$\rho_m^2(\theta_1) \ge \rho_m^2-\sigma_k^2-\frac{K(\theta_1,x_{m0},y_{k0})}{a(\theta_1)}. \tag{2.3.34}$$

Then with the aid of (2.2.30), we obtain

$$\rho_1^2+...+\rho_{m-1}^2+\rho_m^2(\theta_1) > \sigma_1^2+...+\sigma_{k-1}^2, \quad (p=m, \quad q=k-1), \tag{2.3.35}$$

and therefore at the time θ_1 we arrive at the pursuit problem with the pursuers $z_1,...,z_m$ and evaders $y_1,...,y_{k-1}$. Let θ_2 be an arbitrary fixed number satisfying the inequalities:

$$\begin{gathered}\theta_2 > \theta_1, \quad a^2(\theta_2-\theta_1) \ge \frac{|y_q(\theta_1)-z_p(\theta_1)|^2}{(\rho_p(\theta_1)-\sigma_q(\theta_1))^2}, \\ \rho_1^2(\theta_1)+...+\rho_p^2(\theta_1) > \sigma_1^2(\theta_1)+...+\sigma_q^2(\theta_1)+\frac{K(\theta_2-\theta_1,y_q(\theta_1),z_p(\theta_1))}{a(\theta_2-\theta_1)},\end{gathered} \tag{2.3.36}$$

where p and q are the numbers of pursuers and evaders, respectively, which take part in the pursuit problem at time θ_1. Set

$$\begin{gathered} w_p(t) = \frac{\varphi(t)}{a^2(\theta_2-\theta_1)}(y_q(\theta_1)-z_p(\theta_1))+v_q(t), \quad \theta_1 < t \leq \theta_2, \\ w_i \equiv 0, \quad \theta_1 < t \leq \theta_2, \quad i \in \{1,...,m\}\backslash\{p\}. \end{gathered} \tag{2.3.37}$$

Observe that according to (2.2.37) all pursuers except for z_p will not move on the time interval $(\theta_1, \theta_2]$. Let the pursuers use the strategies (2.2.37). Applying the same arguments above, we arrive at the following conclusion.
(ii) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)$ and $z_p(\theta_2) = y_q(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1...,z_{p-1}$ and evaders $y_1,...,y_{q-1}$ under the condition,

$$\rho_1^2(\theta_2)+...+\rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2)+...+\sigma_{q-1}^2(\theta_2). \tag{2.3.38}$$

(ii) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)$ and $z_p(\theta_2) \neq y_q(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1...,z_{p-1}$ and evaders $y_1,...,y_q$ under the condition,

$$\rho_1^2(\theta_2)+...+\rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2)+...+\sigma_q^2(\theta_2). \tag{2.3.39}$$

(iii) If $\rho_p(\theta_1) > \sigma_q(\theta_1)$ then by Lemma 2.2.1 the equality $z_p(\theta_2) = y_q(\theta_2)$ holds. In this case, starting from θ_2 the pursuit problem with the pursuers $z_1...,z_p$ and evaders $y_1,...,y_{q-1}$ is considered under the condition,

$$\rho_1^2(\theta_2)+...+\rho_p^2(\theta_2) > \sigma_1^2(\theta_2)+...+\sigma_{q-1}^2(\theta_2). \tag{2.3.40}$$

Repeated application of this procedure enables us to complete the pursuit for some finite time T since the number of players is finite and decreasing. Thus, we have proved that FPs can complete the pursuit.

(3) Completion of the proof of the theorem: We will now show that the actual pursuers also can complete the pursuit. Define the controls $u_1,...,u_m$ of the pursuers $x_1,...,x_m$ as the controls of the FPs $w_1,...,w_m$. We denote by $F_N(x)$ the projection of the point $x \in \mathbb{R}^n$ on the set N, that is,

$$\min_{y\in N}|x-y| = |x-F_N(x)|. \tag{2.3.41}$$

Note that $F_N(x) = x$ if $x \in N$. It is familiar that for any point $x \in \mathbb{R}^n$ there exist a unique point $F_N(x)$. Moreover, for any $x, y \in \mathbb{R}^n$

$$|F_N(x) - F_N(y)| \leq |x - y|, \tag{2.3.42}$$

and hence the operator $F_N \equiv F_N(x)$ maps any absolutely continuous function $z \equiv z(t)$, $0 \leq t \leq T$, to an absolutely continuous function x such that

$$x(t) = F_N(z(t)), \quad 0 \leq t \leq T, \tag{2.3.43}$$

where T is the time in which pursuit can be completed by the FPs. We define the control $u_i \equiv u_i(t)$ to satisfy

$$x_i(t) = F_N(z_i(t)) \quad 0 \leq t \leq T, \quad i = 1, ..., m. \tag{2.3.44}$$

We first show admissibility of the defined control function $u_i(\cdot)$. Indeed, from (2.2.42), we have almost everywhere on $[0, T]$

$$\begin{aligned} \varphi(t)|u_i(t)| = |\dot{x}_i(t)| &= \lim_{h \to 0} \frac{|x_i(t+h) - x_i(t)|}{|h|}, \\ &= \lim_{h \to 0} \frac{|F_N(z_i(t+h)) - F_N(z_i(t))|}{|h|}, \\ &\leq \lim_{h \to 0} \frac{|z_i(t+h) - z_i(t)|}{|h|}, \\ &= |\dot{z}_i(t)| = \varphi(t)|w_i(t)|. \end{aligned} \tag{2.3.45}$$

Hence the inequality $|u_i(t)| \leq |w_i(t)|$ holds almost everywhere on $[0, T]$ and therefore,

$$\int_0^T |u_i(t)|^2 dt \leq \int_0^T |w_i(t)|^2 dt \leq \rho_i^2. \tag{2.3.46}$$

Since for any evader y_i, $i \in \{1, ..., k\}$, the equality $z_{n_i}(t_i) = y_i(t_i)$ holds for some $t_i \leq T$ and $n_i \in \{1, ..., m\}$, and evader $y_i(t)$ is in N for any $t \geq 0$, in particular, $y_i(t_i) \in N$, then FPs $z_{n_i}(t_i)$ is also in N. Consequently,

$$x_{n_i}(t_i) = F_N(z_{n_i}(t_i)) = z_{n_i}(t_i) = y_i(t_i). \tag{2.3.47}$$

This means differential game $(2.2.1) - (2.2.4)$ can be completed for the time T. The proof of the theorem 2.3.1 is complete.

CHAPTER THREE

METHODOLOGY

In this chapter, we present our proposed method for finding a sufficient conditions of completion of pursuit Differential Game problem described by $(1.2.1)$ with integral constraints on the control function of the pursuers $x_i, i = 1, ..., m$, that is,

$$\int_0^\infty |u_i(t)|^2 dt \leq {\rho_i}^2,$$

and geometric constraints on the control function of the evaders $y_j, j = 1, ..., k$, that is,

$$|v_j(t)| \leq \sigma_j.$$

3.0.3 Admissible Control of the Pursuer

A measurable function $u_i \equiv u_i(t) = (u_{i1}(t), ..., u_{in}(t)), t \geq 0$, is called an admissible control of the pursuer x_i if

$$\int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \tag{3.0.48}$$

where ρ_i, $i = 1, ..., m$, are given positive numbers. We denoted the set of all admissible controls of the pursuer x_i by $S(\rho_i)$.

3.0.4 Admissible Control of the Evader

A measurable function $v_j \equiv v_j(t) = \big(v_{j1}(t), ..., v_{jn}(t)\big), t \geq 0$, is called an admissible control of the evader y_j if

$$|v_j(s)| \leq \sigma_j, \tag{3.0.49}$$

where σ_j, $j = 1, ..., m$, are given positive numbers. We denoted the set of all admissible controls of the evader y_j by $S(\sigma_j)$.

3.0.5 Strategy of the Pursuer

A Borel measurable function $U_i(x_i, y_j, ...y_k, v_1, ..., v_k), U_i : \mathbb{R}^{2k+1} \to \mathbb{R}^n$, is called a strategy of the pursuer x_i if for any control of the evader $v_j \equiv v_j(t)$, $t \geq 0$, the initial value problem

$$\begin{aligned} \dot{x_i} = \varphi(t)U_i(x_i, y_j, ...y_k, v_1(t), ..., v_k(t)) \quad x_i(0) = x_{i0}, \\ \dot{y_j} = \varphi(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, ...k, \end{aligned} \tag{3.0.50}$$

has a unique solution $(x_i(t), y_1(t), ..., y_k(t))$ and the inequality

$$\int_0^\infty |U_i(x_i(s), y_j(s), ...y_k(s), v_1(s), ..., v_k(s))|^2 ds \leq \rho_i^2 \tag{3.0.51}$$

holds.

3.0.6 Completion of Pursuit

We say that pursuit can be completed from the initial position x_{i0}, $\{i = 1, ...m\}$, of pursuers and y_{j0}, $\{j = 1..., k\}$, of evaders for the time T in the game $(1.2.1)$-$(1.2.3)$, if there exist strategies U_i, $i = 1, ...m$, of the pursuers such that for any controls $v_1(\cdot), ...v_k(\cdot)$ of the evaders and numbers $j = 1, 2, ...k$, the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, ..., m\}$, at some time $t_j \in [0, T]$.

3.0.7 Condition for Completion of Pursuit

The method is based on solving the Differential Game $(1.2.1)$ in three phases.

3.0.8 A Game Problem of one Pursuer one Evader

We study an auxiliary Differential Game of one pursuer x and one evader y described by

$$\begin{aligned} P : \dot{x} = \varphi(t)u, \quad x(0) = x_0, \\ E : \dot{y} = \varphi(t)v, \quad y(0) = y_0. \end{aligned} \tag{3.0.52}$$

In this game, the admissibility of the strategy which ensures completion of pursuit is obtained. That is we showed

$$\int_0^\theta |u(t)|^2 dt \leq \rho^2,$$

and

$$x(\theta) = y(\theta).$$

3.0.9 Dummy Pursuers

We introduce the dummy pursuers whose initial positions is the same as the real pursuers. The aim of introducing the dummy pursuers is to complete the pursuit in a finite time. The strategy of dummy pursuers which ensures the completion of pursuit is as follows:

$$\begin{aligned} w_m(t) &= \frac{\varphi(t)}{a^2(\theta_1)}(y_{k0} - x_{m0}) + v_k(t), \quad 0 \leq t \leq \theta_1, \\ w_i(t) &\equiv 0, \quad i = 1, ..., m-1, \quad 0 \leq t \leq \theta_1, \end{aligned} \tag{3.0.53}$$

where θ_1 is any number satisfying inequalities

$$\begin{aligned} a^2(\theta_1) &\geq \frac{|y_{k0} - x_{m0}|^2}{(\rho_m - \sigma_k)^2}, \\ \rho_1^2 + \rho_2^2 + ... + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + ... + \sigma_k^2 &+ \frac{K(\theta_1, x_{m0}, y_{k0})}{a(\theta_1)}. \end{aligned} \tag{3.0.54}$$

Thus, we prove that dummy pursuers can complete the pursuit.

3.0.10 Real Pursuers and Evaders

Finally to solve the game with real pursuers and evaders, in addition to the method indicated above, we use the following mathematical tools in showing the completion of pursuit in a finite time.

Theorem 3.0.2 (The closest point property) *Let H be a Hilbert space and S be a closed convex subset of H. Then $\forall x \in H$, $\exists$ a unique $y \in S$ such that*

$$||x - y|| = \inf_{z \in S} ||x - z||.$$

Lemma 3.0.2 *The following conditions on a real-valued functions f on a compact interval $[a, b]$ are equivalent:*
i. f is absolutely continuous,
ii. f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt, \quad \forall x \in [a, b]$$

.

iii. There exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt, \quad \forall x \in [a, b].$$

Definition 3.0.5 ***Lipschitz Continuous*** *A function* $f : A \to \mathbb{R}^n$, *is said to be L-Lipschitz,* $L \geq 0$, *if* $|f(a) - f(b)| \leq |a - b|$ *for every pair of points* $a, b \in A$.

Lemma 3.0.3 *If a function* f *is Lipschitz on a closed, bounded interval* $[a,b]$, *then it is absolutely continuous on* $[a,b]$.

3.0.11 Cauchy-Schwartz Inequality

Let f and g be any two integrable functions on $[a,b]$, then

$$\int_a^b fg dx \leq \left(\int_a^b f^2 dx \right)^{\frac{1}{2}} \left(\int_a^b g^2 dx \right)^{\frac{1}{2}}.$$

CHAPTER FOUR

RESULTS

This chapter houses the solution of the research problems stated in Section 1.2 which is a modification of the problems considered in work by Ibragimov and Satimov [14]. It also contains some illustrative examples of the application of the theorem.

4.1 STATEMENT OF THE PROBLEM

We consider a Differential Game of fixed duration described by the following equations:

$$\begin{aligned} P &: \dot{x}_i = \varphi(t)u_i, \quad x_i(0) = x_{i_0}, \quad i = 1,2,...,m, \\ E &: \dot{y}_j = \varphi(t)v_j, \quad y_j(0) = y_{j_0}, \quad j = 1,2,...,k, \end{aligned} \tag{4.1.1}$$

where $x_i(t), u_i(t), y_j(t), v_j(t) \in \mathbb{R}^n$, u_i is control parameter of the pursuer x_i, $i = 1,2,...,m$, v_j is that of the evader y_j, $j = 1,2,...,k$, such that,

$$\int_0^\infty |u_i(t)|^2 dt \leq \rho_i^2, \tag{4.1.2}$$

and

$$|v_j(t)| \leq \sigma_j. \tag{4.1.3}$$

Also $\varphi(t)$ is a scalar measurable function that satisfies the following conditions:

$$a(\tau) = \left(\int_0^\tau \varphi^2(t)dt \right)^{\frac{1}{2}} < \infty, \quad \tau > 0, \quad \lim_{\tau \to \infty} a(\tau) = \infty, \tag{4.1.4}$$

it is also required that $x_{i0}, x_i(t), y_{j0}, y_j(t) \in N \subset \mathbb{R}^n$, $t \geq 0$, $i = 1,2,...,m$, $j = 1,2,...,k$, where N is a convex set.

Definition 4.1.1 ***(Admissible Control of the Pursuer)*** *A measurable function* $u_i \equiv u_i(t) = (u_{i1}(t),...,u_{in}(t))$, $t \geq 0$, *is called an admissible control of the pursuer* x_i *if*

$$\int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \tag{4.1.5}$$

where ρ_i, $i = 1,...,m$, *are given positive numbers. We denoted the set of all admissible controls of the pursuer* x_i *by* $S(\rho_i)$.

Definition 4.1.2 ***(Admissible Control of Evader)*** *A measurable function* $v_j \equiv v_j(t) = \big(v_{j1}(t),...,v_{jn}(t)\big)$, $t \geq 0$, *is called an admissible control of the evader* y_j *if*

$$|v_j(s)| \leq \sigma_j, \tag{4.1.6}$$

where σ_j, $j = 1,...,m$, *are given positive numbers. We denoted the set of all admissible controls of the evader* y_j *by* $S(\sigma_j)$.

Definition 4.1.3 ***(Strategy of the Pursuer)*** *A Borel measurable function* $U_i(x_i, y_j, ...y_k, v_1, ..., v_k), U_i : \mathbb{R}^{2k+1} \to \mathbb{R}^n$, *is called a strategy of the pursuer* x_i *if for any control of the evader* $v_j \equiv v_j(t)$, $t \geq 0$, *the initial value problem*

$$\begin{aligned} \dot{x_i} = \varphi(t)U_i(x_i, y_j, ...y_k, v_1(t), ..., v_k(t)) \quad x_i(0) = x_{i0}, \\ \dot{y_j} = \varphi(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, ...k, \end{aligned} \tag{4.1.7}$$

has a unique solution $(x_i(t), y_1(t), ..., y_k(t))$ *and the inequality*

$$\int_0^\infty |U_i(x_i(s), y_j(s), ...y_k(s), v_1(s), ..., v_k(s))|^2 ds \leq \rho_i^2 \tag{4.1.8}$$

holds.

Definition 4.1.4 ***(Completion of Pursuit)*** *We say that pursuit can be completed from the initial position* x_{i0}, $\{i = 1, ...m\}$, *of pursuers and* y_{j0}, $\{j = 1..., k\}$, *of evaders for the time* T *in the game* (4.2.1)-(4.2.4), *if there exist strategies* U_i, $i = 1, ...m$, *of the pursuers such that for any controls* $v_1(\cdot), ...v_k(\cdot)$ *of the evaders and numbers* $j = 1, 2, ...k$, *the equality* $x_i(t_j) = y_j(t_j)$ *holds for some* $i \in \{1, ..., m\}$, *at some time* $t_j \in [0, T]$.

The problem is to find sufficient conditions that guarantee completion of pursuit in the game (4.2.1)-(4.2.4).

4.2 CONDITIONS FOR COMPLETION OF PURSUIT

Suppose that $\varphi \equiv \varphi(t)$, is non-zero in any open interval. The following theorem gives sufficient condition for completion of pursuit.

Theorem 4.2.1 *For pursuit to be completed in the game* (4.1.1)-(4.1.3)*, it is sufficient that*

$$\sum_{i=1}^{m} \rho_i^2 > \sum_{j=1}^{k} \sigma_j^2. \tag{4.2.1}$$

Proof.
(**1**). To prove this theorem, we first consider the following game where $m = k = 1$ described by:

$$\begin{aligned} P &: \dot{x} = \varphi(t)u, \quad x(0) = x_0, \\ E &: \dot{y} = \varphi(t)v, \quad y(0) = y_0, \end{aligned} \tag{4.2.2}$$

where $x(t), y(t) \in \mathbb{R}^n$, u is the control function of the pursuer x, and v is that of the evader y. Assume that $u = u(\cdot) \in S(\rho)$, $v = v(\cdot) \in S(\sigma)$. In this game we ignore constraints on the state equation $x(t)$ and $y(t)$.
We now construct the strategy of the pursuer as follows:

$$u(t) = \begin{cases} \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t), & x_0 \neq y_0, \\ v(t), & x_0 = y_0. \end{cases} \tag{4.2.3}$$

where θ is arbitrary fixed number satisfying,

$$a^2(\theta) \geq \frac{|y_0 - x_0|^2}{(\rho - \sigma\sqrt{\theta})^2}. \tag{4.2.4}$$

Let us now denote the following,

$$\begin{aligned} \rho^2(t) &= \rho^2 - \int_0^t |u(s)|^2 ds, \\ \sigma^2(t) &= \sigma^2 - \int_0^t |v(s)|^2 ds. \end{aligned} \tag{4.2.5}$$

If the pursuer uses the strategy (4.2.3), then the following lemma is true.

Lemma 4.2.1 *If* $\rho^2 > \sigma^2\theta$ *then pursuit can be completed in the game* (4.2.2) *for the time* θ*. Furthermore,*

$$\rho^2(\theta) \geq \rho^2 - \sigma^2\theta - \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.6}$$

Proof

Suppose that, we show that pursuit can be completed in time θ, $(i.e \quad x(\theta) = y(\theta))$. Indeed, if $x_0 = y_0$, we have

$$x(\theta) = x_0 + \int_0^\theta \varphi(t)u(t)dt = y_0 + \int_0^\theta \varphi(t)v(t)dt = y(\theta) \tag{4.2.7}$$

and if $x_0 \neq y_0$, we have

$$\begin{aligned} x(\theta) &= x_0 + \int_0^\theta \varphi(t)\left(\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t)\right)dt \\ &= x_0 + \frac{(y_0 - x_0)}{a^2(\theta)} \int_0^\theta \varphi^2(t)dt + \int_0^\theta \varphi(t)v(t)dt \\ &= x_0 + \frac{(y_0 - x_0)}{a^2(\theta)}.a^2(\theta) + \int_0^\theta \varphi(t)v(t)dt \\ &= x_0 + y_0 - x_0 + \int_0^\theta \varphi(t)v(t)dt \\ &= y_0 + \int_0^\theta \varphi(t)v(t)dt \\ &= y(\theta). \end{aligned} \tag{4.2.8}$$

The strategy is admissible. Indeed, for case
(i) $x_0 = y_0$, we have

$$\int_0^\theta |u(t)|^2 dt = \int_0^\theta |v(t)|^2 dt \leq \int_0^\theta \sigma^2 dt = \sigma^2\theta \leq \rho^2. \tag{4.2.9}$$

(ii) if $x_0 \neq y_0$, then we have

$$\begin{aligned} \int_0^\theta |u(t)|^2 dt &= \int_0^\theta \left|\left(\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t)\right)\right|^2 dt, \\ &\leq \int_0^\theta \left(\frac{\varphi^2(t)}{a^4(\theta)}|y_0 - x_0|^2 + 2\frac{\varphi(t)}{a^2(\theta)}|y_0 - x_0||v(t)| + |v(t)|^2\right)dt, \\ &= \int_0^\theta \frac{\varphi^2(t)}{a^4(\theta)}|y_0 - x_0|^2 dt + 2\int_0^\theta \frac{\varphi(t)}{a^2(\theta)}|y_0 - x_0||v(t)|dt + \int_0^\theta |v(t)|^2 dt, \\ &= \frac{|y_0 - x_0|}{a^4(\theta)} \int_0^\theta \varphi^2(t)dt + 2\frac{|y_0 - x_0|}{a^2(\theta)} \int_0^\theta \varphi(t)|v(t)|dt + \int_0^\theta |v(t)|^2 dt. \end{aligned} \tag{4.2.10}$$

But using Cauchy-Schwartz inequality, we have,

$$|y_0 - x_0| \int_0^\theta \varphi(t)|v(t)|dt \leq |y_0 - x_0| a(\theta)\sigma\sqrt{\theta}, \tag{4.2.11}$$

therefore, by inequality (4.2.11) we have

$$\begin{aligned}\int_0^\theta |u(t)|^2dt &\le \frac{|y_0-x_0|^2}{a^4(\theta)}.a^2(\theta)+\frac{2}{a^2(\theta)}|y_0-x_0|a(\theta)\sigma\sqrt{\theta}+\int_0^\theta |v(t)|^2dt \\ &= \frac{1}{a^2(\theta)}|y_0-x_0|^2+\frac{2}{a(\theta)}|y_0-x_0|\sigma\sqrt{\theta}+\int_0^\theta |v(t)|^2dt.\end{aligned} \tag{4.2.12}$$

But according to (4.2.4) and the condition $v(\cdot)\in S(\sigma)$, we have

$$\frac{1}{a^2(\theta)} \le \frac{(\rho-\sigma\sqrt{\theta})^2}{|y_0-x_0|^2}. \tag{4.2.13}$$

Therefore, from equation (4.2.12) and (4.2.13) we have

$$\begin{aligned}\int_0^\theta |u(t)|^2dt &\le \frac{(\rho-\sigma\sqrt{\theta})^2}{|y_0-x_0|^2}.|y_0-x_0|^2+2\frac{(\rho-\sigma\sqrt{\theta})}{|y_0-x_0|}.|y_0-x_0|\sigma\sqrt{\theta}+\sigma^2\theta \\ &= \rho^2-2\rho\sigma\sqrt{\theta}+\sigma^2\theta+2\rho\sigma\sqrt{\theta}-2\sigma^2\theta+\sigma^2\theta \\ &= \rho^2.\end{aligned} \tag{4.2.14}$$

Hence the strategy is admissible.
To show inequality (4.2.6), we proceed as follows:
From (4.2.12), we have

$$\begin{aligned}\int_0^\theta |u(t)|^2 &\le \frac{1}{a^2(\theta)}|y_0-x_0|^2+\frac{2}{a(\theta)}|y_0-x_0|\sigma\sqrt{\theta}+\int_0^\theta |v(t)|^2dt \\ &= \frac{1}{a(\theta)}\left(\frac{|y_0-x_0|^2}{a(\theta)}+2\sigma\sqrt{\theta}|y_0-x_0|\right)+\int_0^\theta |v(t)|^2dt \\ &= \frac{|y_0-x_0|}{a(\theta)}\left(\frac{|y_0-x_0|}{a(\theta)}+2\sigma\sqrt{\theta}\right)+\int_0^\theta |v(t)|^2dt \\ &\le \frac{|y_0-x_0|}{a(\theta)}\left(\rho+\sigma\sqrt{\theta}\right)+\int_0^\theta |v(t)|^2dt \\ &\le \frac{|y_0-x_0|}{a(\theta)}\left(\rho+\sigma\sqrt{\theta}\right)+\int_0^\theta \sigma^2dt \\ &= \frac{|y_0-x_0|}{a(\theta)}\left(\rho+\sigma\sqrt{\theta}\right)+\sigma^2\theta\end{aligned} \tag{4.2.15}$$

But from the notation

$$\rho^2(\theta)=\rho^2-\int_0^\theta |u(t)|^2dt, \tag{4.2.16}$$

we have

$$\int_0^\theta |u(t)|^2dt=\rho^2-\rho^2(\theta). \tag{4.2.17}$$

Therefore, from (4.2.15) and (4.2.17), we have

$$\rho^2 - \rho^2(\theta) \le \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)} + \sigma^2\theta. \tag{4.2.18}$$

This means

$$\rho^2(\theta) \ge \rho^2 - \sigma^2\theta - \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}, \tag{4.2.19}$$

holds, and hence the proof of the lemma is complete.

Lemma 4.2.2 *If $\rho^2 \le \sigma^2\theta$, then either $x(\theta) = y(\theta)$ or*

$$\sigma^2(\theta) \le \sigma^2 - \rho^2 + \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.20}$$

Proof

Suppose that $\rho^2 \le \sigma^2\theta$, and the following inequality is satisfied:

$$\int_0^\theta |u(t)|^2 dt = \int_0^\theta \left| \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right|^2 dt \le \rho^2. \tag{4.2.21}$$

Then this strategy is admissible and ensure that $x(\theta) = y(\theta)$, as shown in equation (4.2.8), and (4.2.14).

Now suppose that is not the case, then we have

$$\int_0^\theta |u(t)|^2 dt = \int_0^\theta \left| \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right|^2 dt > \rho^2. \tag{4.2.22}$$

But, from (4.2.15) we have

$$\int_0^\theta |u(t)|^2 dt \le \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)} + \int_0^\theta |v(t)|^2 dt. \tag{4.2.23}$$

Hence from equation (4.2.22) and (4.2.23) we have

$$\int_0^\theta |v(t)|^2 dt > \rho^2 - \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.24}$$

But from the notation

$$\sigma^2(\theta) = \sigma^2 - \int_0^\theta |v(t)|^2 dt, \tag{4.2.25}$$

we have

$$\int_0^\theta |v(t)|^2 dt = \sigma^2 - \sigma^2(\theta). \tag{4.2.26}$$

This implies, from equation (4.2.24) and (4.2.26), that

$$\sigma^2 - \sigma^2(\theta) > \rho^2 - \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}, \tag{4.2.27}$$

and so

$$\sigma^2(\theta) < \sigma^2 - \rho^2 + \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.28}$$

This means (4.2.20) is satisfied and hence the proof of the lemma is complete.

(**2**). To prove the main theorem let us introduce the dummy pursuers $z_1, ..., z_m$ whose equation of motion are described by

$$\dot{z}_i = \varphi(t)w_i, \quad z_i(0) = x_{i0}, \quad w_i \in S(\rho_i), \quad i = 1, ..., m, \tag{4.2.29}$$

where w_i is control parameter of the pursuer z_i. Dummy pursuers may move out of the convex set, that is, they move without any state constraint in $\mathbb{R}^n$. The aim of the dummy pursuer is to complete the pursuit in a finite time.

We define the dummy pursuers strategy as follows:

$$w_m(t) = \frac{\varphi(t)}{a^2(\theta_1)}(y_{k0} - x_{m0}) + v_k(t), \quad 0 \le t \le \theta_1, \tag{4.2.30}$$

$$w_i(t) \equiv 0, \quad i = 1, ..., m-1, \quad 0 \le t \le \theta_1, \tag{4.2.31}$$

where θ_1 is any number satisfying inequalities

$$a^2(\theta_1) \ge \frac{|y_{k0} - x_{m0}|^2}{(\rho_m - \sigma_k\sqrt{\theta_1})^2}, \tag{4.2.32}$$

$$\sum_{i=1}^{m} \rho_i^2 > \sum_{j=i}^{k} \sigma_j^2 + \frac{|y_0 - x_0|(\rho + \sigma\sqrt{\theta_1})}{a(\theta_1)}. \tag{4.2.33}$$

Since by assumption $a(t) \to \infty$ as $t \to \infty$, it follows from (4.2.1) that such θ_1 exists. Equations (4.2.30) and (4.2.31) mean that all dummy pursuers, $z_1, ..., z_{m-1}$ do not move on the time interval $[0, \theta_1]$, and only one dummy pursuer, z_m, moves according to (4.2.30).

We will now show that if inequality (4.2.1) holds and the dummy pursuers use the strategies (4.2.30) and (4.2.31), then the pursuit problem with the pursuers $z_1, ..., z_m$ and evaders $y_1, ..., y_k$ is reduced to a pursuit problem with the pursuers $z_1, ..., z_p$ and evaders $y_1, ..., y_q$, for which

$$\sum_{i=1}^{p} \rho_i^2(\theta_1) > \sum_{j=1}^{q} \sigma_j^2(\theta_1)$$

and $p+q<m+k$. Hence, by the time θ_1, the number of players is reduced to $p+q$. Indeed, in the first phase of the Game, if the dummy pursuers use the strategy (4.2.30) and (4.2.31), and if $\rho_m \le \sigma_k\sqrt{\theta_1}$, and the equality $z_m(\theta_1) = y_k(\theta_1)$ holds then the $m^{th}-$ *pursuer* will catch the $k^{th}-$*evader*. Therefore the number of pursuers and evaders reduce to $m-1$ and $k-1$, respectively.

Now a Game problem with pursuers $z_1,...,z_{m-1}$ and evaders $y_1,...,y_{k-1}$ is considered with the condition

$$\sum_{i=1}^{m-1}\rho_i^2 > \sum_{j=1}^{k-1}\sigma_j^2 \quad (p=m-1, q=k-1). \tag{4.2.34}$$

If $z_m(\theta_1) \neq y_k(\theta_1)$, according to (4.2.20), we obtain that

$$\sigma_k^2(\theta_1) \le \sigma_k^2 - \rho_m^2 + \frac{|y_0-x_0|(\rho+\sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.35}$$

Then from equations (4.2.33) and (4.2.35) we obtain

$$\sum_{i=1}^{m-1}\rho_i^2 > \sum_{j=1}^{k}\sigma_1^2 \quad (p=m-1, q=k), \tag{4.2.36}$$

and at the time θ_1, we consider the pursuit problem with the pursuers $z_1,...,z_{m-1}$ and evaders $y_1,...,y_k$. We now turn to the second phase of the Game, that is if $\rho_m > \sigma_k\sqrt{\theta_1}$. Then the pursuer z_m ensurers the equality $z_m(\theta_1) = y_k(\theta_1)$ and according to Lemma 4.3.1

$$\rho_m^2(\theta_1) \ge \rho_m^2 - \sigma_k^2 - \frac{|y_0-x_0|(\rho+\sigma\sqrt{\theta})}{a(\theta)}. \tag{4.2.37}$$

Then from equations (4.2.33) and (4.2.37) we obtain

$$\sum_{i=1}^{m-1}\rho_i^2 + \rho_m^2(\theta_1) > \sum_{j=1}^{k-1}\sigma_j^2 \quad (p=m, q=k-1), \tag{4.2.38}$$

and therefore at the time θ_1 we considered the pursuit problem with the pursuers $z_1,...,z_m$ and evaders $y_1,...,y_{k-1}$. Let θ_2 be an arbitrary fixed number satisfying inequalities

$$\begin{cases} \theta_2 > \theta_1, \\ a^2(\theta_2-\theta_1) \ge \frac{|y_q(\theta_1)-z_p(\theta_1)|^2}{(\rho_p(\theta_1)-\sigma_q(\theta_1)\sqrt{\theta_1})^2}, \\ \rho_1^2(\theta_1)+...+\rho_p^2(\theta_1) > \sigma_1^2(\theta_1)+...+\sigma_q^2(\theta_1)+\frac{|y_0-x_0|(\rho+\sigma\sqrt{\theta})}{a(\theta)}, \end{cases} \tag{4.2.39}$$

where p and q are the numbers of pursuers and evaders, respectively, which take part in the pursuit problem at time θ_1. Define the pursuers strategy as follows:

$$\begin{gathered} w_p(t) = \frac{\varphi(t)}{a^2(\theta_2 - \theta_1)}(y_q(\theta_1) - z_p(\theta_1)) + v_q(t), \quad \theta_1 < t \leq \theta_2, \\ w_i \equiv 0, \quad \theta_1 < t \leq \theta_2, \quad i \in \{1, ..., m\} \backslash \{p\}. \end{gathered} \tag{4.2.40}$$

Observe that according to (4.2.40) all pursuers except for z_p will not move on the time interval $(\theta_1, \theta_2]$. Let the pursuers use the strategies (4.2.40). Applying the same arguments above, we arrive at the following:
(I) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)\sqrt{\theta_1}$ and $z_p(\theta_2) = y_q(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1 ..., z_{p-1}$ and evaders $y_1, ..., y_{q-1}$ under the condition

$$\sum_{i=1}^{p-1} \rho_i^2(\theta_2) > \sum_{j=1}^{q-1} \sigma_j^2(\theta_2). \tag{4.2.41}$$

(II) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)\sqrt{\theta_1}$ and $z_p(\theta_2) \neq y_q(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1 ..., z_{p-1}$ and evaders $y_1, ..., y_q$ under the condition

$$\sum_{i=1}^{p-1} \rho_i^2(\theta_2) > \sum_{j=1}^{q} \sigma_1^2(\theta_2). \tag{4.2.42}$$

(III) If $\rho_p(\theta_1) > \sigma_q(\theta_1)\sqrt{\theta_1}$, then by Lemma 4.2.1 the equality $z_p(\theta_2) = y_q(\theta_2)$ holds. In this case, starting from θ_2 the pursuit problem with the pursuers $z_1 ..., z_p$ and evaders $y_1, ..., y_{q-1}$ is considered under the condition

$$\sum_{i=1}^{p} \rho_i^2(\theta_2) > \sum_{j=1}^{q-1} \sigma_1^2(\theta_2). \tag{4.2.43}$$

Repeated application of this procedure enables us to complete the pursuit for some finite time T since the number of players is finite and decreasing. Thus, we have proved that dummy pursuers can complete the pursuit.
(**3**). We now complete the proof of the theorem by considering the real pursuers and evaders whose the state variables are constrained, that is, do not move outside the closed convex set $\mathbb{N}$. Define the controls $u_1, ..., u_m$ of the pursuers $x_1, ..., x_m$ by the controls of the Dummy pursuers $w_1, ..., w_m$, respectively. We denote by $F_N(x)$ the projection of the point $x \in \mathbb{R}^n$ on the set N, that is,

$$\min_{y \in N} |x - y| = |x - F_N(x)| \tag{4.2.44}$$

Note that $F_N(x) = x$ if $x \in N$. It is well known that for any point $x \in \mathbb{R}^n$, $F_N(x)$ exists and it is unique. Moreover, for any $x, y \in \mathbb{R}^n$ satisfying (4.2.44),

$$|F_N(x) - F_N(y)| \leq |x - y|, \tag{4.2.45}$$

and hence the operator $F_N(\cdot)$ relates any absolutely continuous function $z \equiv z(t)$, $0 \leq t \leq T$, to an absolutely continuous function x such that

$$x(t) = F_N(z(t)), \quad 0 \leq t \leq T, \tag{4.2.46}$$

where T is the time in which pursuit can be completed by the dummy pursuers. We define the control $u_i \equiv u_i(t)$ such that

$$x_i(t) = F_N(z_i(t)), \quad 0 \leq t \leq T \quad i = 1, ..., m. \tag{4.2.47}$$

We first show admissibility of such defined control function $u_i(\cdot)$. Indeed, from (4.2.46), and (4.2.45), we have almost everywhere on $[0, T]$,

$$\begin{aligned} \varphi(t)|u_i(t)| = |\dot{x}_i(t)| &= \lim_{h \to 0} \frac{|x_i(t+h) - x_i(t)|}{|h|} \\ &= \lim_{h \to 0} \frac{|F_N(z_i(t+h)) - F_N(z_i(t))|}{|h|} \\ &\leq \lim_{h \to 0} \frac{|z_i(t+h) - z_i(t)|}{|h|} \\ &= |\dot{z}_i(t)| = \varphi(t)|w_i(t)| \end{aligned} \tag{4.2.48}$$

Hence the inequality $|u_i(t)| \leq |w_i(t)|$ holds almost everywhere on $[0, T]$ and therefore

$$\int_0^T |u_i(t)|^2 dt \leq \int_0^T |w_i(t)|^2 dt \leq \rho_i^2. \tag{4.2.49}$$

This control ensures the completion of the pursuit since for any evader y_i, $i \in \{1, ..., k\}$, the equality $z_{n_i}(t_i) = y_i(t_i)$ holds for some $t_i \leq T$ and $n_i \in \{1, ..., m\}$, and evader $y_i(t)$ is in N for any $t \geq 0$. In particular, $y_i(t_i) \in N$, and therefore dummy pursuers $z_{n_i}(t_i)$ is also in N. Consequently,

$$x_{n_i}(t_i) = F_N(z_{n_i}(t_i)) = z_{n_i}(t_i) = y_i(t_i). \tag{4.2.50}$$

This means Differential Game (4.1.1) – (4.1.4) can be completed for the time T. The proof of the theorem 4.2.1 is complete.

We give a illustrative examples.
Example 1:
We consider a differential game described by the following equations:

$$\begin{aligned} P &: \dot{x}_i(t) = x_i + u_i, \quad x_i(0) = x_{i0}, i = 1, ..., m, \\ E &: \dot{y}_j(t) = y_j + v_j, \quad y_j(0) = y_{j0}, j = 1, ..., k. \end{aligned} \tag{4.2.51}$$

It assumed that $x_{i0} \neq y_{j0}$ for all $i = 1, ..., m, j = 1, ..., k$. The control functions satisfy the mixed constraints (1.2.2). Pursuit is said to be completed if for any numbers $j = 1, ..., k$ the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, ..., k\}$ at some time $t_j \geq 0$. Players may not leave the half space $N = \{x| \quad (x, \lambda) \geq 0, x \in \mathbb{R}^n\}$, where $\lambda \in \mathbb{R}^n$ is a nonzero vector. Since

$$x_i(t) = e^t \overline{x_i}(t) \quad y_j(t) = e^t \overline{y_j}(t),$$

where

$$\overline{x_i}(t) = x_{i0} + \int_0^t e^{-ts} u_i(s) ds, \quad \overline{y_j}(t) = y_{j0} + \int_0^t e^{-ts} v_j(s) ds,$$

then the equality $x_i(t_j) = y_j(t_j)$ is equivalent to ones $\overline{x_i}(t_j) = \overline{y_j}(t_j)$ and inclusion $x_i(t) \in N$ and $y_i(t) \in N$ are equivalent to ones $\overline{x_i}(t) \in N$ and $\overline{y_i}(t) \in N$, respectively. Therefore, differential game described by (4.2.51) is equivalent to the game described by the following equations:

$$\begin{aligned} \overline{x_i}(t_j) &= e^{-t} u_i, \quad \bar{x}_i(0) = x_{i0}, i = 1, ..., m, \\ \overline{y_j}(t_j) &= e^{-t} v_j, \quad \bar{y}_j(0) = y_{j0}, j = 1, ..., k. \end{aligned}$$

Denoting $\varphi(t) = e^{-t}$, we obtain a differential game of the form (4.1.1).
Example 2:
We consider a Differential Game described by the following equations:

$$\begin{aligned} P &: \dot{x}_i(t) = e^t \cdot u_i(t), \quad x_i(0) = x_{i0}, \\ E &: \dot{y}_j(t) = e^t \cdot v_j(t), \quad y_j(0) = y_{j0}. \end{aligned}$$

It is assumed that $x_i(0) \neq y_{j0}$ for all $i = 1, ..., m, \quad j = 1, ..., k$. The control functions satisfy the mixed constraints (4.1.2) and (4.1.3). Let $\rho_i = 6$, $\sigma_j = 1$, $\theta = 2$, and consider the following initial positions $x_{i0} = (0, ..., 0, 3, 0, ...)$, $y_{j0} = (0, ..., 0, 8, 0, ...)$ of the players, where 3 is the *ith* coordinate of the point x_{i0} and number 8 is the *jth* coordinate of the point y_{j0}. Pursuit is said to be completed if for any numbers $j = 1, ..., k$ the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{i, ..., k\}$ at some $t_j \geq 0$. Since $\rho_i > \sigma_j$ then the hypothesis

of the theorem 4.2.1 is satisfied.
Now, we consider a Game of one-pursuer and one-evader described by:

$$P : \dot{x}(t) = e^t \cdot u(t), \quad x(0) = x_0,$$
$$E : \dot{y}(t) = e^t \cdot v(t), \quad y(0) = y_0.$$

Since we already have $\rho^2 > \sigma^2\theta$ we now apply lemma 4.2.1 to solve this problem by showing the following:

i. The strategy $u_i(t) = \frac{e^t}{27}(0,...,0,5,0,...) + v_j(t)$ is indeed admissible.

ii. Pursuit can be completed at time $\theta = 2$.

$$a^2(2) = \int_0^2 e^{2t} dt = \left[\frac{1}{2}e^{2t}\right]_0^2 = \frac{1}{2}(e^4 - e^0) = 27.$$

Also,

$$\begin{aligned}
\frac{|y_{j0} - x_{i0}|^2}{(6-\sqrt{2})^2} &= \frac{|(0,...,0,8,0,...) - (0,...,0,3,0,...)|^2}{(6-\sqrt{2}})^2, \\
&= \frac{5^2}{(6-\sqrt{2})^2}, \\
27 &\geq \frac{5^2}{(6-\sqrt{2})^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
x(2) &= x_0 + \int_0^2 e^t u(t) dt, \\
&= (0,...,0,3,0,...) + \int_0^2 e^t \left(\frac{e^t}{27}(0,...,0,5,0,...) + v(t)\right) dt, \\
&= (0,...,0,3,0,...) + \int_0^2 \frac{e^{2t}}{27}(0,...,0,5,0,...) dt + \int_0^2 e^t v(t) dt, \\
&= (0,...,0,3,0,...) + \frac{(0,...,0,5,0,...)}{27} \int_0^2 e^{2t} dt + \int_0^2 e^t v(t) dt, \\
&= (0,...,0,3,0,...) + \frac{(0,...,0,5,0,...)}{27} .27 + \int_0^2 e^t v(t) dt, \\
&= (0,...,0,8,0,...) + \int_0^2 e^t v(t) dt, \\
&= y(2).
\end{aligned}$$

To show the admissibility, it's sufficient to show

$$\int_0^2 |u(t)|^2 dt \leq 36.$$

That is,

$$\begin{aligned}
\int_0^2 |u(t)|^2 dt &= \int_0^2 \left| \left(\frac{e^t}{27}(...,0,5,0,...) + v(t) \right) \right|^2 dt, \\
&\leq \int_0^2 \frac{e^{2t}}{27^2} |(0,...,0,5,0,...)|^2 dt + 2\int_0^2 \frac{e^t}{27} |(...,0,5,0,...)||v(t)|dt + \int_0^2 |v(t)|^2 dt, \\
&= \frac{|(0,...,0,5,0,...)^2}{27^2} \int_0^2 e^{2t} dt + 2\frac{0,...,0,5,0,...}{27} \int_0^2 e^t |v(t)|dt + \int_0^2 |v(t)|^2 dt, \\
&\leq \frac{|(0,...,0,5,0,...)|^2}{27^2} .27 + 2\frac{|(0,...,0,5,0,...)|}{(\sqrt{27})^2} .\sqrt{27}.1.\sqrt{2} + 2, \\
&= \frac{5^2}{27} + 2\sqrt{2}\frac{5}{\sqrt{27}} + 2, \\
&\leq 5^2.\frac{(6-\sqrt{2})^2}{5^2} + 2\sqrt{2}.5.\frac{(6-\sqrt{2})}{5} + 2, \\
&= 36 - 12\sqrt{2} + 2 + 12\sqrt{2} - 4 + 2, \\
&= 36.
\end{aligned}$$

Hence, the strategy is admissible.

Example 3:
Consider the game problem involving one pursuer and one evader described by

$$\begin{aligned}
P &: \dot{x}(t) = e^{-\lambda t} \cdot u(t), \quad x(0) = x_0, \\
E &: \dot{y}(t) = e^{-\lambda t} \cdot v(t), \quad y(0) = y_0,
\end{aligned}$$

where $\lambda \leq 0$, $y_0 = (0,2,0,...)$, and $x_0 = (0,1,0,...)$, $u(\cdot)$, and $v(\cdot)$ are control functions of the players which satisfy:

$$\int_0^4 |u(t)|^2 dt \leq 25,$$
$$|v(t)| \leq 2, \quad t \in [0,4].$$

Clearly, the hypothesis of the lemma 4.2.1 is satisfied, that is, $\rho \geq \sigma\sqrt{\theta}$ $(25 > 8)$.
Letting $\lambda = 0$,

$$\begin{aligned}
a^2(4) &= \int_0^4 e^{-\lambda t} dt = \int_0^4 dt = 4 \\
a^2(4) &= 4.
\end{aligned}$$

Now according to the lemma 4.2.1 pursuit can be completed in this game problem. To see this, we use the strategy

$$u(t) = \frac{e^{-\lambda t}}{a^2(4)}(y_0 - x_0) + v(t), \quad t \in [0,4] \tag{4.2.52}$$

such that

$$a^2(4) \geq \frac{|y_0 - x_0|^2}{(5 - 2\sqrt{4})^2}.$$

Indeed,

$$\begin{aligned}
\int_0^4 |u(t)|^2 dt &= \int_0^4 \left| \left(\frac{e^{-\lambda t}}{a^2(4)}(y_0 - x_0) + v(t) \right) \right|^2 dt, \\
&\leq \int_0^4 \left(\frac{e^{-2\lambda t}}{a^4(4)}|y_0 - x_0|^2 + 2\frac{e^{-\lambda t}}{a^2(4)}|y_0 - x_0||v(t)| + |v(t)|^2 \right) dt, \\
&= \int_0^4 \frac{e^{-2\lambda t}}{a^4(4)}|y_0 - x_0|^2 dt + 2\int_0^4 \frac{e^{-\lambda t}}{a^2(4)}|y_0 - x_0||v(t)|dt + \int_0^4 |v(t)|^2 dt, \\
&= \frac{|y_0 - x_0|^2}{a^4(4)} \int_0^4 e^{-2\lambda t} dt + 2\frac{|y_0 - x_0|}{a^2(4)} \int_0^4 e^{-\lambda t}|v(t)|dt + \int_0^4 |v(t)|^2 dt, \\
&\leq \frac{|y_0 - x_0|^2}{a^4(4)}.a^2(4) + \frac{2}{a^2(4)}|y_0 - x_0|a(4).2.\sqrt{4} + \int_0^4 |v(t)|^2 dt,
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a^2(4)}|y_0 - x_0|^2 + \frac{2}{a(4)}|y_0 - x_0|.2.\sqrt{4} + \int_0^4 |v(t)|^2 dt, \\
&\leq \frac{1}{|y_0 - x_0|^2}.|y_0 - x_0|^2 + 8\frac{1}{|y_0 - x_0|}.|y_0 - x_0| + 16, \\
&= 1 + 8 + 16, \\
&= 25.
\end{aligned}$$

Hence,

$$\int_0^4 \left| \left(\frac{e^{-\lambda t}}{a^2(4)}(y_0 - x_0) + v(t) \right) \right|^2 dt \leq 25.$$

Now we show that the admissible strategy above guarantees the equality

$$x(4) = y(4).$$

Indeed,

$$\begin{aligned}
x(4) &= x_0 + \int_0^4 e^{-\lambda t}\left(\frac{e^{-\lambda t}}{a^2(4)}(y_0 - x_0) + v(t)\right)dt \\
&= x_0 + \frac{(y_0 - x_0)}{a^2(4)}\int_0^4 e^{-2\lambda t}dt + \int_0^4 e^{-\lambda t}v(t)dt \\
&= x_0 + \frac{(y_0 - x_0)}{a^2(4)}.a^2(4) + \int_0^4 e^{-\lambda t}v(t)dt \\
&= x_0 + y_0 - x_0 + \int_0^4 e^{-\lambda t}v(t)dt \\
&= y_0 + \int_0^4 e^{-\lambda t}v(t)dt \\
&= y(4)
\end{aligned}$$

Hence

$$x(4) = y(4)$$

.

Example 4

Let $\rho = 9, \sigma = 3, \theta = 2$ and $\varphi(t) = \cos(t)$ in the game (4.1.1). We have $\rho \geq \sigma\sqrt{\theta}$ The question is, can pursuit be completed at time $\theta = 2$?

Since the conditions of lemma 4.2.1 is satisfied, then we can find an admissible strategy for completion of pursuit. To achieve this, we first show that at time $\theta = 2$,

i. Using the strategy defined by

$$u(t) = \frac{\cos(t)}{0.8108}(y_0 - x_0) + v(t)$$

such that,

$$a^2(2) \geq \frac{|y_0 - x_0|^2}{(9 - 3\sqrt{2})^2}$$

.

ii. The strategy constructed above is admissible.

Solution:

Now,

$$a^2(2) = \int_0^2 \cos^2 t dt = \int_0^2 \frac{1}{2}(\cos 2t + 1)dt = \left[\frac{1}{4}\sin 2t + \frac{1}{2}t\right]_0^2 = \frac{1}{4}\sin 4 + 1 = 0.8108.$$

To show (i), we only need to have $x(2) = y(2)$.
We have

$$x(2) = x_0 + \int_0^2 \cos t u(t) dt = x_0 + \int_0^2 \cos(t) \left(\frac{\cos t}{0.8108}(y_0 - x_0) + v(t) \right) dt$$
$$= x_0 + \int_0^2 \frac{\cos^2 t}{0.8108}(y_0 - x_0) dt + \int_0^2 \cos t v(t) dt$$
$$= x_0 + \frac{(y_0 - x_0)}{0.8108} \int_0^2 \cos^2 t dt + \int_0^2 \cos t v(t) dt$$
$$= x_0 + \frac{(y_0 - x_0)}{0.8108} 0.8108 + \int_0^2 \cos t v(t) dt$$
$$= y_0 + \int_0^2 \cos t v(t) dt = y(2).$$

Hence

$$x(2) = y(2)$$

.
Lastly, we show the admissibility of the strategy above, that is

$$\int_0^2 |u(t)|^2 dt \leq 81.$$

This is seen below

$$\int_0^2 |u(t)|^2 dt = \int_0^2 \left| \left(\frac{\cos t}{0.8108}(y_0 - x_0) + v(t) \right) \right|^2 dt$$
$$\leq \int_0^2 \frac{\cos^2 t}{(0.8108)^2} |y_0 - x_0|^2 dt + 2 \int_0^2 \frac{\cos t}{0.8108} |y_0 - x_0| |v(t)| dt + \int_0^2 |v(t)|^2 dt$$
$$= \frac{|y_0 - x_0|^2}{(0.8108)^2} \int_0^2 \cos^2 t dt + 2 \frac{|y_0 - x_0|}{0.8108} \int_0^2 \cos t |v(t)| dt + \int_0^2 |v(t)|^2 dt$$
$$\leq \frac{|y_0 - x_0|^2}{a^4(2)} . a^2(2) + 2 \frac{|y_0 - x_0|}{a^2(2)} . a^2(2) . 3 . \sqrt{2} + 18$$
$$= \frac{|y_0 - x_0|^2}{a^2(2)} + 6\sqrt{2} \frac{|y_0 - x_0|}{a(2)} + 18$$
$$\leq |y_0 - x_0|^2 \frac{(9 - 3\sqrt{2})^2}{|y_0 - x_0|^2} + 6\sqrt{2} . \frac{(9 - 3\sqrt{2})}{|y_0 - x_0|} . |y_0 - x_0| + 18$$
$$= 81 - 54\sqrt{2} + 18 + 54\sqrt{2} - 36 + 18$$
$$= 81.$$

Consequently,

$$\int_0^2 |u(t)|^2 dt = \int_0^2 \left| \left(\frac{\cos t}{0.8108}(y_0 - x_0) + v(t) \right) \right|^2 dt \leq 81$$

Hence, the strategy is admissible.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

This chapter gives the summary and conclusion of the entire research work, together with suggestions and recommendations for further research.

5.1 Summary

We studied a Pursuit Differential Game Problem with Multiple Players on a Closed Convex Set with finite number of pursuers and evaders with mixed constraints (integral and geometric) on players' control functions. The dissertation comprises of five chapters:

Chapter one contains a brief introduction of a Differential Game, statement of our research problem, its scope and limitations, and definition of some basic terms used in the dissertation.

In chapter two, we reviewed some related literature on pursuit differential game problem with different type of constraints on the control function of the players, and we present the result of paper by Ibragimov and Satimov [14] in detail from which we formulated our research problem as published but made some elaborations where necessary. Methodology and some mathematical inequalities were presented in chapter three.

Finally, chapter four gives the solution of our research problem, which is an extension of the problems in the main paper reviewed. Theorems on completion of pursuit were stated and proven. In addition, illustrative examples were given.

5.2 Conclusion

We study a fixed duration pursuit Differential Game problem with countably many pursuers and evaders in the space $\mathbb{R}^n$. The controls functions of the pursuers are subject to integral constraints, and the evaders are subject to geometric constraints. Moreover, we constructed an admissible strategy for the pursuer that guarantees the completion of pursuit in a finite time.

The papers [12], [14] and [16] are significant papers with respect to this research work. We derived our research problem from the papers [14] and [16]. Indeed, our research work is a modification of the problems considered in the first two papers. Lastly, we used the idea in the third paper to solve our research problem.

5.3 Recommendations

It should be recalled that all our results are on completion of pursuit. We have not considered the problem of finding value of the game. This could be a future research. In addition to this, solving evasion problem in relation to our research problem could be another research question.

Furthermore, considering the problem with coefficients of the control functions of pursuer and evader to be $a(t)$ and $b(t)$ respectively would suggest many research questions. Different set of constraints on the control functions of the players can also be considered.

REFERENCES

[1] Atamurat, K. Ibragimov, G. and Massimiliano, F. (2016). Simple Motion Pursuit and Evasion Differential Games with Many Pursuers on Manifolds with Euclidean Metric. *Discrete Dynamics in Nature and Society*, 2016: p. 8.

[2] Azimov, A. (1975). Linear Differential Pursuit Game with Integral Constraints on the Control. *Differential'nye Uravneniya*, 11(10), 1723-1731.

[3] Chodun, W. (1989). Differential games of evasion with many pursuers. *Journal of Mathematical Analysis and Application*, 142 , 371-389.

[4] Dockner, E. (2000). *Differential Games in Economics and Management Science*. United State of America.: Cambridge University Press.

[5] Enid, R. (1993). *Optimal Control and the Calculus of Variations*. New York, U.S.A.: Oxford Science Publications.

[6] Friedman, A. (1971). *Differential Games*. New York, U.S.A.: John Wiley and sons.

[7] Ibragimov, G. and Salimi, M. (2009). Pursuit - Evasion Differential Game with Many Inertial Players. *Mathematical problems in Engineering*, 2009: p. 15.

[8] Ibragimov, G. and Hasim R. (2010). Pursuit and Evasion Differential Games in Hilbert Space. *International Game Theory Review*, 12(3), 239-251.

[9] Ibragimov, G. (2004). An n-person Differential Game with Integral Constraints on the Controls of the Players. *Izvestiya Vysshikh Uchebnykh Zavedeni. Matematika*, 48(1), 48-52.

[10] Ibragimov, G., Azamov, A. and Khakestari, M. (2010) Solution of a Linear Pursuit-Evasion Game with Integral Constraints. *AZIAM Journal*, 52(E), 59-75.

[11] Ibragimov, G., Salimi, M. and Amini, M. (2012). Evasion from Many Pursuers in Simple Motion Differential Game with Integral Constraints. *European Journal of Operational Research*, 218(2), 505-511.

[12] Ibragimov, G. (2005). Optimal Pursuit with Countably Many Pursuers and one Evader, *Differential Equations*, 41(5), 627- 635.

[13] Ibragimov, G. (2002). A Game Problem on a Closed Convex Set. *Serberian Advances in Mathematics*, 12(3), 16-31, 2002.

[14] Ibragimov, G. and Satimov, N. (2012). A Multiplayer Pursuit Differential Game on a Closed Convex Set with Integral Constraints. *Abstract and Applied Analysis*, 2012, p. 12.

[15] Ibragimov, G., Abd Rasid, N., Kuchkarov, A. and Ismail, F. (2015). Multi Pursuer Differential Game of Optimal Approach with Integral Constraints on Controls of Players. *Taiwanese Journal of Mathematics*, 19(3), 963-976.

[16] Ibragimov, G. Siegmond, S. Salimi, M. and Sharifi, S. (2015). On a fixed Duration Pursuit Differential Game With Geometric and Integral Constraint. *Dynamic Games and Application*, under revision.

[17] Ibragimov, G. (1998). A Game of Optimal Pursuit of one Object by Several. *Journal of Applied Mathematics and Mechanics*, 62(2), 187-192.

[18] Isaac, R. (1965). *Differential Games*. New York, NY, U.S.A.: John Willey and Sons.

[19] Ivanov, R. and Ladyaev, Yu. (1981). Optimality of Pursuit Time in Differential Game of Many Objects with Simple Motion. *Trudy MIAN USSR., Russia*, 158, 87-97.

[20] Ivanov, R. (1980). Simple Pursuit-Evasion on a Compactum. *Doklady Akademii Nauk SSSR*, 254(6), 1318-1321.

[21] Krasovskii, N. (1985). *Control of a Dynamical System*. Nauka, Moscow.

[22] Leong, W. and Ibragimov, G. (2008). A Multiperson Pursuit Problem on a Closed Convex set in Hilbert Space. *Far East Journal of Applied Mathematics*, 33, 205-214.

[23] Levchenko, A. and Pashkov, A. (1985). A Game of Optimal Pursuit of one Non-inertial Object by Two Inertial Objects. *Prikl. Mekhan*, 49, 536 - 547.

[24] Littlewood, J. (1953). *A Mathemician's Misccellany*. London, Methuen and Co. Ltd.

[25] Mabizela, S. (2006). *Notes on Measure Theory and Integration*. Department of Mathematics (Pure and Applied) Rhodes University , South Africa.

[26] Pontryagin, L. (2004). *Selected works*, Moscow, Russia: MAKS Press.

[27] Rakhmamov, A., Alias, I. and Ibragimov, G. (2016). Evasion Differential Game of Infinitely Many Evaders from Many Pursuers in Hilbert Space. *Dynamic Games and Apllication*, 6(2), 1-13.

[28] Satimov, N. Rikhsiev, B. and Khamdamov, A. (1982). On a Pursuit Problem for Linear Differential and *n*-person Games with Integral Constraints. *Mathematics of the USSR-Sbornik*, 46(4), 456-469.

[29] Subbotin, A. and Chentsov, A. (1981). *Optimization of Guaranteed Result in Control Problems*. Nauka, Moscow.

[30] Vagin, D. and Petrov, N. (2001). The Problem of the Pursuit of a Group of Rigidly Coordinated Evaders. *Journal of Computer and System Sciences*, 40(5), 749 - 753.

[31] Wah, L. and Ibragimov, G. (2008). A Multiperson Pursiut Problem on a Closed Convex Set in Hilbert Space. *Academic Journal*, 33(2), 205-21.

[32] Weihong, H. and Leong, C. (2010). A Stochastic Differential Game of Capitalism. *Journal of Mathematical Economics*, 16, 552-561.

[33] Williams, J. (2009). *System Dynamics*. McGraw-Hill.