MATHEMATICAL MODEL FOR THE TRANSMISSION DYNAMICS OF AVIAN INFLUENZA WITH VACCINATION FOR SUSCEPTIBLE BIRDS

AHMED, Jiddah Ahmed

MSC/MTH/17/1001

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BY

AHMED, Jiddah Ahmed
(MSC/MTH/17/1001)

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, SCHOOL OF PHYSICAL SCIENCES, IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF MASERS OF SCIENCE IN MATHEMATICS OF THE MODIBO ADAMA UNIVERSITY OF TECHNOLOGY, YOLA

FEBRUARY, 2020

DECLARATION

I hereby declare that this thesis was written by me and it is a record of my own research work. It has not been presented before in any previous application for a higher degree. All references cited have been duly acknowledged.

_____ _____

AHMAD, Jiddah Ahmed Date

DEDICATION

This project is dedicated to Almighty Allah for his bounty mercy on me.

APPROVAL

This thesis entitled ‘Mathematical model for the transmission dynamics of avian influenza with vaccination for susceptible birds’ meets the regulations governing the award of Masters of the Modibbo Adama University of Technology, Yola and is approved for its contribution to knowledge and literary presentation.

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Dr. S. Musa Date
(Supervisor)

_____

_____

(Internal Examiner) Date

_____

_____

(External Examiner) Date

_____

_____

Dr. A. M. Alkali Date
(Head of the Department)

_____

_____

Prof. A.A. Adebayo Date
(Dean, School of Postgraduate Studies)

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ABSTRACT

The model by Bimal and Durgesh has been extended to account for vaccination for susceptible birds and transmission to human by infected dead birds. The existence and uniqueness of the solution, invariant region and positivity of the solution were established. The basic reproduction number, R_0 was obtained using next generation matrix method. The local and global stability of the disease free equilibrium were analysed using Routh-Hurwitz stability criteria and the theorem by Castillo-Chavez respectively. The disease free equilibrium was found to be locally asymptotically stable under some specified conditions and globally asymptotically stable if the vaccine efficacy is 100%. Behaviour of the solutions were also illustrated by numerical simulations with different parameter values.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Avian influenza or bird flu is a contagious disease of animals caused by influenza A virus that normally infects mostly birds and less commonly, pigs (Arora & Arora, 2008). Avian influenza is a highly contagious disease of poultry that continues to spread across the globe in bird populations. Occasionally, transmission of a highly pathogenic avian influenza virus (HPAIV) from infected poultry to humans results in a severe public health crisis (Abdullah *et al*., 2017).

The first description of avian influenza (AI) dates back to 1878 in northern Italy, when Perroncito described it as a contagious disease of poultry associated with high mortality. The disease, termed ‘‘fowl plague’’, was initially confused with the acute septicemic form of fowl cholera. However, in 1880, soon after its first description, Rivolta and Delprato showed it to be different from fowl cholera, based on clinical and pathological properties (Blanca & Sanjay, 2009).

Avian influenza, listed by the World Organization for Animal Health (OIE), has become a disease of great importance for animal and human health. Several aspects of the disease lack scientific information, which has hampered the management of some recent crises. Millions of animals have died, and concern is growing over the loss of human lives and management of the pandemic potential (Ilaria & Stefano, 2006).

Infection with avian influenza A viruses in birds (wild and domestic) causes two main forms of diseases that are distinguished by low and high extremes of virulence, namely low pathogenic avian influenza (LPAI) and highly pathogenic avian influenza (HPAI). The ‘low pathogenic’ form may go undetected and usually causes only mild symptoms such as ruffled feathers and a drop in egg production in domestic poultry. However the high pathogenic form spreads more rapidly through flocks of poultry. The mortality can approach 100%, often within 48 hours (Kimbir, Aboiyar & Okoloet, 2014).

In 2016, highly pathogenic avian influenza A(H5N1) virus caused continued outbreaks and was detected in poultry and wild birds. Sporadic transmission to humans was observed in Egypt, with 10 reported cases of which 3 died. Transmission patterns were similar as in previous years: cases were associated with close contact with infected poultry. Between 2003 and 2016, WHO reported 856 human cases due to influenza A (H5N1), including 452 deaths (ECDPC, 2016).

The outbreaks caused by the high pathogenicity avian influenza virus (HPAIV) strains of subtype H5N1 of Asian origin have caused animal and human disease and mortality in several countries of Southeast Asia, such as Bangladesh, Cambodia, China, India, Indonesia, Laos, Myanmar, Thailand and Viet Nam. In some countries, HPAIV H5N1 has become endemic in domestic poultry, and caused the death or slaughter of 250 million birds, including potential reservoir species (Nelson & Silva, 2012).

Avian influenza A subtype H5N1 has been reported to have emerged in Hong Kong in 1997. From late 2003 to early 2004, the virus reached endemic levels among poultry in several south-east Asian countries and spread to Europe and Africa during 2005, with H5N1 virus infected birds discovered in more than 50 countries. The 2004-2007 outbreaks in various countries have highlighted the highly pathogenic avian influenza (HPAI) subtype H5N1 virus as the cause of a major epidemic, with potentially vast repercussions on economics, public health and society at large (Yong, 2007).

From December 1999 through April 2003, more than 50 million birds died or were depopulated after HPAI infection in the European Union, causing severe economic losses to the private and public sectors. These losses suggest that the strategies and control measures used to combat the disease need improvement, from disease control and animal welfare perspectives. However, the mutation to virulence is unpredictable and may occur very soon after the virus is introduced to poultry or after the LPAI virus has circulated in domestic birds for several months (Ilaria & Stefano, 2006).

Birds infected with avian influenza virus shed large quantities of virus in their faeces as well as in their saliva and nasal secretions. Shedding occurs in the first two weeks of infection. The infected droppings or other secretions from both symptomatic and asymptomatic migratory waterfowl will enter water environments where the birds gather (WHO, 2005). Infected birds shed influenza virus in their saliva, nasal secretions and faeces. Susceptible birds become infected when they have contact with contaminated secretions or excretions or with surfaces that are contaminated with secretions or excretions from infected birds (Kimbir, Aboiyar & Okoloet, 2014).

Control strategies based on a combination of stamping out, movement restrictions and emergency vaccination could maximize eradication efforts in certain situations. Controlled elimination of infected poultry, movement restrictions, improved hygiene and

biosecurity, and appropriate surveillance should result in a significant decrease of viral contamination of the environment. These measures should be taken whether or not vaccination is part of the overall strategy. Vaccination is an additional measure aimed primarily at a reduction of viral replication and viral shedding. The scientific basis for the use of a vaccination strategy is the induction of a protective immunity in the target population. A good vaccination program would raise the levels of protective flock immunity and increase the resistance to infection. An exposure to AI virus may not lead to infection in the vaccinated birds or, if infection were to occur, the clinical presentation should be less severe and viral shedding reduced in terms of amount and duration. Moreover, the risk of human exposure to AI viruses with zoonotic potential and the consequent human cases may be reduced by vaccinating poultry (OIE, 2007).

In May 2017, an outbreak due to highly pathogenic avian influenza A(H5N8) was documented at a commercial poultry farm in Zimbabwe. This outbreak required the culling of over 70,000 birds. There is a risk of spread to South African poultry industry through transmission by wild migratory birds (NICD, 2017).

On present evidence, the vast majority of human cases have acquired their infection following direct contact with infected live or dead poultry. Exposure might also occur when the virus is inhaled through dust and possibly through contact with surfaces contaminated with the virus. Infected poultry excrete virus in their saliva and faeces. WHO is aware of recent concerns that the virus could also spread to humans through contact with contaminated poultry products. To date, no epidemiological data suggest that the disease can be transmitted to humans through properly cooked food In backyard production settings, the practices of home slaughtering, defeathering, and eviscerating, related to the marketing of live birds, create opportunities for further and extensive exposure to potentially contaminated parts of poultry. These practices pose a significant risk of infection. From the information currently available, a large number of confirmed human cases acquired their infection during the slaughtering and subsequent handling of diseased or dead birds prior to cooking (WHO, 2005).

March 2015, H5N1 strains have become enzootic in a number of avian species across wide geographical areas, with at least 784 laboratory-confirmed human infections with H5N1 AIVs and 429 deaths across 16 countries, amounting to a case fatality rate (CFR) of 55% (Shuo, Yuhai, Gary, Gregory, George & Shoujun, 2015).

Bimal and Durgesh (2016) formulated a modified SEIQR model that accounted for vaccination of susceptible human. They did not consider the vector population and human infection from dead birds which play vital roles in the dynamics of the virus as well the control of the virulence of the disease.

1.2 Statement of the Problem

Avian influenza, listed by the World Organization for Animal Health (OIE), has become a disease of great importance for animal and human health. Millions of animals have died, and concern is growing over the loss of human lives and management of the pandemic potential (Ilaria & Stefano, 2006). Avian influenza remains a life burden virus with a high fatality rate always greater than 52% (WHO, 2018; Shuo *et al.*, 2019). This prompt Bimal and Durgesh (2016) to formulate a modified SEIQR model that takes into consideration vaccination for susceptible birds and transmission to human by infected dead birds. Their model did not account for vaccination for susceptible birds and transmission to human by infected dead birds. However, a large number of confirmed human cases acquired the infection during the slaughtering and subsequent handling of dead birds prior to cooking (WHO 2005). A good vaccination program would raise the levels of protective immunity and increase the resistance to infection (OIE, 2007). This study therefore intends to modify the model due to Bimal and Durgesh (2016) by incorporating compartment for vaccinated birds and infected dead birds.

1.3 Aim and Objectives of the Study

The aim of this study is to modify the model due to Bimal and Dugesh (2016) by incorporating vaccination for susceptible birds and infection from infected dead birds. Thus, the objectives are as follows:

i. obtain disease free equilibrium state.
ii. obtain the basic reproduction number.
iii. carry out local and global stability analysis for the disease free equilibrium state.
iv. carry out numerical simulation using MATLAB R2015

1.4 Significance of the Study

This work will contribute significantly in educating the poultry farmers on how to manage their poultry against avian influenza. It will also help individual protect themselves from being infected.

1.5 Scope of the Study

This research will be restricted to the modification of the model due to Bimal and Durgesh (2016), accounting for vaccination of susceptible birds and infection of human from dead birds. A system of non-linear ordinary differential equations will be formed and method of linearization will be used to study the stability of the Disease Free Equilibrium Points of the model followed by numerical simulation.

1.6 Operational Definition of Terms

1.6.1 Susceptible individuals: individuals who can be infected but have not yet contacted avian influenza virus.

1.6.2 Exposed individuals: Individuals who have contacted avian influenza virus but not yet infectious.

1.6.3 Infectious individuals: Individuals that have contacted avian influenza and can transmit the virus.

1.6.4 Quarantine individuals: Individuals that are isolated as a result been infected with avian influenza.

1.6.5 Recovered individuals: Individuals who have recovered from avian influenza.

1.6.6 Vaccinated individuals: Individuals that are vaccinated against avian influenza.

1.6.7 Susceptible birds: Birds that can be infected but have not yet contacted avian influenza virus.

1.6.8 Exposed birds: Birds that have contacted avian influenza virus but not yet infectious.

1.69 Infectious birds: Birds that have contacted avian influenza and can transmit it

1.6.10 Vaccinated birds: Birds that are vaccinated against avian influenza

CHAPTER TWO

LITERATURE REVIEW

2.1 Mathematical Models on Avian Influenza

Inyama (2009) developed a simple SI model for the transmission dynamics of bird flu among birds and humans. The model assumes that there is no migration of birds in the susceptible bird population immediately the disease starts. The model formulated is analyzed using dynamical systems theory. The analysis of the steady state and its stability show that the system will be stable if there is a bound on the growth (birth) of birds in the community. The endemic flu steady state was also considered. It was shown that the endemic flu steady state will be asymptotically stable if there is a bound on the infection transmission rate from birds to birds, and recommended that effects should also be made to reduce the growth rate of birds (chickens) to bring about disease-free state. Chickens in poultry firms should all be killed immediately the disease is detected and the owners should be compensated. This will help to reduce the spread and bring about a disease-free situation.

Mohamed and Abdesslam (2008), formulated an SIR model for the both human and birds population, the disease free stability analysis was carried out using the linearization method and found to be stable if the eigenvalues of the Jacobian matrix have negative real part or if the reproduction number $R_0<1$. They also found that the dynamics of the disease is mainly determined by the average number of adequate contacts of a human susceptible with infected birds. This parameter constitutes an essential key to preventive strategies against pandemics.

Kimbir, Aboiyar and Okoloet (2014) modified an SEIR model, incorporating the culling of infected birds and isolation of infected humans with avian influenza. Deterministic model for the transmission dynamics of avian influenza was formulated showed that the existing biological feasible region is positively – invariant and attracting. The numerical simulation showed that any control strategy aimed at reducing the infection transmission will go a long way in eradication avian influenza infection. The simulation shows that the proportion of infected humans peaks and drops gradually as the proportion of recovered humans increases, it also shows that shows that the proportion of susceptible humans gradually decreases and remains stable as time increases while the proportion of

infected, isolated and recovered humans gradually increases and remains stable over time in the presence of culling of infected birds and isolation of infected humans.

Collazo-Rivera and Cruz-Aponte (2015) developed a model that specifically targets the spread of avian influenza H5N1 to investigate possible alternatives to mitigate the spread using measures such as treatment, immunization and as for the importance to educate the public or the community about the conduct of the epidemic in humans and how to prevent the spread of the disease. They discovered that at least a 30% vaccination coverage reduces the morbidity of the epidemic significantly. Which dynamics of disease spreading affect an outbreak among cities? The connectivity of the cities, especially where the epidemic starts shape the overall morbidity of a country's epidemic. The mechanism that is effective to contain a potential avian flu epidemic among cities is that if vaccination is not administered democratically and epidemic cannot be contained in the whole country having a lower morbidity over all but an epidemic on cities that have been neglected.

Pranav, David, Satish and Sharif (2013) simulated the introduction of H5N1 by wild birds and their contact with poultry through a stochastic continuous-time mathematical model. Results showed that reducing contact between wild birds and domestic poultry, and increasing the culling rate of infected domestic poultry communities will reduce the probability of outbreaks. Poultry communities that shared habitat with wild birds or those in districts with previous outbreaks were more likely to suffer an outbreak. These results indicate that wild birds can introduce HPAI to domestic poultry and that limiting their contact at shared habitats together with swift culling of infected domestic poultry can greatly reduce the likelihood of HPAI outbreaks.

Puntani and Jiraporn (2016) formulated a model and take into account the age structure of avian influenza patients. They separated the population into two groups such as human and birds. Age structure of human population is separated into two classes; juvenile and adult human. The equations are constructed for each class. The stability conditions for the disease free equilibrium state and disease endemic equilibrium states were determined. The basic reproduction number, R_0 was determined and concluded that if R_0 <1, the desease free equilibriumstate is stable and the endemic equilibrium is stable where R_0 >1.

2.2 Epidemiology of Avian Influenza

Bird flu also called avian influenza is a major contributor to mortality and morbidity throughout the world. It is a respiratory infection in humans and birds caused by an RNA virus in the family of orthomyxoviridae (WHO, 2005)

Avian influenza (AI) is by far the most dangerous disease linking animal and human wellbeing today. Ten years ago, AI was a disease of poultry and wild birds of limited significance. Today, the emergence of a strain that can infect humans through bird-to-human transmission and kill nearly 60% of those infected, has changed this perspective. But the real danger that this pathogen poses to the human health comes from its potential to change into an extremely virulent human-to-human transmittable pandemic strain (Maia, 2013).

The vaccination of poultry against HPAIV is a control measure in endemic regions and may be important in eradication programs, in order to prevent the destruction of large numbers of flocks, to reduce the number of outbreaks and the circulation of virus in a country or region, or also to be used as insurance against economic losses resulting from outbreaks. Official mass vaccination campaigns of poultry against HPAI have been conducted in several countries, including Hong Kong, China, Viet Nam, Indonesia, Egypt, Côte d'Ivoire, Pakistan and Mexico (Nelson & Stefano 2012).

CHAPTER THREE

METHODOLOGY

3.0 Introduction

This section presents the existing and modified models and discussed the methods used in studying and analyzing the modified model.

3.1 Existing Model

We reviewed the existing model by Bimal and Durgesh (2016) and account for vaccination of susceptible bird and infection of humans from dead birds.

3.1.1 Existing model assumptions

The model is based on the following assumptions

i. The disease is transmitted in a closed environment. There is no emigration or immigration.
ii. The susceptible human and birds become exposed before becoming infectious.
iii. Individuals either die by infection or natural death.
iv. It is also assumed that infected individuals either die or recover.

Table 3.1: Variables and parameters of the existing model

Variables	Description
$S(t)$	the number of Susceptible human at time t.
$E(t)$	the number of Exposed human at time t.
$I(t)$	the number of Infected human at time t.
$Q(t)$	the number of Quarantined human at time t.
$R(t)$	the number of Recovered human at time t.
$V(t)$	the number of Vaccinated human at time t.
$N_h(t)$	Total number of human at time t.
$N_b(t)$	Total number of birds at time t.
$S_b(t)$	the number of Susceptible birds at time t.
$E_b(t)$	the number of Exposed birds at time t.
$I_b(t)$	the number of Infected birds at time t.
B	The human birth rate.
B_b	The bird birth rate
β	The rate at human become exposed through human to human
β_{bh}	The rate at which human become exposed from contact with birds.
μ	The human natural Mortality rate.
ρ	The rate of transmission from vaccinated to susceptible human
σ	The rate of transmission from susceptible to vaccinated human
ε	The rate of transmission from recovered to susceptible human

η	The rate of transmission from exposed to infected human
δ	the human Avian influenza Mortality rate
γ	The rate of transmission from infected to quarantined human
ξ	The rate of transmission from infected to recovered human
α	The rate of transmission from quarantined to recovered human
β_b	Birth rate of birds
μ_b	the birds Avian influenza Mortality rate
χ	The rate of transmission from exposed to infected human
δ_b	the birds Avian influenza Mortality rate

3.1.2 Existing model diagram

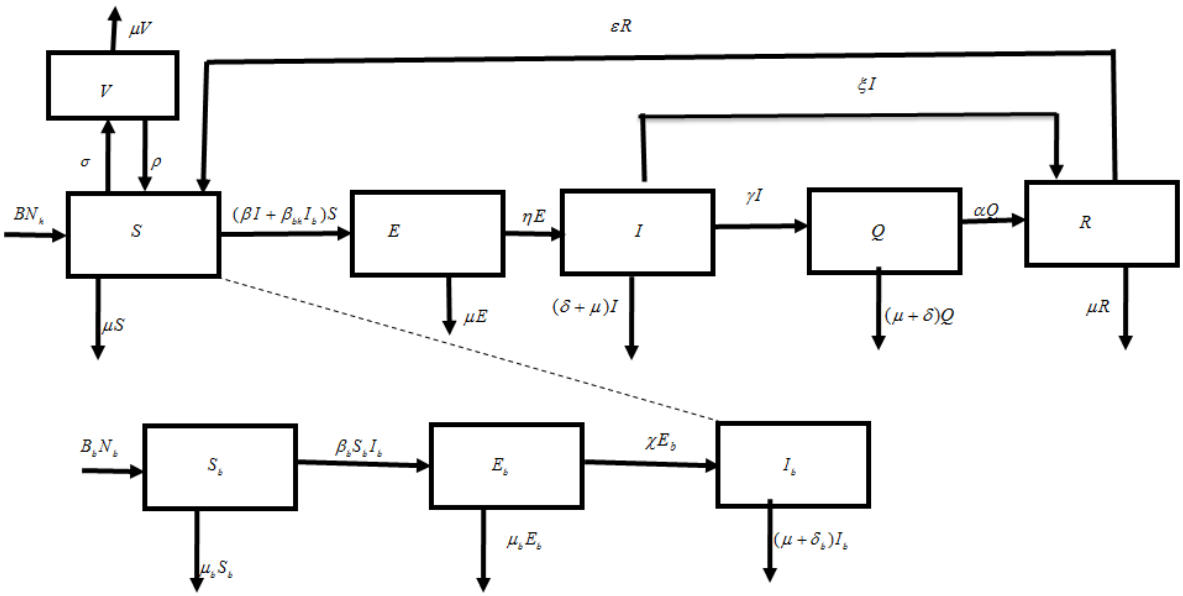

Figure 3.1: Schematic diagram of the existing model

3.1.3 Existing model equation

$$\frac{dS}{dt} = BN_h - \beta SI - \beta_{bh} SI_b + \rho V - \sigma S - \mu S + \varepsilon R \tag{3.1}$$

$$\frac{dE}{dt} = \beta SI + \beta_{bh} SI_b - (\mu + \eta)E \tag{3.2}$$

$$\frac{dI}{dt} = \eta E - (\mu + \delta + \xi + \gamma)I \tag{3.3}$$

$$\frac{dQ}{dt} = \gamma I - (\mu + \delta + \alpha)Q \tag{3.4}$$

$$\frac{dR}{dt} = \alpha Q + \xi I - (\mu + \varepsilon)R \tag{3.5}$$

$$\frac{dV}{dt} = \sigma S - (\mu + \rho)V \tag{3.6}$$

$$\frac{dS_b}{dt} = B_b N_b - \beta_b S_b I_b - \mu_b S_b \tag{3.7}$$

$$\frac{dE_b}{dt} = \beta_b S_b I_b - \mu_b E_b - \chi E_b \tag{3.8}$$

$$\frac{dI_b}{dt} = \chi E_b - (\mu_b + \delta_b) I_b \tag{3.9}$$

$$N_h(t) = S(t) + E(t) + I(t) + Q(t) + V(t) \tag{3.10}$$

$$N_b(t) = S_b(t) + E_b(t) + I_b(t) \tag{3.11}$$

3.1.3 Description of the existing model

In the Model developed by Bimal and Durgesh(2016), human and birds population were considered. The human population is divided into six classes: Susceptible $S(t)$, Exposed $E(t)$, Infected $I(t)$, Quarantined $Q(t)$, Recovered $R(t)$ and Vaccinated $V(t)$. The class of susceptible human S increases by birth at a rate B. The class reduces due to the progression of individuals to the exposed class at the rate βSI, and $\beta_{bh} SI_b$, to the vaccination class at the rate σS, as natural death at the rate μS. The Exposed class grows as a result of incoming of individuals from susceptible class at the rate βSI, and $\beta_{bh} SI_b$. The class reduces as a result of progression of individuals to the infected class at the rate ηE and reduces due to the natural death at the rate μE. The Infected class grows as a result of incoming individuals from Exposed class at the rate ηE and reduces due to the progression of individuals to the Recovered class at the rate ξI and due to the Avian influenza Mortality rate δI and due to the natural death at the rate μI. The quarantined class grows as a result of progression of individuals from infected class at the rate γI and reduces due to the progression of individuals to the Recovered class at the rate αI and as a result of natural death at the rate μI. The recovered class grows as a result of successful treatment and cure of infected individuals at the rate αI and reduces as a result of natural death at the rate αI. The vaccinated class grows as a result of vaccination of susceptible individuals at the rate σV and reduces to the susceptible class due to vaccination duration at the rate ρV

Similarly the birds' population is divided into three classes: Susceptible $S_b(t)$, Exposed $E_b(t)$ and Infected $I_b(t)$. The class of susceptible birds S_b increases by birth at a rate $B_b N_b$. The class reduces due to the progression of individuals to the exposed class at the rate $\beta_b SI_b$ and reduces as natural death at the rate $\mu_b S$. The Exposed class grows as a result of incoming of individuals from susceptible class at the rate $\beta_b SI_b$. The class reduces as a result of progression of individuals to the infected class at the rate χE and reduces due to the natural death at the rate $\mu_b E$. The Infected class grows as a result of incoming individuals from Exposed class at the rate χE and reduces due to the Avian influenza Mortality rate $\delta_b I$ and due to the natural death at the rate $\mu_b I$.

3.2 The Modified Model

Below are the assumptions, description, diagram and model equations of the formulated modified model.

3.2.1 Assumptions of the modified model

The modified model has the following assumptions

i. The disease is transmitted in a closed environment. There is no emigration or immigration.
ii. The susceptible human and birds become exposed before becoming infectious.
iii. Individuals either die by infection or natural death.
iv. It is also assumed that infected individuals either die or recover.
v. Susceptible birds are vaccinated at the rate τ
vi. A susceptible can also be infected if there is an effective contact between susceptible human and infected dead birds.
vii. The bird vaccine efficacy is at the rate ϕ
viii. A bird whose vaccine fails will move to the expose class/

Table 3.2: Variables of the modified model

Variables	**Description**
$S_H(t)$	the number of Susceptible human at time t.
$E_H(t)$	the number of Exposed human at time t.
$I_H(t)$	the number of Infected human at time t.
$Q_H(t)$	the number of Isolated human at time t.
$R_H(t)$	the number of Recovered human at time t.
$V_H(t)$	the number of Vaccinated human at time t.
$N_H(t)$	Total number of human at time t.
$N_B(t)$	Total number of birds at time t.
$S_B(t)$	the number of Susceptible birds at time t.
$E_B(t)$	the number of Exposed birds at time t.
$I_B(t)$	the number of Infected birds at time t.
$D_B(t)$	the number of Infected dead birds at time t.

Table 3.3: Parameters of the modified model

Parameters	Description
Λ_H	The human constant recruitment rate.
β_H	The rate at human become exposed through human to human
β_{BH}	The rate at which human become exposed from contact with infected birds.
β_{DH}	The rate at which human become exposed from contact with infected dead birds.
μ_H	The human natural mortality rate.
ρ	The rate of transmission from vaccinated to susceptible human
σ	The rate of transmission from susceptible to vaccinated human
ε	The rate of transmission from recovered to susceptible human
η	The rate of transmission from exposed to infected human
δ_1	the infected human Avian influenza mortality rate
δ_2	the quarantined human Avian influenza mortality rate
γ	The rate of transmission from infected to isolated human
ξ	The rate of transmission from infected to recovered human
α	The rate of transmission from isolated to recovered human
Λ_B	The bird recruitment rate
ϕ	Birds vaccine efficacy
β_B	The rate at which birds become exposed
μ_B	the birds natural death rate

χ	The rate of transmission from exposed to infected birds
δ_B	the birds Avian influenza mortality rate
ψ	The rate of transmission from vaccinated to susceptible birds
τ	The rate of transmission from susceptible to vaccinated birds
π	The rate at which infected dead birds are properly handled

3.2.2 Modified model diagram

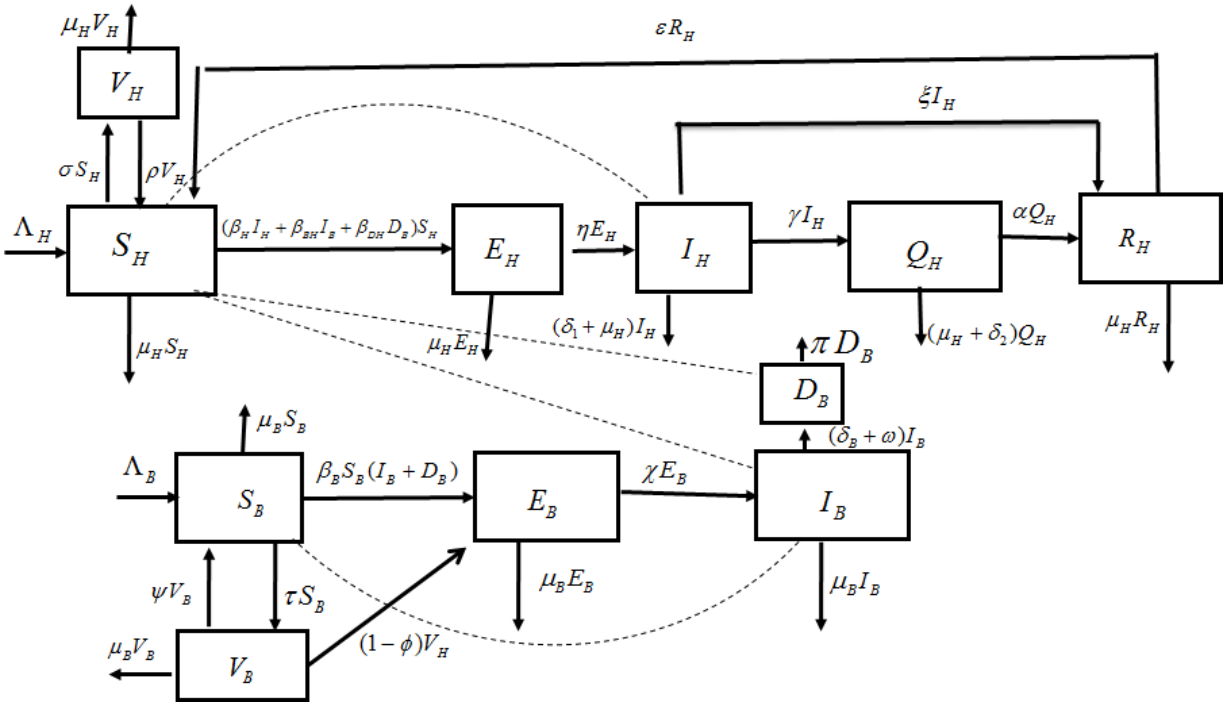

Figure 3.2: Schematic diagram of the modified model

3.2.2 Modified model equations

$$\frac{dS_H}{dt} = \Lambda_H - (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H \tag{3.12}$$

$$\frac{dE_H}{dt} = (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H - (\mu_H + \eta) E_H \tag{3.13}$$

$$\frac{dI_H}{dt} = \eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H \tag{3.14}$$

$$\frac{dQ_H}{dt} = \gamma I_H - (\mu_H + \delta_2 + \alpha) Q_H \tag{3.15}$$

$$\frac{dR_H}{dt} = \alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H \tag{3.16}$$

$$\frac{dV_H}{dt} = \sigma S_H - (\mu_H + \rho) V_H \tag{3.17}$$

$$\frac{dS_B}{dt} = \Lambda_B - \beta_B S_B (I_B + D_B) + \psi V_B - \tau S_B - \mu_B S_B \tag{3.18}$$

$$\frac{dE_B}{dt} = \beta_B S_B (I_B + D_B) - (\mu_B + \chi) E_B + (1 - \phi) V_B \tag{3.19}$$

$$\frac{dI_B}{dt} = \chi E_B - (\mu_B + \delta_B + \omega) I_B \tag{3.20}$$

$$\frac{dV_B}{dt} = \tau S_B - (\mu_B + \psi) V_B - (1 - \phi) V_B = 0 \tag{3.21}$$

$$\frac{dD_B}{dt} = (\delta_B + \omega) I_B - \pi D_B \tag{3.22}$$

$$N_B(t) = S_B(t) + E_B(t) + I_B(t) + Q_B(t) + R_B(t) \tag{3.23}$$

$$N_B(t) = S_B(t) + E_B(t) + I_B(t) + V_B(t) \tag{3.24}$$

3.2.3 Model description of the modified model

In the process to modify the model due to Birmal and Durgeshi (2016), the modified model consists of eleven compartments. Six for human while the remaining five are for birds. The human population is divided into six classes: Susceptible $S_H(t)$, Exposed $E_H(t)$, Infected $I_H(t)$, Isolated $Q_H(t)$, Recovered $R_H(t)$ and Vaccinated $V_H(t)$. The class of susceptible human S_H increases by recruitment rate Λ_H The class reduces due to the progression of individuals to the exposed class at the rate $(\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H$ and to the vaccination class at the rate σS_H, as natural death at the rate $\mu_H S_H$, it also increase by number of human that lost their vaccine immunity ρV_H and those that have fully recover at rate εR_H. The Exposed class grows as a result of incoming of individuals from susceptible class at the rate $(\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H$. The class reduces as a result of progression of individuals to the infected class at the rate ηE_H and reduces due to the natural death at the rate $\mu_H E_H$. The Infected class grows as a result of incoming individuals from Exposed class at the rate ηE_h and reduces due to the progression of individuals to the isolated and Recovered class at the rate γI_H and ξI_H respectively and due to the Avian influenza mortality rate $\delta_1 I_H$ and due to the natural death at the rate $\mu_H I_H$ The Isolated class grows as a result of progression of individuals from infected class at the rate γI_H and reduces due to the progression of individuals to the Recovered class at the rate αQ_H, as a result of Avian influenza induced death at the rate $\delta_2 Q_H$ and as a result of natural death at the rate $\mu_H Q_H$. The recovered class grows as a result of successful treatment and cure of infected individuals at the rate ξI_H and αQ_H, and reduces by returning back to the susceptible class at the rate εR_H and as a result of natural death at the rate $\mu_H R_H$.

The vaccinated class grows as a result of vaccination of susceptible individuals at the rate σS_H and reduces to the susceptible class due to vaccination duration at the rate ρV_H and due to natural mortality rate at $\mu_H V_H$

Similarly the birds' population is divided into five classes: Susceptible $S_B(t)$, Exposed $E_B(t)$ and Infected $I_B(t)$, Death $D_B(t)$ and Vaccination $V_B(t)$. The class of susceptible birds S_B increases by recruitment rate Λ_B and by the number of those that lost their

vaccine immunity at the rate ψV_B The class reduces due to the progression of individuals to the exposed and vaccinated classes at the rate $\beta_B(I_B + D_B)S_B$ and τS_B and as natural death at the rate $\mu_B S_B$. The Exposed class grows as a result of incoming of individuals from susceptible class at the rate $\beta_B(I_B + D_B)S_B$. The class reduces as a result of progression of individuals to the infected class at the rate χE_B and reduces due to the natural death at the rate $\mu_B E_B$. The Infected class grows as a result of incoming individuals from Exposed class at the rate χE_B and reduces due to the Avian influenza mortality, natural mortality and selective slaughtering at the rates $\delta_B I_B$, $\mu_B I_B$ and ωI_B respectively. The vaccinated class grows as a result of vaccinating the susceptible birds at the rate τS_B and reduce to the susceptible class at a rate ψV_B and natural death at $\mu_B V_B$. The Death class grows as a result avian influenza mortality and culling of infected birds at $\delta_B I_B$ and ωI_B respectively.

3.3 Method of Model Analysis

The following methods were adopted in the course of this research work:

(i) Equilibrium States

The Equilibrium state for the model was obtained by setting the model equations to be zero.i.e.

$$\frac{dS_H}{dt} = \frac{dE_H}{dt} = \frac{dI_H}{dt} = \frac{dQ_H}{dt} = \frac{dR_H}{dt} = \frac{dV_H}{dt} = \frac{dS_B}{dt} = \frac{dE_B}{dt} = \frac{dI_B}{dt} = \frac{dD_B}{dt} = \frac{dV_B}{dt} = 0$$

(ii) Basic Reproduction Number

Diekmann, and Heesterbeek (2000) defined the basic reproduction number, R_0, as the average number of secondary infections caused by an infectious individual during his/her entire life as an infectious person. In this model, we adopted the method of the Next Generation Matrix to compute our reproduction number. We call FV^{-1}, the next generation matrix for the model and set the reproduction number $R_0 = \rho(FV^{-1})$ where $F = \left[\frac{\partial F_i(x_0)}{\partial x_j}\right]$

and $V = \left[\frac{\partial V_i(x_0)}{\partial x_j}\right]$ for i ≥ 1 for the number of compartments and $1 \leq j \leq m$ for the infected

compartments only. $\rho(FV^{-1})$ denotes the spectral radius of the matrix A. F and V are m×m matrices, where m is the number of infected classes (Diekmann & Heesterbeek, 2000).

(iii) Stability Analysis

Routh-Hurwitz stability criterion was used to obtain the steady state of the model. The Routh-Hurwitz criterion states that a necessary and sufficient condition that the equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0,$ (with real coefficients) have only roots of negative real part if the values of the determinants of the matrices are all positive, $D_1 = a_1 > 0,$

$$D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0,\ D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0 \text{ where}$$

$$D_K = \begin{vmatrix} a_1 & a_3 & . & . & . & . \\ 1 & a_2 & a_4 & . & . & . \\ 0 & a_1 & a_3 & . & . & \\ 0 & 1 & a_2 & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_k \end{vmatrix} > 0 \text{ is called the Hurwitz matrix}$$

For quadratic and cubic polynomials, these conditions reduce to:

$n = 2,\ a_1 > 0,\ a_2 > 0$

$n = 3\ a_1 > 0,\ a_2 > 0,\ a_1 a_2 > 0.$

(iv) Numerical solutions

In this study, we used MATLAB R2015a inbuilt scheme of ode45 as the scientific package for all numerical solutions.

(v) Source of Data

The study adopted theoretical data from specified literatures in an event where real data is not tenable.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, the analytical result of the model was presented. we established the disease free equilibrium (DFE) and the reproduction number of the disease was obtained. We also analyzed both local and global stability of the disease free equilibrium point of the model, as well as positivity of the solution of the model and carry out numerical simulation using MATLAB R2015a.

4.2 Analytical Results

4.2.1 The existence and the uniqueness of the model solution

The validity and authenticity of any mathematical model depends on whether the given system of equation has a solution and if the solution exists is unique. We use the theorem of existence and uniqueness of solution of ode by Derrick and Grossman (1776), the existence and uniqueness of the model equation given by (3.12) to (3.22).

Consider the system of equation below

$$\left.\begin{aligned} x_1^1 &= f_1(t, x_1, x_2, \ldots x_n), x_1(t_0) = x_{10} \\ x_2^1 &= f_1(t, x_1, x_2, \ldots x_n), x_2(t_0) = x_{20} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n^1 &= f_n(t, x_1, x_2, \ldots x_n), x_n(t_0) = x_{n0} \end{aligned}\right\} \quad (4.1)$$

We may write equation (4.1) in compact form as

$$x^1 = f_1(t, x), x_1(t_0) = x_0 \quad (4.2)$$

Theorem 1

Let D denote the region

$$\left|t - t_o\right| \le a, \left\|x - x_o\right\| \le b, \quad x = (x_1, x_2, \ldots, x_n),\ x_o = (x_{10}, x_{20}, \ldots, x_{n0}) \tag{4.3}$$

And suppose that $F(t, x)$ satisfies the Lipchitz condition

$$\left\| f(t, x_1) - f(t, x_2) \right\| \le k \left\| x_1 - x_2 \right\| \tag{4.4}$$

Whenever the pairs (t, x_1) and (t, x_2) belong to D^1, where K is a positive constant. Then there is a constant $\delta > 0$ such that there exist a unique continues vector solution of $\underline{X}(t)$ of system (4.2) in the interval $\left|t - t_o\right| \le \delta$. It is important to note that the condition (4.4) is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}, i, j = 1, 2, \ldots$ be continues and bounded in D^1.

Theorem 2

Let D denote the region defined in $1 \le \varepsilon \le R$, such that $1 \le \varepsilon \le R$, and $0 < R < \infty$, hold then there exist a unique solution of equation (3.12) to (3.22)of the model equations which is bounded in the region D.

Proof

Let

$$f_1 = \Lambda_H - k_1 S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H$$

$$f_2 = k_1 S_H - (\mu_H + \eta) E_H$$

$$f_3 = \eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H$$

$$f_4 = \gamma I_H - (\mu_H + \delta_2 + \alpha) Q_H$$

$$f_5 = \alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H$$

$$f_6 = \sigma S_H - (\mu_H + \rho) V_H$$

$$f_7 = \Lambda_B - k_2 S_B + \psi V_B - \tau S_B - \mu_B S_B$$

$$f_8 = k_2 S_B - (\mu_B + \chi) E_B + (1 - \phi) V_B$$

$$f_9 = \chi E_B - (\mu_B + \delta_B + \omega) I_B$$

$$f_{10} = \tau S_B - (\mu_B + \psi) V_B - (1 - \phi) V_B$$

$$f_{11} = (\delta_B + \omega) I_B - \pi D_B$$

where $k_1 = (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B),\ k_2 = \beta_B (I_B + D_B)$

It is sufficient to show that $\frac{\partial f_i}{\partial x_j}, i, j = 1, 2, 3, \ldots, 10$ are continuous.

Let $S_H = x_1, V_H = x_2, E_H = x_3, G_H = x_4, I_H = x_5, J_H = x_6, R_H = x_7, D_H = x_8, S_b = x_9 and I_b = x_{10}$

Consider the partial derivatives below

For f_1

$$f_1 = \Lambda_H - k_1 S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H$$

$$\left|\frac{\partial f_1}{\partial S_H}\right| = \left|-(\mathrm{k}_1 + \sigma + \mu_H)\right| < \infty$$

$$\left|\frac{\partial f_1}{\partial E_H}\right| = \left|\frac{\partial f_1}{\partial I_H}\right| = \left|\frac{\partial f_1}{\partial Q_H}\right| = \left|\frac{\partial f_1}{\partial S_B}\right| = \left|\frac{\partial f_1}{\partial E_B}\right| = \left|\frac{\partial f_1}{\partial I_B}\right| = \left|\frac{\partial f_1}{\partial V_B}\right| = \left|\frac{\partial f_1}{\partial D_B}\right| 0 < \infty$$

$$\left|\frac{\partial f_1}{\partial R_H}\right| = \left|\varepsilon\right| < \infty, \qquad \left|\frac{\partial f_1}{\partial V_H}\right| = \left|\rho\right| < \infty$$

For f_2

$$\mathrm{f}_2 = k_1 S_H - (\mu_H + \eta) E_H$$

$$\left|\frac{\partial f_2}{\partial S_H}\right| = \left|k_1\right| < \infty$$

$$\left|\frac{\partial f_2}{\partial E_H}\right| = \left|-(\mu_H + \eta)\right| < \infty$$

$$\left|\frac{\partial f_2}{\partial I_H}\right| = \left|\frac{\partial f_2}{\partial Q_H}\right| = \left|\frac{\partial f_2}{\partial R_H}\right| = \left|\frac{\partial f_2}{\partial V_H}\right| = \left|\frac{\partial f_2}{\partial S_B}\right| = \left|\frac{\partial f_2}{\partial E_B}\right| = \left|\frac{\partial f_2}{\partial I_B}\right| = \left|\frac{\partial f_2}{\partial V_B}\right| = \left|\frac{\partial f_2}{\partial D_B}\right| 0 < \infty$$

For f_3

$$f_3 = \eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H$$

$$\left|\frac{\partial f_3}{\partial E_H}\right| = \left|\eta\right| < \infty$$

$$\left|\frac{\partial f_3}{\partial I_H}\right| = \left|-(\mu_H + \delta_1 + \xi + \gamma)\right| < \infty$$

$$\left|\frac{\partial f_3}{\partial S_H}\right| = \left|\frac{\partial f_3}{\partial Q_H}\right| = \left|\frac{\partial f_3}{\partial R_H}\right| = \left|\frac{\partial f_3}{\partial V_H}\right| = \left|\frac{\partial f_3}{\partial S_B}\right| = \left|\frac{\partial f_3}{\partial E_B}\right| = \left|\frac{\partial f_3}{\partial I_B}\right| = \left|\frac{\partial f_3}{\partial V_B}\right| = \left|\frac{\partial f_3}{\partial D_B}\right| 0 < \infty$$

For f_4,

$$f_4 = \gamma I_H - (\mu_H + \delta_2 + \alpha) Q_H$$

$$\left|\frac{\partial f_4}{\partial I_H}\right| = \left|\gamma\right| < \infty$$

$$\left|\frac{\partial f_4}{\partial Q_H}\right| = \left|-(\mu_H + \delta_2 + \alpha)\right| < \infty$$

$$\left|\frac{\partial f_4}{\partial S_H}\right| = \left|\frac{\partial f_4}{\partial E_H}\right| = \left|\frac{\partial f_4}{\partial R_H}\right| = \left|\frac{\partial f_4}{\partial V_H}\right| = \left|\frac{\partial f_4}{\partial S_B}\right| = \left|\frac{\partial f_4}{\partial E_B}\right| = \left|\frac{\partial f_4}{\partial I_B}\right| = \left|\frac{\partial f_4}{\partial V_B}\right| = \left|\frac{\partial f_4}{\partial D_B}\right| 0 < \infty$$

For f_5

$$f_5 = \alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H$$

$$\left|\frac{\partial f_5}{\partial I_H}\right| = \left|\alpha\right| < \infty$$

$$\left|\frac{\partial f_5}{\partial Q_H}\right| = \left|\xi\right| < \infty$$

$$\left|\frac{\partial f_5}{\partial R_H}\right| = \left|-(\mu_H + \varepsilon)\right| < \infty$$

$$\left|\frac{\partial f_5}{\partial S_H}\right| = \left|\frac{\partial f_5}{\partial E_H}\right| = \left|\frac{\partial f_5}{\partial V_H}\right| = \left|\frac{\partial f_5}{\partial S_B}\right| = \left|\frac{\partial f_5}{\partial E_B}\right| = \left|\frac{\partial f_5}{\partial I_B}\right| = \left|\frac{\partial f_5}{\partial V_B}\right| = \left|\frac{\partial f_5}{\partial D_B}\right| 0 < \infty$$

For f_6,

$$f_6 = \sigma S_H - (\mu_H + \rho) V_H$$

$$\left|\frac{\partial f_6}{\partial S_H}\right| = |\sigma| < \infty$$

$$\left|\frac{\partial f_6}{\partial V_H}\right| = |-(\mu_H + \rho)| < \infty$$

$$\left|\frac{\partial f_6}{\partial E_H}\right| = \left|\frac{\partial f_6}{\partial I_H}\right| = \left|\frac{\partial f_6}{\partial Q_H}\right| = \left|\frac{\partial f_6}{\partial R_H}\right| = \left|\frac{\partial f_6}{\partial S_B}\right| = \left|\frac{\partial f_6}{\partial E_B}\right| = \left|\frac{\partial f_6}{\partial I_B}\right| = \left|\frac{\partial f_6}{\partial V_B}\right| = \left|\frac{\partial f_6}{\partial D_B}\right| 0 < \infty$$

For f_7 ,

$$f_7 = \Lambda_B - k_2 S_B + \psi V_B - \tau S_B - \mu_B S_B$$

$$\left|\frac{\partial f_7}{\partial S_B}\right| = |-(k_2 + \tau + \mu_B)| < \infty$$

$$\left|\frac{\partial f_7}{\partial V_B}\right| = |\psi| < \infty$$

$$\left|\frac{\partial f_7}{\partial S_H}\right| = \left|\frac{\partial f_7}{\partial E_H}\right| = \left|\frac{\partial f_7}{\partial I_H}\right| = \left|\frac{\partial f_7}{\partial Q_H}\right| = \left|\frac{\partial f_7}{\partial R_H}\right| = \left|\frac{\partial f_7}{\partial V_H}\right| = \left|\frac{\partial f_7}{\partial E_B}\right| = \left|\frac{\partial f_7}{\partial I_B}\right| = \left|\frac{\partial f_7}{\partial D_B}\right| 0 < \infty$$

For f_8

$$f_8 = k_2 S_B - (\mu_B + \chi) E_B + (1 - \phi) V_B$$

$$\left|\frac{\partial f_8}{\partial S_B}\right| = |k_2| < \infty$$

$$\left|\frac{\partial f_8}{\partial E_B}\right| = |-(\mu_B + \chi)| < \infty$$

$$\left|\frac{\partial f_8}{\partial V_B}\right| = |(1 - \phi)| < \infty$$

$$\left|\frac{\partial f_8}{\partial S_H}\right| = \left|\frac{\partial f_8}{\partial E_H}\right| = \left|\frac{\partial f_8}{\partial I_H}\right| = \left|\frac{\partial f_8}{\partial Q_H}\right| = \left|\frac{\partial f_8}{\partial R_H}\right| = \left|\frac{\partial f_8}{\partial V_H}\right| = \left|\frac{\partial f_8}{\partial I_B}\right| = \left|\frac{\partial f_8}{\partial D_B}\right| 0 < \infty$$

For f_9

$$f_9 = \chi E_B - (\mu_B + \delta_B + \omega) I_B$$

$$\left|\frac{\partial f_9}{\partial E_B}\right| = |\chi| < \infty$$

$$\left|\frac{\partial f_9}{\partial I_B}\right| = |-(\mu_B + \delta_B + \omega)| < \infty$$

$$\left|\frac{\partial f_9}{\partial S_H}\right| = \left|\frac{\partial f_9}{\partial E_H}\right| = \left|\frac{\partial f_9}{\partial I_H}\right| = \left|\frac{\partial f_9}{\partial Q_H}\right| = \left|\frac{\partial f_9}{\partial R_H}\right| = \left|\frac{\partial f_9}{\partial V_H}\right| = \left|\frac{\partial f_9}{\partial S_B}\right| = \left|\frac{\partial f_9}{\partial V_B}\right| = \left|\frac{\partial f_9}{\partial D_B}\right| 0 < \infty$$

$For \quad f_{10}$

$$f_{10} = \tau S_B - (\mu_B + \psi)V_B - (1-\phi)V_B$$

$$\left|\frac{\partial f_{10}}{\partial S_B}\right| = |\tau| < \infty$$

$$\left|\frac{\partial f_{10}}{\partial V_B}\right| = |-(\mu_B + \psi - 1 + \phi)| < \infty$$

$$\left|\frac{\partial f_{10}}{\partial S_H}\right| = \left|\frac{\partial f_{10}}{\partial E_H}\right| = \left|\frac{\partial f_{10}}{\partial I_H}\right| = \left|\frac{\partial f_{10}}{\partial Q_H}\right| = \left|\frac{\partial f_{10}}{\partial R_H}\right| = \left|\frac{\partial f_{10}}{\partial V_H}\right| = \left|\frac{\partial f_{10}}{\partial E_B}\right| = \left|\frac{\partial f_{10}}{\partial I_B}\right| = \left|\frac{\partial f_{10}}{\partial D_B}\right| 0 < \infty$$

$For \quad \mathrm{f}_{11}$

$$\mathrm{f}_{11} = (\delta_B + \omega)I_B - \pi D_B$$

$$\left|\frac{\partial f_{11}}{\partial I_B}\right| = |\delta_B + \omega| < \infty$$

$$\left|\frac{\partial f_{11}}{\partial D_B}\right| = |-\pi| < \infty$$

$$\left|\frac{\partial f_{11}}{\partial S_H}\right| = \left|\frac{\partial f_{11}}{\partial E_H}\right| = \left|\frac{\partial f_{11}}{\partial I_H}\right| = \left|\frac{\partial f_{11}}{\partial Q_H}\right| = \left|\frac{\partial f_{11}}{\partial R_H}\right| = \left|\frac{\partial f_{11}}{\partial V_H}\right| = \left|\frac{\partial f_{11}}{\partial S_B}\right| = \left|\frac{\partial f_{11}}{\partial E_B}\right| = \left|\frac{\partial f_{11}}{\partial V_B}\right| = 0 < \infty$$

Clearly all these partial derivatives are continuous and bounded. Hence by Theorem 2, there exists a unique solution of equation (3.12)-(3.22) in the region D.

4.2.2 Invariant region

Lemma 4.1.3

Let $(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b)$ be the solution of the model equations (3.12) to (3.22) with the initial conditions and biological feasible region given by the set

$$\Omega = \Omega_H \cup \Omega_B \text{ where } \Omega_H = \left\{ (S_H, E_H, I_H, Q_H, R_H, V_H) \in R_+^6 : N_H \leq \frac{\Lambda_H}{\mu_H} \right\}$$

$$\Omega_b = \left\{ (S_B, \mathrm{E}_B, \mathrm{I}_B, \mathrm{V}_B, \mathrm{D}_B) \in R_+^5 : N_B \leq \frac{\Lambda_B}{\mu_B} \right\}$$

Proof

Adding equation (3.12 to 3.17)

$$\frac{N_H(t)}{dt} = \frac{S_H(t)}{dt} + \frac{E_H(t)}{dt} + \frac{I_H(t)}{dt} + \frac{Q_H(t)}{dt} + \frac{R_H(t)}{dt} + \frac{V_H(t)}{dt} \tag{4.5}$$

$$= \Lambda_H - (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H + \rho V_H - \sigma S_H - \mu S_H + \varepsilon R_H$$
$$+ (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H - (\mu + \eta) E_H + \eta E_H - (\mu + \delta_1 + \xi + \gamma) I_H$$
$$+ \gamma I_H - (\mu + \delta_2 + \alpha) Q_H + \alpha Q_H + \xi I_H - (\mu + \varepsilon) R_H + \sigma S_H - (\mu + \rho) V_H$$

$$= \Lambda_H - \mu_H S_H - \mu_H E_H - \delta_1 I_H - \mu_H I_H - \mu_H \delta_2 Q_H - \delta_2 Q_H - \mu_H R_H - \mu_H V_H$$

$$= \Lambda_H - \mu_H (S_H + E_H + I_H + Q_H + R_H + V_H) - \delta_1 I_H - \delta_2 Q_H$$

$$\frac{N_H(t)}{dt} = \Lambda_H - \mu_H (N_H) - \delta_1 I_H - \delta_2 Q_H$$

$$\frac{N_H(t)}{dt} \leq \Lambda_H - \mu_H (N_H)$$

$$\frac{N_H(t)}{dt} + \mu_H (N_H) \leq \Lambda_H \tag{4.6}$$

Using integrating factor (IF) to integrate

$$N_H I.F \leq \Lambda_H \int I.F\, dt$$

and $IF = \ell^{\int \mu_H dt} = \ell^{\mu_H T}$

$$\therefore N_H \ell^{\mu_H t} \leq \Lambda_H \int \ell^{\mu_H t} dx$$

$$N_H \ell^{\mu_H t} \leq \Lambda_H . \frac{\ell^{\mu_H t}}{\mu_H} + c$$

$$N_H \leq \left(\Lambda_H . \frac{\ell^{\mu_H t}}{\mu_H} + c \right) \ell^{-\mu_H t}$$

$$N_H(t) \leq \frac{\Lambda_H}{\mu_H} + c\,\ell^{-\mu_H t} \tag{4.7}$$

$at\ t = 0,$

$$N_H(0) \leq \frac{\Lambda_H}{\mu_H} + c$$

$$\Rightarrow\ N_H(0) - \frac{\Lambda_H}{\mu_H} \leq c$$

By substituting the value of c into (4.7)

$$N_H(t) \leq \frac{\Lambda_H}{\mu_H} + \left(N_H(0) - \frac{\Lambda_H}{\mu_H} \right) \ell^{-\mu_H t} \tag{4.8}$$

As $t \to \infty$, $N_H(t) \to \frac{\Lambda_H}{\mu_H}$

As $t \to \infty$, the population size $N_H(t) \to \frac{\Lambda_H}{\mu_H}$ which implies that

$0 \leq N_H(t) \leq \frac{\Lambda_H}{\mu_H}$. Thus the feasible solution set of the system equation of the model enter and remain in the region

$$\Omega_H = \left\{ \left(S_H, +E_H, I_H, Q_H, R_H V_H \right) \in \Re_+^6 : N_H \leq \frac{\Lambda_H}{\mu_H} \right\} \tag{4.9}$$

Similarly

Adding equation (3.18-3.22)

$$\frac{N_B(t)}{dt} = \frac{S_B(t)}{dt} + \frac{E_B(t)}{dt} + \frac{I_B(t)}{dt} + \frac{V_B(t)}{dt}$$

$$\frac{N_B(t)}{dt} = \Lambda_B - \beta_B S_B (I_B + D_B) + \psi V_B - \tau S_B - \mu_B S_B + \beta_B S_B (I_B + D_B)$$
$$-(\mu_B + \chi) E_B + (1 - \phi) V_B + \chi E_B - (\mu_B + \delta_B + \omega) I_B + \tau S_B$$
$$-(\mu_B + \psi) V_B - (1 - \phi) V_B + (\delta_B + \omega) I_B - \pi D_B$$

$$\frac{N_B(t)}{dt} = \Lambda_B - \mu_B S_B - \mu_B E_B - \mu_B I_B - \mu_B V_B - \pi D_B$$

$$= \Lambda_B - \mu_B(S_B + E_B + I_B + V_B) - \pi D_B$$

$$\frac{N_B(t)}{dt} = \Lambda_B - \mu_B(N_B) - \pi D_B$$

$$\frac{N_B(t)}{dt} \leq \Lambda_B - \mu_B(N_B)$$

$$\frac{N_B(t)}{dt} + \mu_B(N_B) \leq \Lambda_B$$

Using integrating factor (IF) to integrate

$$N_B I.F \leq \Lambda_B \int I.F\, dt$$

and $IF = \ell^{\int \mu_H dt} = \ell^{\mu_H T}$

$$\therefore N_B \ell^{\mu_B t} \leq \Lambda_B \int \ell^{\mu_B t} dx$$

$$N_B \ell^{\mu_B t} \leq \Lambda_B . \frac{\ell^{\mu_B t}}{\mu_B} + c$$

$$N_B \leq \left(\Lambda_B . \frac{\ell^{\mu_B t}}{\mu_B} + c \right) \ell^{-\mu_B t}$$

$$N_B(t) \leq \frac{\Lambda_B}{\mu_B} + c\,\ell^{-\mu_B t} \qquad (4.10)$$

$at\ t = 0,$

$$N_B(0) \leq \frac{\Lambda_B}{\mu_B} + c$$

$$\Rightarrow N_H(0) - \frac{\Lambda_H}{\mu_H} \leq c$$

By substituting the value of c into (4.10)

$$N_B(t) \leq \frac{\Lambda_B}{\mu_B} + \left(N_B(0) - \frac{\Lambda_B}{\mu_B} \right) \ell^{-\mu_B t} \qquad (4.11)$$

As $t \to \infty$, $N_B(t) \to \frac{\Lambda_B}{\mu_B}$

As $t \to \infty$, the population size $N_B(t) \to \frac{\Lambda_B}{\mu_B}$ which implies that $0 \le N_B(t) \le \frac{\Lambda_B}{\mu_B}$. Thus the feasible solution set of the system equation of the model enter and remain in the region:

$$\Omega_B = \left\{ (S_B, +E_B, I_B, V_B) \in \Re_+^4 : N_B \le \frac{\Lambda_B}{\mu_B} \right\} \qquad (4.12)$$

4.2.3 Positivity of solution of the model

Since the model given by (3.12 to 3.22) monitors human and bird population, it is significant to show that all the state variables in the model is non negative for all time.

Theorem 3: For non negative initial conditions of the model equations given by (3.12 to 3.22), the solution $(S_H, V_H, E_H, Q_H, I_H, J_H, R_H, D_H, S_b, I_b)$ of the model equations (3.12 to 3.22) are all non negative for all time $t \ge 0$

Proof:

From equation (3.12),

$$\frac{dS_H}{dt} = \Lambda_H + \rho V_H + \varepsilon R_H - (k_1 + \sigma + \mu_H) S_H$$

$$\text{where } k_1 = (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B)$$

$$\frac{dS_H}{dt} \ge -(k_1 + \sigma + \mu_H) S_H$$

$$\frac{dS_H}{S_H} \ge -(k_1 + \sigma + \mu_H) dt$$

Integrating both sides we have

$$\ln S_H \ge -(k_1 + \sigma + \mu_H) t + c_1$$

Taking the exponent of both side

$$S_H \ge \ell^{-(k_1 + \sigma + \mu_H) t + c_1}$$

$$S_H(t) \geq C_1 \ell^{-(k_1+\sigma+\mu_H)t}$$

At $t = 0$

$$S_H(0) \geq C_1$$

$$\therefore S_H(t) \geq S_H(0) \ell^{-(k_1+\sigma+\mu_H)t}$$

$$S_H(0) > 0 \quad \Rightarrow \quad S_H(t) > 0$$

From equation (3.13),

$$\frac{dE_H}{dt} = k_1 S_H - (\mu_H + \eta) E_H$$

$$\frac{dE_H}{dt} \geq -(\mu_H + \eta) E_H$$

$$\frac{dE_H}{E_H} \geq -(\mu_H + \eta) dt$$

Integrating both sides we have

$$\ln E_H \geq -(\mu_H + \eta)t + c_2$$

Taking the exponent of both side

$$E_H \geq \ell^{-(\mu_H+\eta)t+c_2}$$

$$E_H(t) \geq C_2 \ell^{-(\mu_H+\eta)t}$$

At $t = 0$

$$E_H(0) \geq C_2$$

$$\therefore E_H(t) \geq E_H(0) \ell^{-(\mu_H+\eta)t}$$

$$E_H(0) > 0 \quad \Rightarrow \quad E_H(t) > 0$$

From equation (3.14),

$$\frac{dI_H}{dt} = \eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H$$

$$\frac{dI_H}{dt} \geq -(\mu_H + \delta_1 + \xi + \gamma) I_H$$

$$\frac{dI_H}{I_H} \geq -(\mu_H + \delta_1 + \xi + \gamma) dt$$

Integrating both sides we have

$$\ln I_H \geq -(\mu_H + \delta_1 + \xi + \gamma)t + c_3$$

Taking the exponent of both side

$$I_H \geq \ell^{-(\mu_H + \delta_1 + \xi + \gamma)t + c_3}$$

$$I_H(t) \geq C_3 \ell^{-(\mu_H + \delta_1 + \xi + \gamma)t}$$

At $t = 0$

$$I_H(0) \geq C_3$$

$$\therefore I_H(t) \geq I_H(0) \ell^{-(\mu_H + \delta_1 + \xi + \gamma)t}$$

$$I_H(0) > 0 \quad \Rightarrow \quad I_H(t) > 0$$

From equation (3.15),

$$\frac{dQ_H}{dt} = \gamma I_H - (\mu_H + \delta_2 + \alpha) Q_H$$

$$\frac{dQ_H}{dt} \geq -(\mu_H + \delta_2 + \alpha) Q_H$$

$$\frac{dQ_H}{Q_H} \geq -(\mu_H + \delta_2 + \alpha) dt$$

Integrating both sides we have

$$\ln Q_H \geq -(\mu_H + \delta_2 + \alpha)t + c_4$$

Taking the exponent of both side

$$Q_H \geq \ell^{-(\mu_H+\delta_2+\alpha)t+c_4}$$

$$Q_H(t) \geq C_4 \ell^{-(\mu_H+\delta_2+\alpha)t}$$

$$\text{At } t = 0$$

$$Q_H(0) \geq C_4$$

$$\therefore Q_H(t) \geq Q_H(0) \ell^{-(\mu_H+\delta_2+\alpha)t}$$

$$Q_H(0) > 0 \quad \Rightarrow \quad Q_H(t) > 0$$

From equation (3.16),

$$\frac{dR_H}{dt} = \alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H$$

$$\frac{dR_H}{dt} \geq -(\mu_H + \varepsilon) R_H$$

$$\frac{dR_H}{R_H} \geq -(\mu_H + \varepsilon) dt$$

Integrating both sides we have

$$\ln R_H \geq -(\mu_H + \varepsilon)t + c_5$$

Taking the exponent of both side

$$R_H \geq \ell^{-(\mu_H+\varepsilon)t+c_5}$$

$$R_H(t) \geq C_5 \ell^{-(\mu_H+\varepsilon)t}$$

$$\text{At } t = 0$$

$$R_H(0) \geq C_5$$

$$\therefore S_H(t) \geq S_H(0) \ell^{-(k_1+\sigma+\mu_H)t}$$

$$R_H(0) > 0 \quad \Rightarrow \quad R_H(t) > 0$$

From equation (3.17),

$$\frac{dV_H}{dt} = \sigma S_H - (\mu_H + \rho)V_H$$

$$\frac{dV_H}{dt} \geq -(\mu_H + \rho)V_H$$

$$\frac{dV_H}{V_H} \geq -(\mu_H + \rho)dt$$

Integrating both sides we have

$$\ln V_H \geq -(\mu_H + \rho)t + c_6$$

Taking the exponent of both side

$$V_H \geq \ell^{-(\mu_H + \rho)t + c_6}$$

$$V_H(t) \geq C_6 \ell^{-(\mu_H + \rho)t}$$

$$\text{At } t = 0$$

$$V_H(0) \geq C_6$$

$$\therefore V_H(t) \geq V_H(0)\ell^{-(\mu_H + \rho)t}$$

$$V_H(0) > 0 \quad \Rightarrow \quad V_H(t) > 0$$

From equation (3.18),

$$\frac{dS_B}{dt} = \Lambda_B + \psi V_B - (k_2 + \tau + \mu_B)S_B$$

$$\text{where } k_2 = \beta_B(I_B + D_B)$$

$$\frac{dS_B}{dt} \geq -(k_2 + \tau + \mu_B)S_B$$

$$\frac{dS_B}{S_B} \geq -(k_2 + \tau + \mu_B)dt$$

Integrating both sides we have

$$\ln S_B \geq -(k_2 + \tau + \mu_B)t + c_7$$

Taking the exponent of both side

$$S_B \geq \ell^{-(k_2+\tau+\mu_B)t+c_7}$$

$$S_B(t) \geq C_7 \ell^{-(k_2+\tau+\mu_B)t}$$

$$\text{At } t = 0$$

$$S_B(0) \geq C_7$$

$$\therefore S_B(t) \geq S_B(0)\ell^{-(k_2+\tau+\mu_B)t}$$

$$S_B(0) > 0 \quad \Rightarrow \quad S_B(t) > 0$$

From equation (3.19),

$$\frac{dE_B}{dt} = k_2 S_B + (1-\phi)V_B - (\mu_B + \chi)E_B$$

$$\frac{dE_B}{dt} \geq -(\mu_B + \chi)E_B$$

$$\frac{dE_B}{E_B} \geq -(\mu_B + \chi)dt$$

Integrating both sides we have

$$\ln E_B \geq -(\mu_B + \chi)t + c_8$$

Taking the exponent of both side

$$E_B \geq \ell^{-(\mu_B+\chi)t+c_8}$$

$$E_B(t) \geq C_8 \ell^{-(\mu_B+\chi)t}$$

$$\text{At } t = 0$$

$$E_B(0) \geq C_8$$

$$\therefore E_B(t) \geq E_B(0)^{\ell^{-(\mu_B+\chi)t}}$$

$$E_B(0) > 0 \quad \Rightarrow \quad E_B(t) > 0$$

From equation (3.20),

$$\frac{dI_B}{dt} = \chi E_B - (\mu_B + \delta_B + \omega) I_B$$

$$\frac{dE_B}{dt} \geq -(\mu_B + \delta_B + \omega) I_B$$

$$\frac{dI_B}{I_B} \geq -(\mu_B + \delta_B + \omega) dt$$

Integrating both sides we have

$$\ln I_B \geq -(\mu_B + \delta_B + \omega)t + c_9$$

Taking the exponent of both side

$$I_B \geq \ell^{-(\mu_B+\delta_B+\omega)t+c_9}$$

$$I_B(t) \geq C_9 \ell^{-(\mu_B+\delta_B+\omega)t}$$

At $t = 0$

$$I_B(0) \geq C_9$$

$$\therefore I_B(t) \geq I_B(0)^{\ell^{-(\mu_B+\delta_B+\omega)t}}$$

$$I_B(0) > 0 \quad \Rightarrow \quad I_B(t) > 0$$

From equation (3.21),

$$\frac{dV_B}{dt} = \tau S_B - (\mu_B + \psi - 1 + \phi) V_B$$

$$\frac{dV_B}{dt} \geq -(\mu_B + \psi - 1 + \phi) V_B$$

$$\frac{dV_B}{V_B} \geq -(\mu_B + \psi - 1 + \phi)dt$$

Integrating both sides we have

$$\ln V_B \geq -(\mu_B + \psi - 1 + \phi)t + c_{10}$$

Taking the exponent of both side

$$V_B \geq \ell^{-(\mu_B + \psi - 1 + \phi)t + c_{10}}$$

$$V_B(t) \geq C_{10}\ell^{-(\mu_B + \psi - 1 + \phi)t}$$

$$\text{At } t = 0$$

$$V_B(0) \geq C_{10}$$

$$\therefore V_B(t) \geq V_B(0)\ell^{-(\mu_B + \psi - 1 + \phi)t}$$

$$V_B(0) > 0 \quad \Rightarrow \quad V_B(t) > 0$$

From equation (3.22),

$$\frac{dD_B}{dt} = (\delta_B + \omega)I_B - \pi D_B$$

$$\frac{dD_B}{dt} \geq -\pi D_B$$

$$\frac{dD_B}{D_B} \geq -\pi dt$$

Integrating both sides we have

$$\ln D_B \geq -\pi t + c_{11}$$

Taking the exponent of both side

$$D_B \geq \ell^{-\pi t + c_{11}}$$

$$D_B(t) \geq C_{11}\ell^{-\pi}$$

At $t = 0$

$D_B(0) \geq C_{11}$

$\therefore D_B(t) \geq D_B(0)^{\ell - \pi t}$

$D_B(0) > 0 \Rightarrow D_B(t) > 0$

4.2.4 Existence of disease free equilibrium state

The equilibrium state for the system is obtained by setting the model equation to zero. i.e.

$$\frac{dS_H}{dt} = \frac{dE_H}{dt} = \frac{dI_H}{dt} = \frac{dQ_H}{dt} = \frac{dR_H}{dt} = \frac{dV_H}{dt} = \frac{dS_B}{dt} = \frac{dE_B}{dt} = \frac{dI_B}{dt} = \frac{dD_B}{dt} = \frac{dV_B}{dt} = 0 \quad (4.13)$$

Thus, at equilibrium, equation (3.12) to (3.22) becomes

$$\Lambda_H - (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H = 0 \quad (4.14)$$

$$(\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H - (\mu_H + \eta) E_H = 0 \quad (4.15)$$

$$\eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H = 0 \quad (4.16)$$

$$\gamma I_H - (\mu_H + \delta_2 + \alpha) Q_H = 0 \quad (4.17)$$

$$\alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H = 0 \quad (4.18)$$

$$\sigma S_H - (\mu_H + \rho) V_H = 0 \quad (4.19)$$

$$\Lambda_B - \beta_B S_B (I_B + D_B) + \psi V_B - \tau S_B - \mu_B S_B = 0 \quad (4.20)$$

$$\beta_B S_B (I_B + D_B) - (\mu_B + \chi) E_B + (1 - \phi) V_B = 0 \quad (4.21)$$

$$\chi E_B - (\mu_B + \delta_B + \omega) I_B = 0 \quad (4.22)$$

$$\tau S_B - (\mu_B + \psi) V_B - (1 - \phi) V_B = 0 \quad (4.23)$$

$$(\delta_B + \omega) I_B - \pi D_B \quad (4.24)$$

At disease free,

$$E_H = I_H = Q_H = R_H = E_B = I_B = D_B = 0 \quad (4.25)$$

Substituting equation (4.25) into equation (4.14) and (4.19), we have

$$\Lambda_H - (\sigma + \mu) S^*_H + \rho V^*_H = 0 \quad (4.26)$$

$$\sigma S^*_H - (\mu + \rho) V^*_H = 0 \quad (4.27)$$

$$V^*_H = \frac{\sigma S^*_H}{(\mu_H + \rho)}$$

$$V^*_H = \left(\frac{\sigma}{(\mu_H + \rho)}\right) S^*_H \tag{4.28}$$

Substituting (4.28) into (4.26)

$$\Lambda_H - (\sigma + \mu) S^*_H + \rho \left(\frac{\sigma}{(\mu_H + \rho)}\right) S^*_H = 0$$

$$\Lambda_H - \left((\sigma + \mu) - \frac{\rho\sigma}{(\mu_H + \rho)}\right) S^*_H = 0$$

$$\left(\frac{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}{(\mu_H + \rho)}\right) S^*_H = \Lambda_H$$

$$S^*_H = \frac{\Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} \tag{4.29}$$

Substitute (4.29) into (4.28) to get V_H

$$V^*_H = \left(\frac{\sigma}{(\mu_H + \rho)}\right)\left(\frac{\Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}\right)$$

$$V^*_H = \frac{\sigma\Lambda_H}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} \tag{4.30}$$

Substituting equation (4.25) into equation (4.20) and (4.23), we have

$$\Lambda_B - (\tau + \mu_B) S^*_B + \psi V^*_B = 0 \tag{4.31}$$

$$\tau S^*_B - (\mu + \psi) V^*_B - (1 - \phi) V^*_B = 0 \tag{4.32}$$

$$\tau S^*_B - (\mu + \psi + 1 - \phi) V^*_B = 0$$

$$V^*_B = \left(\frac{\tau}{(\mu + \psi + 1 - \phi)}\right) S^*_B \tag{4.33}$$

Substituting equation (4.33) into (4.31)

$$\Lambda_B - (\tau + \mu_B) S^*_B + \left(\frac{\psi\tau}{(\mu + \psi + 1 - \phi)}\right) S^*_B = 0$$

$$\left((\tau + \mu_B) - \frac{\psi \tau}{(\mu + \psi + 1 - \phi)} \right) S^*_B = \Lambda_B$$

$$\left((\tau + \mu_B) - \frac{\psi \tau}{(\mu + \psi + 1 - \phi)} \right) S^*_B = \Lambda_B$$

$$\left(\frac{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau}{(\mu + \psi + 1 - \phi)} \right) S^*_B = \Lambda_B$$

$$S^*_B = \frac{\Lambda_B (\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau} \tag{4.34}$$

Substitute (4.34) into (4.33)

$$V^*_B = \left(\frac{\tau}{(\mu + \psi + 1 - \phi)} \right) \left(\frac{\Lambda_B (\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau} \right)$$

$$V^*_B = \left(\frac{\tau \Lambda_B}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau} \right) \tag{4.35}$$

Therefore, the disease -free equilibrium state is

$$E_O \left(S^*_H, E^*_H, I^*_H, Q^*_H, R^*_H, V^*_H, S^*_B, E^*_B, I^*_B, V^*_B, D^*_B, \right) =$$

$$\left(\frac{\Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, 0, 0, 0, 0, \frac{\sigma \Lambda_H}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, \frac{\Lambda_B (\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau}, 0, 0, \frac{\tau \Lambda_B}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi \tau}, 0 \right) \tag{4.36}$$

The equilibrium state exist for $(\sigma + \mu)(\mu_H + \rho) > \rho\sigma$ and $(\tau + \mu_B)(\mu + \psi + 1 - \phi) > \psi \tau$

4.2.5 Reproduction number

The basic reproduction number is denoted by R_0. It is an important parameter that is used to study the behaviour of epidemiological models. In order to obtain the basic reproduction number R_0

We apply the next generation matrix technique by Diekman and Heesterbeck (2002) to obtain the basic reproduction number of the model by considering the equations of the infected compartments of the system (3.2), (3.3), (3.4), (3.8), (3.9) and (3.10) in chapter three.

Let F_i be the rate of appearance of new infection in the compartment i and V_i be the rate of transfer of individuals out of i, given the disease free equilibrium, then R_0 of the model is the spectral radius (largest Eigen values) of the next generation matrix denoted by $G = FV^{-1}$. The R_0 is given by $R_0 = \rho(FV^{-1})$.

$$F_i(x) = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H \\ 0 \\ 0 \\ \beta_B S_B (I_B + D_B) \\ 0 \\ 0 \end{bmatrix} \tag{4.37}$$

Evaluating the Jacobean matrix of $F(x)$

$$\frac{\partial F_i}{\partial x_i} = \begin{bmatrix} \frac{\partial F_1}{\partial E_H} & \frac{\partial F_1}{\partial I_H} & \frac{\partial F_1}{\partial Q_H} & \frac{\partial F_1}{\partial E_B} & \frac{\partial F_1}{\partial I_B} & \frac{\partial F_1}{\partial D_B} \\ \frac{\partial F_2}{\partial E_H} & \frac{\partial F_2}{\partial I_H} & \frac{\partial F_2}{\partial Q_H} & \frac{\partial F_2}{\partial E_B} & \frac{\partial F_2}{\partial I_B} & \frac{\partial F_2}{\partial D_B} \\ \frac{\partial F_3}{\partial E_H} & \frac{\partial F_3}{\partial I_H} & \frac{\partial F_3}{\partial Q_H} & \frac{\partial F_3}{\partial E_B} & \frac{\partial F_3}{\partial I_B} & \frac{\partial F_3}{\partial D_B} \\ \frac{\partial F_4}{\partial E_H} & \frac{\partial F_4}{\partial I_H} & \frac{\partial F_4}{\partial Q_H} & \frac{\partial F_4}{\partial E_B} & \frac{\partial F_4}{\partial I_B} & \frac{\partial F_4}{\partial D_B} \\ \frac{\partial F_5}{\partial E_H} & \frac{\partial F_5}{\partial I_H} & \frac{\partial F_5}{\partial Q_H} & \frac{\partial F_5}{\partial E_B} & \frac{\partial F_5}{\partial I_B} & \frac{\partial F_5}{\partial D_B} \\ \frac{\partial F_6}{\partial E_H} & \frac{\partial F_6}{\partial I_H} & \frac{\partial F_6}{\partial Q_H} & \frac{\partial F_6}{\partial E_B} & \frac{\partial F_6}{\partial I_B} & \frac{\partial F_6}{\partial D_B} \end{bmatrix} \tag{4.38}$$

Substituting equation (4.37) into equation (4.38) and evaluating at disease free equilibrium E_0, we obtain

$$F = \begin{bmatrix} 0 & \beta_H S_H & 0 & 0 & \beta_{BH} S_H & \beta_{DH} S_H \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_B S_B & \beta_B S_B \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.39}$$

Evaluating F at the disease free equilibrium point

$$F = \begin{bmatrix} 0 & \frac{\beta_H \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} & 0 & 0 & \frac{\beta_{BH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} & \frac{\beta_{DH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta_B \Lambda_B (\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} & \frac{\beta_B \Lambda_B (\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \frac{\beta_H \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, B_2 = \frac{\beta_{BH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}$$

Let

$$B_3 = \frac{\beta_{DH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} B_4 = \frac{\beta_B \Lambda_B (\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}$$

Then F becomes;

$$F = \begin{bmatrix} 0 & B_1 & 0 & 0 & B_2 & B_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_4 & B_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.40}$$

$$V_i(x) = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} (\mu_H + \eta) E_H \\ (\mu_H + \delta_1 + \xi + \gamma) I_H - \eta E_H \\ (\mu_H + \delta_2 + \alpha) Q_H - \gamma I_H \\ (\mu_B + \chi) E_B - (1 - \phi) V_B \\ (\mu_B + \delta_B + \omega) I_B - \chi E_B \\ \pi D_B - (\delta_B + \omega) I_B \end{bmatrix} \tag{4.41}$$

Evaluating the Jacobean matrix of $V(x)$

$$\frac{\partial V_i}{\partial x_i} = \begin{bmatrix} \frac{\partial V_1}{\partial E_H} & \frac{\partial V_1}{\partial I_H} & \frac{\partial V_1}{\partial Q_H} & \frac{\partial V_1}{\partial E_B} & \frac{\partial V_1}{\partial I_B} & \frac{\partial V_1}{\partial D_B} \\ \frac{\partial V_2}{\partial E_H} & \frac{\partial V_2}{\partial I_H} & \frac{\partial V_2}{\partial Q_H} & \frac{\partial V_2}{\partial E_B} & \frac{\partial V_2}{\partial I_B} & \frac{\partial V_2}{\partial D_B} \\ \frac{\partial V_3}{\partial E_H} & \frac{\partial V_3}{\partial I_H} & \frac{\partial V_3}{\partial Q_H} & \frac{\partial V_3}{\partial E_B} & \frac{\partial V_3}{\partial I_B} & \frac{\partial V_3}{\partial D_B} \\ \frac{\partial V_4}{\partial E_H} & \frac{\partial V_4}{\partial I_H} & \frac{\partial V_4}{\partial Q_H} & \frac{\partial V_4}{\partial E_B} & \frac{\partial V_4}{\partial I_B} & \frac{\partial V_4}{\partial D_B} \\ \frac{\partial V_5}{\partial E_H} & \frac{\partial V_5}{\partial I_H} & \frac{\partial V_5}{\partial Q_H} & \frac{\partial V_5}{\partial E_B} & \frac{\partial V_5}{\partial I_B} & \frac{\partial V_5}{\partial D_B} \\ \frac{\partial V_6}{\partial E_H} & \frac{\partial V_6}{\partial I_H} & \frac{\partial V_6}{\partial Q_H} & \frac{\partial V_6}{\partial E_B} & \frac{\partial V_6}{\partial I_B} & \frac{\partial V_6}{\partial D_B} \end{bmatrix} \tag{4.42}$$

Substituting equation (4.41) into equation (4.42) and evaluating at disease free equilibrium E_0, we obtain

$$V = \begin{bmatrix} (\mu_H + \eta) & 0 & 0 & 0 & 0 & 0 \\ -\eta & (\mu_H + \delta_1 + \xi + \gamma) & 0 & 0 & 0 & 0 \\ 0 & -\gamma & (\mu_H + \delta_2 + \alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mu_B + \chi) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\mu_B + \delta_B + \omega) & 0 \\ 0 & 0 & 0 & 0 & -(\delta_B + \omega) & \pi \end{bmatrix} \tag{4.43}$$

Let $A_1 = (\mu_H + \eta)$, $A_2 = (\mu_H + \delta_1 + \xi + \gamma)$, $A_3 = (\mu_H + \delta_2 + \alpha)$, $A_4 = (\mu_B + \chi)$, $A_5 = (\mu_B + \delta_B + \omega)$, $A_6 = (\delta_B + \omega)$

Then V becomes

$$V = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ -\eta & A_2 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_5 & 0 \\ 0 & 0 & 0 & 0 & -A_6 & \pi \end{bmatrix} \tag{4.44}$$

Thus, we evaluate the determinant of V

$$\det(V) = A_1 A_2 A_3 A_4 A_5 \pi \tag{4.45}$$

Matrix of the Co-factors of V is

$$\begin{bmatrix} \pi A_2A_3A_4A_5 & \eta\pi A_3A_4A_5 & \pi\eta\gamma A_4A_5 & 0 & 0 & 0 \\ 0 & \pi A_1A_3A_4A_5 & \pi\gamma A_1A_4A_5 & 0 & 0 & 0 \\ 0 & 0 & \pi A_1A_2A_4A_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi A_1A_2A_3A_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi A_1A_2A_3A_4 & -A_1A_2A_3A_4A_6 \\ 0 & 0 & 0 & 0 & 0 & A_1A_2A_3A_4A_5 \end{bmatrix} \quad (4.46)$$

Then the adjoint of V gives

$$adj(V) = \begin{bmatrix} \pi A_2A_3A_4A_5 & 0 & 0 & 0 & 0 & 0 \\ \pi\eta A_3A_4A_5 & \pi A_1A_3A_4A_5 & 0 & 0 & 0 & 0 \\ \pi\eta\gamma A_4A_5 & \pi\gamma A_1A_4A_5 & \pi A_1A_2A_4A_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi A_1A_2A_3A_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi A_1A_2A_3A_4 & 0 \\ 0 & 0 & 0 & 0 & -A_1A_2A_3A_4A_6 & A_1A_2A_3A_4A_5 \end{bmatrix} \quad (4.47)$$

Hence, $V^{-1} = \dfrac{adj(V)}{\det(V)}$

$$V^{-1} = \begin{bmatrix} \frac{1}{A_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{\eta}{A_1A_2} & \frac{1}{A_2} & 0 & 0 & 0 & 0 \\ \frac{\eta\gamma}{A_1A_2A_3} & \frac{\gamma}{A_2A_3} & \frac{1}{A_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{A_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{A_5} & 0 \\ 0 & 0 & 0 & 0 & -\frac{A_6}{\pi A_5} & \frac{1}{\pi} \end{bmatrix} \quad (4.48)$$

Hence, we obtain the matrix $G = FV^{-1}$ by multiplying equation (4.40) and equation (4.48) to obtain

$$FV^{-1} = \begin{bmatrix} 0 & B_1 & 0 & 0 & B_2 & B_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_4 & B_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{A_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{\eta}{A_1 A_2} & \frac{1}{A_2} & 0 & 0 & 0 & 0 \\ \frac{\eta\gamma}{A_1 A_2 A_3} & \frac{\gamma}{A_2 A_3} & \frac{1}{A_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{A_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{A_5} & 0 \\ 0 & 0 & 0 & 0 & -\frac{A_6}{\pi A_5} & \frac{1}{\pi} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\eta B_1}{A_1 A_2} & \frac{B_1}{A_2} & 0 & 0 & \frac{\pi B_2 - A_6 B_3}{\pi A_5} & \frac{B_3}{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-B_4(A_6 - \pi)}{\pi A_5} & \frac{B_3}{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.49}$$

Therefore, we evaluate the characteristic equation $\left|FV^{-1} - \lambda I\right| = 0$ of equation (4.49) to get

$$\left|FV^{-1} - \lambda I\right| = \begin{vmatrix} \frac{\eta B_1}{A_1 A_2} - \lambda & \frac{B_1}{A_2} & 0 & 0 & \frac{\pi B_2 - A_6 B_3}{\pi A_5} & \frac{B_3}{\pi} \\ 0 & 0 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 - \lambda & \frac{-B_4(A_6 - \pi)}{\pi A_5} & \frac{B_3}{\pi} \\ 0 & 0 & 0 & 0 & 0 - \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 - \lambda \end{vmatrix} = 0$$

$$\lambda^6 - \frac{\eta B_1}{A_1 A_2}\lambda^5 = 0 \tag{4.50}$$

$$\lambda^5\left(\lambda - \frac{\eta B_1}{A_1 A_2}\right) = 0$$

$$\lambda^5 = 0 \quad \text{or} \quad \left(\lambda - \frac{\eta B_1}{A_1 A_2}\right) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0 \text{ and } \lambda_6 = \frac{\eta B_1}{A_1 A_2} \tag{4.51}$$

The reproduction number is the largest Eigen value, that is

$$R_0 = \frac{\eta B_1}{A_1 A_2} \tag{4.52}$$

Where $B_1 = \frac{\beta_H \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}$, $A_1 = (\mu + \eta)$ and $A_2 = (\mu + \delta_1 + \xi + \gamma)$,

Then the reproduction number becomes

$$R_0 = \frac{\eta \beta_H \Lambda_H (\mu_H + \rho)}{(\sigma + \mu_H)(\mu_H + \rho) - \rho\sigma} \cdot \frac{1}{(\mu_H + \eta)(\mu_H + \delta_1 + \xi + \gamma)}$$

$$R_0 = \frac{\eta \beta_H \Lambda_H (\mu_H + \rho)}{\left((\sigma + \mu_H)(\mu_H + \rho) - \rho\sigma\right)(\mu_H + \eta)(\mu_H + \delta_1 + \xi + \gamma)} \tag{4.53}$$

For the R_0 to exist, $(\sigma + \mu_H)(\mu_H + \rho) > \rho\sigma$

For the R_0 to be less than 1,

$$\left((\sigma + \mu_H)(\mu_H + \rho) - \rho\sigma\right)(\mu_H + \eta)(\mu_H + \delta_1 + \xi + \gamma) > \eta \beta_H \Lambda_H (\mu_H + \rho) \tag{4.54}$$

4.2.6 Local stability of disease free equilibrium point

Theorem 4

The local stability of the disease free equilibrium point of model equations given by (3.12 to 3.22) can be established by showing that all the eigenvalues of the Jacobian matrix of the linearized system evaluated at E_0 are negative is locally asymptotically stable (LAS).

Let

$$F_1 = \Lambda_H - (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H \tag{4.55}$$

$$F_2 = (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H - (\mu_H + \eta) E_H \tag{4.56}$$

$$F_3 = \eta E_H - (\mu_H + \delta_1 + \xi + \gamma) I_H \tag{4.57}$$

$$F_4 = \gamma I_H - (\mu_H + \delta_2 + \alpha)Q_H \tag{4.58}$$

$$F_5 = \alpha Q_H + \xi I_H - (\mu_H + \varepsilon)R_H \tag{4.59}$$

$$F_6 = \sigma S_H - (\mu_H + \rho)V_H \tag{4.60}$$

$$F_7 = \Lambda_B - \beta_B S_B(I_B + D_B) + \psi V_B - \tau S_B - \mu_B S_B \tag{4.61}$$

$$F_8 = \beta_B S_B(I_B + D_B) - (\mu_B + \chi)E_B + (1-\phi)V_B \tag{4.62}$$

$$F_9 = \chi E_B - (\mu_B + \delta_B + \omega)I_B \tag{4.63}$$

$$F_{10} = \tau S_B - (\mu_B + \psi)V_B - (1-\phi)V_B \tag{4.64}$$

$$F_{11} = (\delta_B + \omega)I_B - \pi D_B \tag{4.65}$$

Thus, the Jacobean matrix J for the system (4.55) to (4.65) is given by

$$J = \begin{bmatrix}
\frac{\partial F_1}{\partial S_H} & \frac{\partial F_1}{\partial E_H} & \frac{\partial F_1}{\partial I_H} & \frac{\partial F_1}{\partial Q_H} & \frac{\partial F_1}{\partial R_H} & \frac{\partial F_1}{\partial V_H} & \frac{\partial F_1}{\partial S_B} & \frac{\partial F_1}{\partial E_B} & \frac{\partial F_1}{\partial I_B} & \frac{\partial F_1}{\partial V_B} & \frac{\partial F_1}{\partial D_B} \\
\frac{\partial F_2}{\partial S_H} & \frac{\partial F_2}{\partial E_H} & \frac{\partial F_2}{\partial I_H} & \frac{\partial F_2}{\partial Q_H} & \frac{\partial F_2}{\partial R_H} & \frac{\partial F_2}{\partial V_H} & \frac{\partial F_2}{\partial S_B} & \frac{\partial F_2}{\partial E_B} & \frac{\partial F_2}{\partial I_B} & \frac{\partial F_2}{\partial V_B} & \frac{\partial F_2}{\partial D_B} \\
\frac{\partial F_3}{\partial S_H} & \frac{\partial F_3}{\partial E_H} & \frac{\partial F_3}{\partial I_H} & \frac{\partial F_3}{\partial Q_H} & \frac{\partial F_3}{\partial R_H} & \frac{\partial F_3}{\partial V_H} & \frac{\partial F_3}{\partial S_B} & \frac{\partial F_3}{\partial E_B} & \frac{\partial F_3}{\partial I_B} & \frac{\partial F_3}{\partial V_B} & \frac{\partial F_3}{\partial D_B} \\
\frac{\partial F_4}{\partial S_H} & \frac{\partial F_4}{\partial E_H} & \frac{\partial F_4}{\partial I_H} & \frac{\partial F_4}{\partial Q_H} & \frac{\partial F_4}{\partial R_H} & \frac{\partial F_4}{\partial V_H} & \frac{\partial F_4}{\partial S_B} & \frac{\partial F_4}{\partial E_B} & \frac{\partial F_4}{\partial I_B} & \frac{\partial F_4}{\partial V_B} & \frac{\partial F_4}{\partial D_B} \\
\frac{\partial F_5}{\partial S_H} & \frac{\partial F_5}{\partial E_H} & \frac{\partial F_5}{\partial I_H} & \frac{\partial F_5}{\partial Q_H} & \frac{\partial F_5}{\partial R_H} & \frac{\partial F_5}{\partial V_H} & \frac{\partial F_5}{\partial S_B} & \frac{\partial F_5}{\partial E_B} & \frac{\partial F_5}{\partial I_B} & \frac{\partial F_5}{\partial V_B} & \frac{\partial F_5}{\partial D_B} \\
\frac{\partial F_6}{\partial S_H} & \frac{\partial F_6}{\partial E_H} & \frac{\partial F_6}{\partial I_H} & \frac{\partial F_6}{\partial Q_H} & \frac{\partial F_6}{\partial R_H} & \frac{\partial F_6}{\partial V_H} & \frac{\partial F_6}{\partial S_B} & \frac{\partial F_6}{\partial E_B} & \frac{\partial F_6}{\partial I_B} & \frac{\partial F_6}{\partial V_B} & \frac{\partial F_6}{\partial D_B} \\
\frac{\partial F_7}{\partial S_H} & \frac{\partial F_7}{\partial E_H} & \frac{\partial F_7}{\partial I_H} & \frac{\partial F_7}{\partial Q_H} & \frac{\partial F_7}{\partial R_H} & \frac{\partial F_7}{\partial V_H} & \frac{\partial F_7}{\partial S_B} & \frac{\partial F_7}{\partial E_B} & \frac{\partial F_7}{\partial I_B} & \frac{\partial F_7}{\partial V_B} & \frac{\partial F_7}{\partial D_B} \\
\frac{\partial F_8}{\partial S_H} & \frac{\partial F_8}{\partial E_H} & \frac{\partial F_8}{\partial I_H} & \frac{\partial F_8}{\partial Q_H} & \frac{\partial F_8}{\partial R_H} & \frac{\partial F_8}{\partial V_H} & \frac{\partial F_8}{\partial S_B} & \frac{\partial F_8}{\partial E_B} & \frac{\partial F_8}{\partial I_B} & \frac{\partial F_8}{\partial V_B} & \frac{\partial F_8}{\partial D_B} \\
\frac{\partial F_9}{\partial S_H} & \frac{\partial F_9}{\partial E_H} & \frac{\partial F_9}{\partial I_H} & \frac{\partial F_9}{\partial Q_H} & \frac{\partial F_9}{\partial R_H} & \frac{\partial F_9}{\partial V_H} & \frac{\partial F_9}{\partial S_B} & \frac{\partial F_9}{\partial E_B} & \frac{\partial F_9}{\partial I_B} & \frac{\partial F_9}{\partial V_B} & \frac{\partial F_9}{\partial D_B} \\
\frac{\partial F_{10}}{\partial S_H} & \frac{\partial F_{10}}{\partial E_H} & \frac{\partial F_{10}}{\partial I_H} & \frac{\partial F_{10}}{\partial Q_H} & \frac{\partial F_{10}}{\partial R_H} & \frac{\partial F_{10}}{\partial V_H} & \frac{\partial F_{10}}{\partial S_B} & \frac{\partial F_{10}}{\partial E_B} & \frac{\partial F_{10}}{\partial I_B} & \frac{\partial F_{10}}{\partial V_B} & \frac{\partial F_{10}}{\partial D_B} \\
\frac{\partial F_{11}}{\partial S_H} & \frac{\partial F_{11}}{\partial E_H} & \frac{\partial F_{11}}{\partial I_H} & \frac{\partial F_{11}}{\partial Q_H} & \frac{\partial F_{11}}{\partial R_H} & \frac{\partial F_{11}}{\partial V_H} & \frac{\partial F_{11}}{\partial S_B} & \frac{\partial F_{11}}{\partial E_B} & \frac{\partial F_{11}}{\partial I_B} & \frac{\partial F_{11}}{\partial V_B} & \frac{\partial F_{11}}{\partial D_B}
\end{bmatrix} \tag{4.66}$$

Substituting equations (4.55) - (4.65) into equation (4.66) and evaluating at the disease free equilibrium, we obtain

$$
J = \begin{bmatrix}
-(\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B + \sigma + \mu_H) S_H & 0 & -\beta S_H & 0 & \varepsilon & \rho & 0 & 0 & -\beta_{BH} S_H & 0 & -\beta_{DH} S_H \\
(\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H & -(\mu_H + \eta) & \beta_H S_H & 0 & 0 & 0 & 0 & 0 & \beta_{BH} S_H & 0 & \beta_{DH} S_H \\
0 & \eta & -(\mu_H + \delta_1 + \xi + \gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma & -(\mu_H + \delta_2 + \alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi & \alpha & -(\mu_H + \varepsilon) & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma & 0 & 0 & 0 & 0 & -(\mu_H + \rho) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\beta_B (I_B + D_B) - \tau - \mu_B & 0 & -\beta_B S_B & \psi & -\beta_B S_B \\
0 & 0 & 0 & 0 & 0 & 0 & \beta_B (I_B + D_B) & -(\mu_B + \chi) & \beta_B S_B & (1-\phi) & \beta_B S_B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & -(\mu_B + \delta_B + \omega) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & -(\mu_B + \psi + 1 - \phi) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\delta_B + \omega) & 0 & -\pi
\end{bmatrix}
$$

$$
J(E_0) = \begin{bmatrix}
-(\sigma + \mu_H) & 0 & -\beta_H S_H & 0 & \varepsilon & \rho & 0 & 0 & -\beta_{BH} S_H & 0 & -\beta_{DH} S_H \\
0 & -(\mu_H + \eta) & \beta_H S_H & 0 & 0 & 0 & 0 & 0 & \beta_{BH} S_H & 0 & \beta_{DH} S_H \\
0 & \eta & -(\mu_H + \delta_1 + \xi + \gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma & -(\mu_H + \delta_2 + \alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi & \alpha & -(\mu_H + \varepsilon) & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma & 0 & 0 & 0 & 0 & -(\mu_H + \rho) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\tau + \mu_B) & 0 & -\beta_B S_B & \psi & -\beta_B S_B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu_B + \chi) & \beta_B S_B & (1-\lambda) & \beta_B S_B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & -(\mu_B + \delta_B + \omega) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & -(\mu_B + \psi + 1 - \lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\delta_B + \omega) & 0 & -\pi
\end{bmatrix} \qquad (4.67)
$$

Let

$$C_1 = (\sigma + \mu_H), C_2 = \beta_H S_H = \frac{\beta_H \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, C_3 = \beta_{BH} S_H = \frac{\beta_{BH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma},$$

$$C_4 = \beta_{DH} S_H = \frac{\beta_{DH} \Lambda_H (\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, C_5 = (\mu_H + \eta), C_6 = (\mu_H + \delta_1 + \xi + \gamma), C_7 = (\mu_H + \delta_2 + \alpha),$$

$$C_8 = (\mu_H + \varepsilon), C_9 = (\mu_H + \rho), C_{10} = (\tau + \mu_B), C_{11} = \beta_B S_B =, \frac{\beta_B \Lambda_B (\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}, \quad (4.69)$$

$$C_{12} = -(\mu_B + \chi), C_{13} = (1 - \lambda) C_{14} = -(\mu_B + \delta_B + \omega) C_{15} = (\mu_B + \psi + 1 - \phi), C_{16} = (\delta_B + \omega).$$

The matrix becomes

$$J(E_0) = \begin{bmatrix} -c_1 & 0 & -c_2 & 0 & \varepsilon & \rho & 0 & 0 & -c_3 & 0 & -c_4 \\ 0 & -c_5 & c_2 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & c_4 \\ 0 & \eta & -c_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & -c_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & \alpha & -c_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & -c_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_{10} & 0 & -c_{11} & \psi & -c_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{12} & c_{11} & c_{13} & c_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & -c_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & -c_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16} & 0 & -\pi \end{bmatrix} \quad (4.70)$$

Given $\left| J(E_O) - \lambda I \right| = 0$ (4.71)

Substituting equation (4.70) into equation (4.71), we obtain

$$\begin{vmatrix} -C_1 - \lambda & 0 & -C_2 & 0 & \varepsilon & \rho & 0 & 0 & -c_3 & 0 & -c_4 \\ 0 & -c_5 - \lambda & c_2 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & c_4 \\ 0 & \eta & -c_6 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & -c_7 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & \alpha & -c_8 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & -c_9 - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_{10} - \lambda & 0 & -c_{11} & \psi & -c_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{12} - \lambda & c_{11} & c_{13} & c_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & -c_{14} - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & -c_{15} - \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16} & 0 & -\pi - \lambda \end{vmatrix} \quad (4.72)$$

The eigenvalues are

$$\lambda_1 = -c_7 \quad \Rightarrow \lambda_1 < 0 \text{ since } c_1 > 0$$

$$\lambda_2 = -c_8 \Rightarrow \lambda_2 < 0 \text{ since } c_8 > 0 \quad -c_8 < 0$$

$$\lambda_3 = -\tfrac{1}{2}c_1 - \tfrac{1}{2}c_9 - \tfrac{1}{2}\sqrt{c_1^{\,2} - 2c_1c_9 + c_9^{\,2} + 4\sigma\rho}$$

$$\lambda_4 = -\tfrac{1}{2}c_1 - \tfrac{1}{2}c_9 + \tfrac{1}{2}\sqrt{c_1^{\,2} - 2c_1c_9 + c_9^{\,2} + 4\sigma\rho}$$

$$\lambda_5 = -\tfrac{1}{2}c_5 - \tfrac{1}{2}c_6 - \tfrac{1}{2}\sqrt{c_5^{\,2} - 2c_5c_6 + c_9^{\,2} + 4\eta c_2}$$

$$\lambda_6 = -\tfrac{1}{2}c_5 - \tfrac{1}{2}c_6 + \tfrac{1}{2}\sqrt{c_5^{\,2} - 2c_5c_6 + c_9^{\,2} + 4\eta c_2}$$

And the roots of

$$Z^5 + Z^4(c_{10} + c_{12} + c_{14} + c_{15} + \pi)$$
$$+ Z^3(c_{10}c_{12} + c_{10}c_{14} + c_{10}c_{15} + c_{12}c_{14} + c_{12}c_{15} + c_{14}c_{15} + \pi c_{10} + \pi c_{12} + \pi c_{14} + \pi c_{15} - \tau\psi\chi c_{11})$$
$$Z^2(-\tau\psi c_{12} - \tau\psi c_{14} - \tau\psi\pi - \chi c_{10}c_{11} - \chi c_{11}c_{15}\chi c_{11}c_{16}\chi\pi c_{11} + c_{10}c_{12}c_{14}$$
$$+ c_{10}c_{12}c_{15} + \pi c_{10}c_{12} + c_{10}c_{14}c_{15} + \pi c_{10}c_{14} + c_{12}c_{14}c_{15} + \pi c_{10}c_{15} + \pi c_{12}c_{14} + \pi c_{12}c_{15} + \pi c_{14}c_{15})$$
$$- Z(\chi c_{10}c_{11}c_{15} + \chi c_{10}c_{11}c_{15} + \chi\pi c_{10}c_{11} + \chi c_{11}c_{15}c_{16} + \pi\chi c_{11}c_{15} - c_{10}c_{12}c_{14}c_{15} - \pi c_{10}c_{12}c_{14}$$
$$- \pi c_{10}c_{12}c_{15} - \pi c_{10}c_{14}c_{15} - \pi c_{12}c_{14}c_{15} - \tau\chi\psi c_{11} - \tau\chi c_{11}c_{13} + \tau\psi c_{12}c_{14} + \tau\psi c_{12}c_{17} + \pi\tau\psi c_{14})$$
$$+ \tau\chi\psi c_{11}$$

Let $b_1 = (c_{10} + c_{12} + c_{14} + c_{15} + \pi)$,

$$b_2 = (c_{10}c_{12} + c_{10}c_{14} + c_{10}c_{15} + c_{12}c_{14} + c_{12}c_{15} + c_{14}c_{15} + \pi c_{10} + \pi c_{12} + \pi c_{14} + \pi c_{15} - \tau\psi\chi c_{11})$$

$$b_3 = \begin{pmatrix} -\tau\psi c_{12} - \tau\psi c_{14} - \tau\psi\pi - \chi c_{10}c_{11} - \chi c_{11}c_{15}\chi c_{11}c_{16}\chi\pi c_{11} + c_{10}c_{12}c_{14} + c_{10}c_{12}c_{15} \\ + \pi c_{10}c_{12} + c_{10}c_{14}c_{15} + \pi c_{10}c_{14} + c_{12}c_{14}c_{15} + \pi c_{10}c_{15} + \pi c_{12}c_{14} + \pi c_{12}c_{15} + \pi c_{14}c_{15} \end{pmatrix}$$

$$b_4 = \begin{pmatrix} (\chi c_{10}c_{11}c_{15} + \chi c_{10}c_{11}c_{15} + \chi\pi c_{10}c_{11} + \chi c_{11}c_{15}c_{16} + \pi\chi c_{11}c_{15} - c_{10}c_{12}c_{14}c_{15} - \pi c_{10}c_{12}c_{14} \\ - \pi c_{10}c_{12}c_{15} - \pi c_{10}c_{14}c_{15} - \pi c_{12}c_{14}c_{15} - \tau\chi\psi c_{11} - \tau\chi c_{11}c_{13} + \tau\psi c_{12}c_{14} + \tau\psi c_{12}c_{17} + \pi\tau\psi c_{14}) \end{pmatrix}$$

$$b_5 = (\tau\chi\psi c_{11})$$

Then the characteristic polynomial becomes

$$Z^5 + b_1 Z^4 + b_2 Z^3 + b_3 Z^2 - b_4 Z + b_5 \tag{4.73}$$

Using Routh Hurwitz stability for the polynomial

Forming the Routh Hurwitz matrices

$$D_1 = [b_1], \qquad \det D_1 = b_1$$

$$D_2 = \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}, \qquad \det D_2 = b_1 b_2$$

$$D_3 = \begin{bmatrix} b_1 & 1 & 0 \\ b_3 & b_2 & b_1 \\ 0 & 0 & b_3 \end{bmatrix} \qquad \det D_3 = b_1 b_2 b_3 - b_3^2$$

$$D_4 = \begin{bmatrix} b_1 & 1 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 \\ 0 & b_4 & b_3 & b_1 \\ 0 & 0 & 0 & b_4 \end{bmatrix} \qquad \det D_4 = b_1 b_2 b_3 b_4 - b_1^2 b_4^2 - b_3^2 b_4$$

$$D_5 = \begin{bmatrix} b_1 & 1 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ 0 & 0 & b_5 & b_4 & b_3 \\ 0 & 0 & 0 & 0 & b_5 \end{bmatrix}$$

$$\det D_5 = b_1 b_2 b_3 b_4 b_5 + 2 b_1 b_4 b_6^2 + b_2 b_3 b_5^2 - \left(b_1^2 b_4^2 b_5 + b_1 b_2 b_5^2 + b_3^2 b_4 b_5^2 + b_5^3 \right)$$

For the disease free equilibrium to be stable, the following conditions must hold

For the Eigen values;

i. For the $\lambda_3 < 0$, $c_1^2 + c_9^2 + 4\sigma\rho > 2c_1 c_9$

ii. For the $\lambda_4 < 0$, $c_1^2 + c_9^2 + 4\sigma\rho < 2c_1 c_9$

iii. For the $\lambda_5 < 0$, $c_5^2 + c_9^2 + 4\eta c_2 > 2c_5 c_6$

iv. For the $\lambda_6 < 0$, $c_5^2 + c_9^2 + 4\eta c_2 < 2c_5 c_6$

For the Routh Hurwitz matrices;

v. $b_1, b_2, b_3 > 0$

vi. $b_1 b_2 b_3 > b_3^{\,2}$

vii. $b_1 b_2 b_3 b_4 > b_1^{\,2} b_2^{\,2} + b_3^{\,2} b_4$

viii. $b_1 b_2 b_3 b_4 b_5 + 2 b_1 b_4 b_5^{\,2} + b_2 b_3 b_5^{\,2} > b_1^{\,2} b_4^{\,2} b_5 + b_1 b_2^{\,2} b_5^{\,2} + b_3^{\,2} b_4 b_5^{\,2} b_5^{\,3}$

4.2.7 Global stability of disease free equilibrium point

To establish the global stability of the disease free equilibrium of the model using the theorem by Castillo-chavez *et al* (2002). The conditions H_1 and H_2 must be satisfied.

$H_1 : \frac{dx}{dt} = H(x, 0), \mathrm{x}^0$ is globally asymptotically stable (GA

$H_2 : G(\mathrm{x}, \mathrm{z}) = pz - \hat{G}(x, z), \hat{G}(x, z) \geq 0$ for $(x, z) \in \Omega,$ where $P = \Delta_z G(\mathrm{x}^0, 0)$ is an $M - matrix$ (the off diagonal elements of P are non-negative) and is also Jacobian of $G(x, z)$

We write the model equation given by (3.12-3.22) as

$$\frac{dx}{dt} = H(X, Z)$$

$$\frac{dz}{dt} = G(X, Z), G(X, 0) = 0$$

Where $X = (S_H, R_H, V_H, S_B, V_B) \in \mathbb{R}^5$ denotes the number of un-infected individuals and $Z = (E_H, I_H, Q_H, E_B, I_B, D_B) \in \mathbb{R}^6$ denotes the number of infected individuals.

$E_o = (\mathrm{x}^0, 0)$ denotes the DFE of the system.

Take $(E_H, I_H, Q_H, E_B, I_B, D_B)$ and evaluated at

$$E_O(S_H, V_H, S_B, V_B) = \left(\frac{\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, \frac{\sigma\Lambda_H}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, \frac{\Lambda_B(\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}, \frac{\tau\Lambda_B}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} \right)$$

.

If the system satisfies the conditions H_1 and H_2 above, then according to Castillo-Chaves (2002), the following theorems holds.

Theorem: The fixed point $E_0 = (x^0, 0)$ is a globally asymptotically stable (GAS) provided that $R_0 < 1$, (L.A.S) and that conditions H_1 and H_2 are satisfied.

Proof: from (4.73), the two functions $H(x,z)$ and $G(x,z)$ are given by

$$H(x,z) = \begin{bmatrix} \Lambda_H - (\beta I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H + \rho V_H - \sigma S_H - \mu_H S_H + \varepsilon R_H \\ \alpha Q_H + \xi I_H - (\mu_H + \varepsilon) R_H \\ \sigma S_H - (\mu + \rho) V_H \\ \Lambda_B - \beta_B S_B (I_B + D_B) + \psi V_B - \tau S_B - \mu_B S_B \\ \tau S_B - (\mu + \psi) V_B - (1 - \phi) V_B \end{bmatrix} \quad (4.74)$$

$$G(x,z) = \begin{bmatrix} (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B) S_H - (\mu + \eta) E_H \\ \eta E_H - (\mu + \delta_1 + \xi + \gamma) I_H \\ \gamma I_H - (\mu + \delta_2 + \alpha) Q_H \\ \beta_B S_B (I_B + D_B) - (\mu_B + \chi) E_B + (1 - \phi) V_B \\ \chi E_B - (\mu_B + \delta_B + \omega) I_B \\ (\delta_B + \omega) I_B - \pi D_B \end{bmatrix} \quad (4.75)$$

We then consider the reduced system $\frac{dx}{dt} = H(x,0)$ from condition (1)

$$H(x,0) = \begin{bmatrix} \Lambda_H - (\sigma + \mu_H) S_H + \rho V_H \\ 0 \\ \sigma S_H - (\mu + \rho) V_H \\ \Lambda_B - (\tau + \mu_B) S_B + \psi V_B \\ \tau S_B - (\mu + \psi) V_B - (1 - \phi) V_B \end{bmatrix} \quad (4.76)$$

From the first equation of the system (4.76)

$$\frac{dS_H}{dt} = \Lambda_H - (\sigma + \mu_H) S_H + \rho V_H$$

$$\frac{dS_H}{dt} + (\sigma + \mu_H) S_H = \Lambda_H + \rho V_H$$

Using integrating factor (IF) to integrate

$$S_H IF = (\Lambda_H + \rho V_H) \int IF \, dt$$

and $IF = \ell^{\int (\sigma + \mu_H) dt} = \ell^{(\sigma + \mu_H) t}$

$$\therefore S_H \ell^{(\sigma+\mu_H)t} = (\Lambda_H + \rho V_H)\int \ell^{(\sigma+\mu_H)t} dx$$

$$S_H \ell^{(\sigma+\mu_H)t} = (\Lambda_H + \rho V_H).\frac{\ell^{(\sigma+\mu_H)t}}{\sigma+\mu_H} + c$$

$$S_H = \left[(\Lambda_H + \rho V_H).\frac{\ell^{(\sigma+\mu_H)t}}{\sigma+\mu_H} + c\right]\ell^{-(\sigma+\mu_H)t}$$

$$S_H = \left[(\Lambda_H + \rho V_H).\frac{\ell^{(\sigma+\mu_H)t}}{\sigma+\mu_H} + c\right]\ell^{-(\sigma+\mu_H)t}$$

$$S_H(t) = \frac{\Lambda_H + \rho V_H}{\sigma+\mu_H} + c\,\ell^{-(\sigma+\mu_H)t} \qquad (4.77)$$

$at\ t = 0,$

$$S_H(0) = \frac{\Lambda_H + \rho V_H}{\sigma+\mu_H} + c$$

$$\Rightarrow c = S_H(0) - \frac{\Lambda_H + \rho V_H}{\sigma+\mu_H}$$

By substituting the c into (4.77)

$$S_H(t) = \frac{(\Lambda_H + \rho V_H)}{\sigma+\mu_H} + \left[S_H(0) - \frac{(\Lambda_H + \rho V_H)}{\sigma+\mu_H}\right]\ell^{-(\sigma+\mu_H)t}$$

As $t \to \infty,\ S_H(t) \to \frac{(\Lambda_H + \rho V_H)}{\sigma+\mu_H}$

$$S_H(t) \to \frac{\Lambda_H}{\sigma+\mu_H} + \frac{\rho V_H}{\sigma+\mu_H}$$

But $V_H = \frac{\sigma\Lambda_H}{(\sigma+\mu)(\mu_H+\rho) - \rho\sigma}$

$$S_H(t) \to \frac{\Lambda_H}{\sigma+\mu_H} + \frac{\rho\sigma\Lambda_H}{(\sigma+\mu_H)\left((\sigma+\mu_H)(\mu_H+\rho) - \rho\sigma\right)}$$

$$\therefore \text{As } t \to \infty,\ S_H(t) \to \frac{\left((\sigma+\mu_H)(\mu_H+\rho) - \rho\sigma\right)\Lambda_H + \rho\sigma\Lambda_H}{(\sigma+\mu_H)\left((\sigma+\mu_H)(\mu_H+\rho) - \rho\sigma\right)}$$

From the third equation of the system (4.76)

$$\frac{dV_H}{dt} = \sigma S_H - (\mu_H + \rho)V_H$$

$$\frac{dV_H}{dt} + (\mu_H + \rho)V_H = \sigma S_H$$

Using integrating factor (IF) to integrate

$$V_H IF = \sigma S_H \int IF\, dt$$

and $IF = \ell^{\int(\mu_H+\rho)dt} = \ell^{(\mu_H+\rho)t}$

$$\therefore V_H \ell^{(\mu_H+\rho)t} = \sigma S_H \int \ell^{(\mu_H+\rho)t} dx$$

$$V_H \ell^{(\mu_H+\rho)t} = \sigma S_H \cdot \frac{\ell^{(\mu_H+\rho)t}}{(\mu_H+\rho)} + c$$

$$V_H = \left(\sigma S_H \cdot \frac{\ell^{(\mu_H+\rho)t}}{(\mu_H+\rho)} + c\right)\ell^{-(\mu_H+\rho)t}$$

$$V_H(t) = \frac{\sigma S_H}{\mu_H+\rho} + c\,\ell^{-(\mu_H+\rho)t} \qquad (4.78)$$

$at\ t = 0,$

$$V_H(0) = \frac{\sigma S_H}{\mu_H+\rho} + c$$

$$\Rightarrow c = V_H(0) - \frac{\sigma S_H}{\mu_H+\rho}$$

By substituting the value of c into (4.78)

$$V_H(t) = \frac{\sigma S_H}{\mu_H+\rho} + \left(S_H(0) - \frac{\sigma S_H}{\mu_H+\rho}\right)\ell^{-(\mu_H+\rho)t}$$

As $t \to \infty$, $V_H(t) \to \frac{\sigma S_H}{\mu_H+\rho}$

But $S_H = \frac{\Lambda_H(\mu_H+\rho)}{(\sigma+\mu)(\mu_H+\rho) - \rho\sigma}$

:. As $t \to \infty$, $V_H(t) \to \left(\frac{\sigma\Lambda_H(\mu_H+\rho)}{(\mu_H+\rho)\left((\sigma+\mu)(\mu_H+\rho) - \rho\sigma\right)}\right)$

Similarly from the forth equation of the system (4.76),

$$\frac{dS_B}{dt} = \Lambda_B - (\tau+\mu_B)S_B + \psi V_B$$

$$\frac{dS_B}{dt} + (\tau+\mu_B)S_B = \Lambda_B + \psi V_B$$

Using integrating factor (IF) to integrate

$$S_B IF = (\Lambda_B + \psi V_B)\int IF\, dt$$

and $IF = \ell^{\int(\tau+\mu_B)dt} = \ell^{(\tau+\mu_B)t}$

$$\therefore S_B \ell^{(\tau+\mu_B)t} = (\Lambda_B + \psi V_B)\int \ell^{((\tau+\mu_B)t} dx$$

$$S_B \ell^{(\tau+\mu_B)t} = (\Lambda_B + \psi V_B).\frac{\ell^{(\tau+\mu_B)t}}{(\tau+\mu_B)} + c$$

$$S_B = \left[(\Lambda_B + \psi V_B).\frac{\ell^{(\tau+\mu_B)t}}{(\tau+\mu_B)} + c\right]\ell^{-(\tau+\mu_B)t}$$

$$S_B = \left[(\Lambda_B + \psi V_B).\frac{\ell^{(\tau+\mu_B)t}}{(\tau+\mu_B)} + c\right]\ell^{-(\tau+\mu_B)t}$$

$$S_B(t) = \frac{\Lambda_B + \psi V_B}{(\tau+\mu_B)} + c\,\ell^{-(\tau+\mu_B)t} \qquad (4.79)$$

$at\ t = 0,$

$$S_B(0) = \frac{\Lambda_B + \psi V_B}{(\tau+\mu_B)} + c$$

$$\Rightarrow c = S_B(0) - \frac{\Lambda_B + \psi V_B}{(\tau+\mu_B)}$$

By substituting the value of c into (4.79)

$$S_B(t) = \frac{(\Lambda_B + \psi V_B)}{(\tau+\mu_B)} + \left[S_B(0) - \frac{(\Lambda_B + \psi V_B)}{(\tau+\mu_B)}\right]\ell^{-(\tau+\mu_B)t}$$

As $t \to \infty$, $S_B(t) \to \frac{\Lambda_B + \psi V_B}{(\tau+\mu_B)}$

$$S_B(t) \to \frac{\Lambda_B}{(\tau+\mu_B)} + \frac{\psi V_B}{(\tau+\mu_B)}$$

But $V_B = \frac{\tau\Lambda_B}{(\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau}$

$$S_B(t) \to \frac{\Lambda_B}{(\tau+\mu_B)} + \frac{\psi}{(\tau+\mu_B)}\left[\frac{\tau\Lambda_B}{(\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau}\right]$$

$$S_B(t) \to \frac{\Lambda_B}{(\tau+\mu_B)} + \frac{\tau\psi\Lambda_B}{(\tau+\mu_B)\left((\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau\right)}$$

$$S_B(t) \to \frac{\Lambda_B\left((\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau\right)+\tau\psi\Lambda_B}{(\tau+\mu_B)\left((\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau\right)}$$

$$\therefore \text{As } t\to\infty,\ S_B(t)\to \frac{\Lambda_B\left((\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau\right)+\tau\psi\Lambda_B}{(\tau+\mu_B)\left((\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau\right)}$$

From the last equation of the system (4.76)

$$\frac{dV_B}{dt}=\tau S_B-(\mu_B+\psi+1-\phi)V_B$$

$$\frac{dV_B}{dt}=\tau S_B-(\mu_B+\psi+1-\phi)V_B$$

$$\frac{dV_B}{dt}+(\mu_B+\psi+1-\phi)V_B=\tau S_B$$

Using integrating factor (IF) to integrate

$$V_B IF=\tau S_B\int IF\,dt$$

and $IF=\ell^{\int(\mu_B+\psi+1-\phi)dt}=\ell^{(\mu_B+\psi+1-\phi)t}$

$$\therefore V_B\ell^{(\mu_B+\psi+1-\phi)t}=\tau S_B\int\ell^{(\mu_B+\psi+1-\phi)t}dx$$

$$V_B\ell^{(\mu_B+\psi+1-\phi)t}=\tau S_B\cdot\frac{\ell^{(\mu_B+\psi+1-\phi)t}}{(\mu_B+\psi+1-\phi)}+c$$

$$V_B=\left(\tau S_B\cdot\frac{\ell^{(\mu_B+\psi+1-\phi)t}}{(\mu_B+\psi+1-\phi)}+c\right)\ell^{-(\mu_B+\psi+1-\phi)t}$$

$$V_B(t)=\frac{\tau S_B}{(\mu_B+\psi+1-\phi)}+c\,\ell^{-(\mu_B+\psi+1-\phi)t} \qquad (4.80)$$

$at\ t=0,$

$$V_B(0)=\frac{\tau S_B}{(\mu_B+\psi+1-\phi)}+c$$

$$\Rightarrow c=V_B(0)-\frac{\tau S_B}{(\mu_B+\psi+1-\phi)}$$

By substituting the value of c into (4.80)

$$V_H(t)=\frac{\tau S_B}{(\mu_B+\psi+1-\phi)}+\left(S_B(0)-\frac{\tau S_B}{(\mu_B+\psi+1-\phi)}\right)\ell^{-(\mu_B+\psi+1-\phi)t}$$

As $t \to \infty$, $V_B(t) \to \dfrac{\tau S_B}{(\mu_B + \psi + 1 - \phi)}$

But $S_B = \dfrac{\Lambda_B(\mu_B + \psi + 1 - \theta)}{(\tau + \mu_B)(\mu_B + \psi + 1 - \phi) - \psi\tau}$

$\therefore$ As $t \to \infty$, $V_B(t) \to \dfrac{\tau\Lambda_B(\mu_B + \psi + 1 - \theta)}{(\mu_B + \psi + 1 - \phi)\left((\tau + \mu_B)(\mu_B + \psi + 1 - \phi) - \psi\tau\right)}$

Convergence of x^0 is therefore global in Ω. This implies that

$$x^0 = \left(\frac{\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, 0, \frac{\sigma\Lambda_H}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, \frac{\Lambda_B(\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}, \frac{\tau\Lambda_B}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} \right)$$

is globally asymptotically equilibrium of $\dfrac{dx}{dt} = H(x, 0)$

We now compute $H(x, z) = PZ - \hat{G}(x, z)$ and show that $\hat{G}(x, z) \geq 0$

$$\therefore J[G(x, z)] = P \tag{4.81}$$

Where J is the Jacobian of $G(x, z)$ taken in $(E_H, I_H, Q_H, E_B, I_B, D_B)$ and evaluated at

$$E_0 = \left(\frac{\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, 0, 0, 0, 0, \frac{\sigma\Lambda_H}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma}, \frac{\Lambda_B(\mu + \psi + 1 - \lambda)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}, 0, 0, \frac{\tau\Lambda_B}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau}, 0 \right)$$

Then (4.51) gives

$$P = \begin{bmatrix} -(\mu + \eta) & \dfrac{\beta_H\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} & 0 & 0 & \dfrac{\beta_{BH}\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} & \dfrac{\beta_{DH}\Lambda_H(\mu_H + \rho)}{(\sigma + \mu)(\mu_H + \rho) - \rho\sigma} \\ \eta & -(\mu + \delta_1 + \xi + \gamma) & 0 & 0 & 0 & 0 \\ 0 & \gamma & -(\mu + \delta_2 + \alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\mu_B + \chi) & \dfrac{\beta_B\Lambda_B(\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} & \dfrac{\beta_B\Lambda_B(\mu + \psi + 1 - \phi)}{(\tau + \mu_B)(\mu + \psi + 1 - \phi) - \psi\tau} \\ 0 & 0 & 0 & \chi & -(\mu_B + \delta_B + \omega) & 0 \\ 0 & 0 & 0 & 0 & (\delta_B + \omega) & -\pi \end{bmatrix}$$

$$\therefore PZ = \begin{bmatrix} -(\mu+\eta) & \frac{\beta_H \Lambda_H (\mu_H+\rho)}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} & 0 & 0 & \frac{\beta_{BH}\Lambda_H(\mu_H+\rho)}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} & \frac{\beta_{DH}\Lambda_H(\mu_H+\rho)}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} \\ \eta & -(\mu+\delta_1+\xi+\gamma) & 0 & 0 & 0 & 0 \\ 0 & \gamma & -(\mu+\delta_2+\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\mu_B+\chi) & \frac{\beta_B\Lambda_B(\mu+\psi+1-\phi)}{(\tau+\mu_B)(\mu+\psi+1-\phi)-\psi\tau} & \frac{\beta_B\Lambda_B(\mu+\psi+1-\phi)}{(\tau+\mu_B)(\mu+\psi+1-\phi)-\psi\tau} \\ 0 & 0 & 0 & \chi & -(\mu_B+\delta_B+\omega) & 0 \\ 0 & 0 & 0 & 0 & (\delta_B+\omega) & -\pi \end{bmatrix} \begin{bmatrix} E_H \\ I_H \\ Q_H \\ E_B \\ I_B \\ D_B \end{bmatrix} \quad (4.82)$$

$$\Rightarrow PZ = \begin{bmatrix} -(\mu+\eta)E_H + \frac{\beta_H\Lambda_H(\mu_H+\rho)I_H}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} + \frac{\beta_{BH}\Lambda_H(\mu_H+\rho)I_B}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} + \frac{\beta_{BH}\Lambda_H(\mu_H+\rho)D_B}{(\sigma+\mu)(\mu_H+\rho)-\rho\sigma} \\ \eta E_H - (\mu+\delta_1+\xi+\gamma)I_H \\ \gamma I_H - (\mu+\delta_2+\alpha)Q_H \\ -(\mu_B+\chi)E_B + \frac{\beta_B\Lambda_B(\mu+\psi+1-\phi)I_B}{(\tau+\mu_B)(\mu_B+\psi+1-\phi)-\psi\tau} + \frac{\beta_B\Lambda_B(\mu+\psi+1-\phi)D_B}{(\tau+\mu_B)(\mu+\psi+1-\phi)-\psi\tau} \\ \chi E_B - (\mu_B+\delta_B+\omega)I_B \\ (\delta_B+\omega)I_B - \pi D_B \end{bmatrix}$$

$$G(X,Z) = PZ - G(X,Z)$$

$$\begin{bmatrix} -(\mu+\eta)E_H + \frac{\beta_H B_H N_H I_H}{\sigma+\mu_H} + \frac{\beta_{BH} B_H N_H I_B}{\sigma+\mu_H} + \frac{\beta_{DH} B_H N_H D_B}{\sigma+\mu_H} \\ \eta E_H - (\mu+\delta_1+\xi+\gamma)I_H \\ \gamma I_H - (\mu+\delta_2+\alpha)Q_H \\ -(\mu_B+\chi)E_B + \frac{\beta_B B_B N_B I_B}{\tau+\mu_B} + \frac{\beta_B B_B N_B D_B}{\tau+\mu_B} \\ \chi E_B - (\mu_B+\delta_B+\omega)I_B \\ (\delta_B+\omega)I_B - \pi D_B \end{bmatrix} - \begin{bmatrix} (\beta_H I_H + \beta_{BH} I_B + \beta_{DH} D_B)S_H - (\mu+\eta)E_H \\ \eta E_H - (\mu+\delta_1+\xi+\gamma)I_H \\ \gamma I_H - (\mu+\delta_2+\alpha)Q_H \\ \beta_B S_B (I_B + D_B) - (\mu_B+\chi)E_B + (1-\phi)V_B \\ \chi E_B - (\mu_B+\delta_B+\omega)I_B \\ (\delta_B+\omega)I_B - \pi D_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (\phi-1)V_B \\ 0 \\ 0 \end{bmatrix}$$

$G(X,Z) = 0$ if and only if $\phi = 1$ meaning the vaccine efficacy is 100%. $G(X,Z)$ cannot be greater than zero since $0 \le \phi \le 1$. Therefore, the Disease free equilibrium state is globally asymptotically stable if the vaccine efficacy $\phi = 1$. Otherwise it is unstable.

4.3 Simulation Results

We perform some numerical experiments using MATLAB R2015a ode45 to study the behaviour of the system and the effects of vaccination of birds and proper handling of infected dead birds. The initial condition for each plot and parameter values are presented in table 4.1

Table 4.1: Variables and parameters values used for computational results

Variables	**Values**	**Reference**
$S_H(t)$	8000000	Assumed
$E_H(t)$	200	Assumed
$I_H(t)$	15	Assumed
$Q_H(t)$	10	Assumed
$R_H(t)$	3	Assumed
$V_H(t)$	5000	Assumed
$S_B(t)$	500000	Assumed
$E_B(t)$	400	Assumed
$I_B(t)$	300	Assumed
$V_B(t)$	300000	Assumed
$D_B(t)$	300	Assumed
Λ_H	800000	Assumed
β_H	0.0012	Bimal and Durgesh (2016)
β_{BH}	0.0002	Bimal and Durgesh (2016)
β_{DH}	0.005	Assumed
μ_H	0.02	Bimal and Durgesh (2016)
ρ	0.02	Bimal and Durgesh (2016)
σ	0.1	Bimal and Durgesh (2016)
ε	0.05	Bimal and Durgesh (2016)
η	0.5	Bimal and Durgesh (2016)
δ_1	0.6	Bimal and Durgesh (2016)
δ_2	0.36	Assumed
γ	0.05	Bimal and Durgesh (2016)
ξ	0.08	Bimal and Durgesh (2016)
α	0.6	Bimal and Durgesh (2016)

Λ_B	800000	Assumed
β_B	0.5	Bimal and Durgesh (2016)
μ_B	0.01	Bimal and Durgesh (2016)
χ	0.6	Bimal and Durgesh (2016)
δ_B	0.5	Bimal and Durgesh (2016)
ω	0.45	Kimbir *et al* (2014)

4.3.1 Effect of bird vaccine on infected human

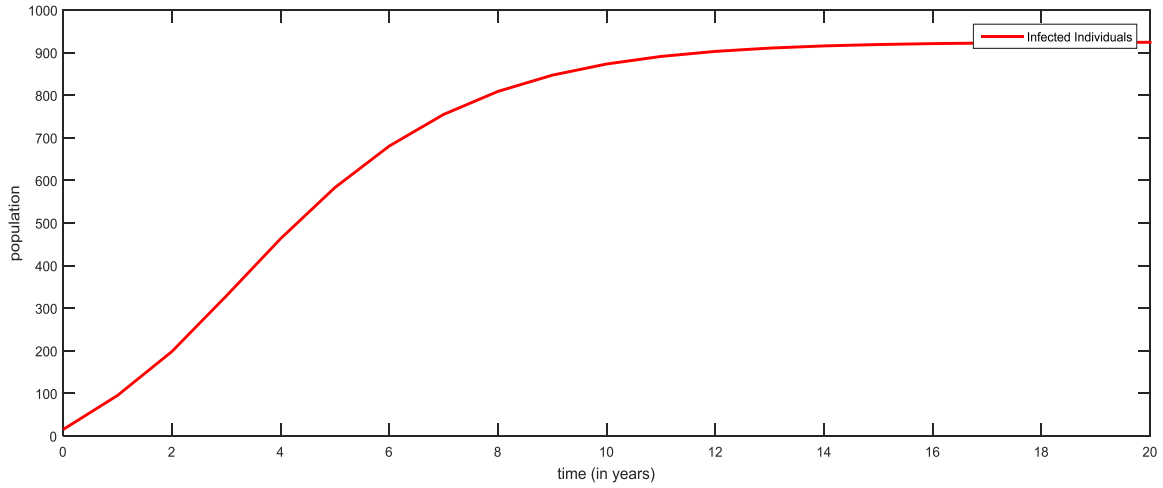

Figure 4.1a: Simulation results for infected human without vaccination of birds ($\tau = 0$)

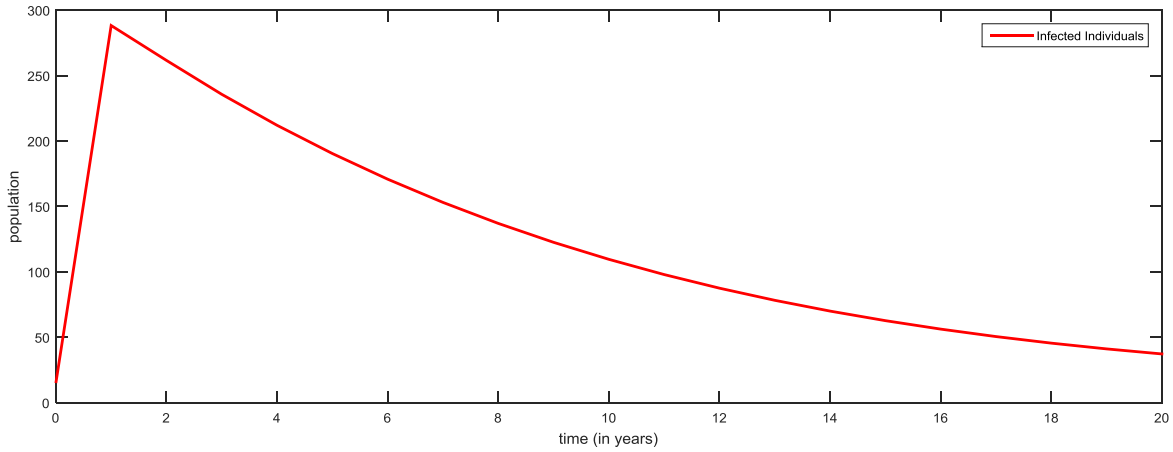

Figure 4.1b: Simulation results for infected human with vaccination of birds $(\tau = 0.5)$

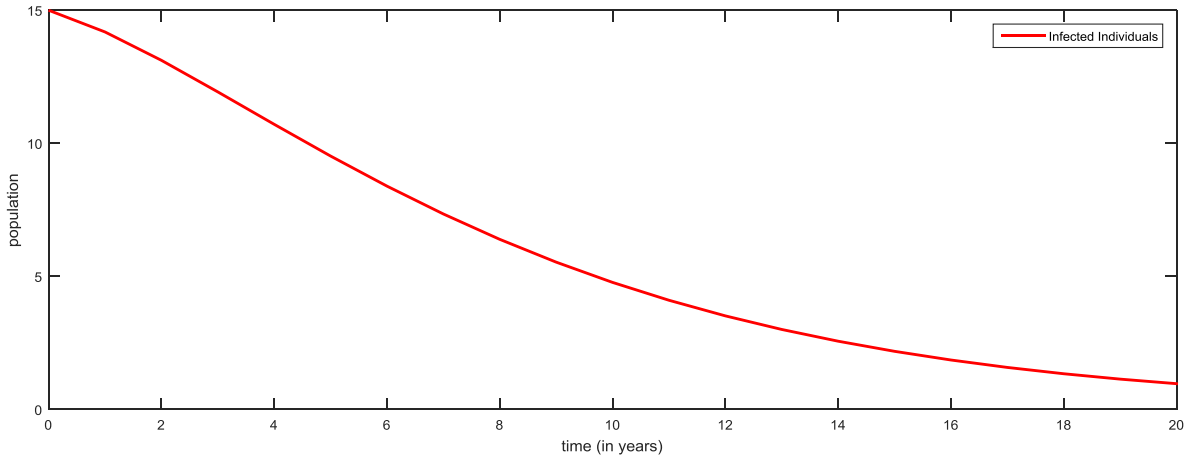

Figure 4.1c: Simulation results for infected human with vaccination of birds $(\tau = 0.9)$

4.3.2 Effect of bird vaccine on infected birds

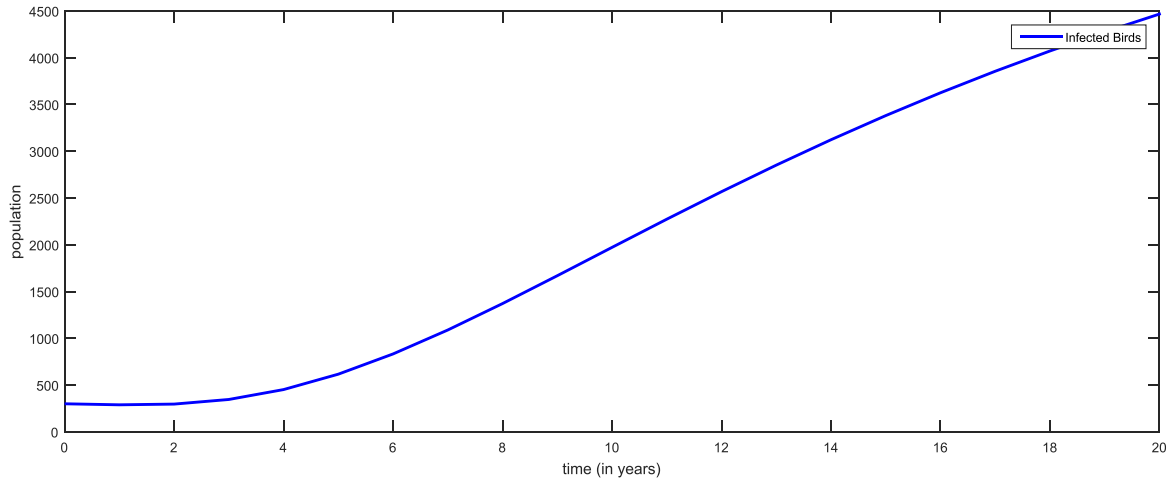

Figure 4.2a: Simulation results for infected birds without vaccination of birds $(\tau = 0)$

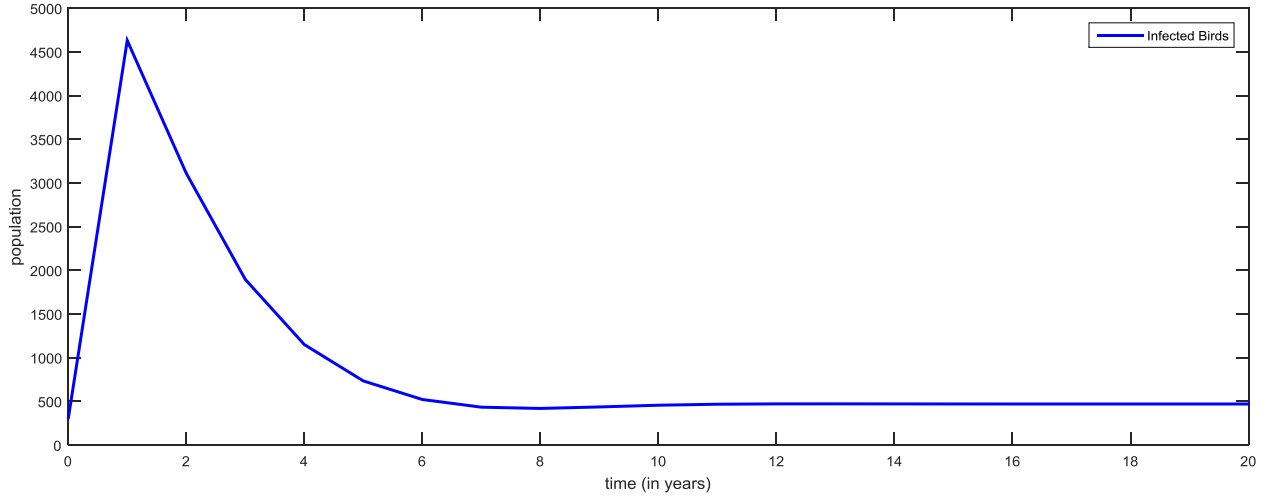

Figure 4.2b: Simulation results for infected birds with vaccination of birds $(\tau = 0.5)$

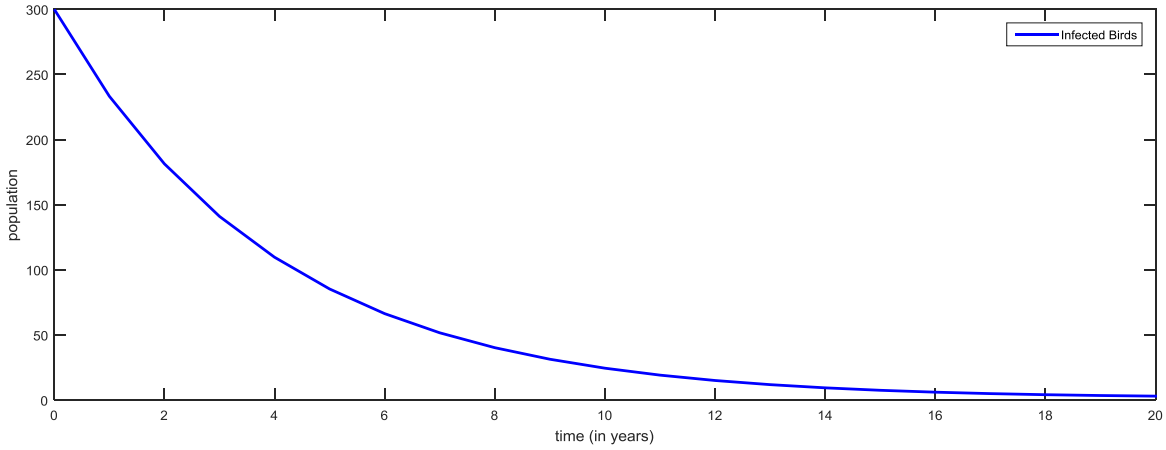

Figure 4.2c: Simulation results for infected birds with vaccination of birds $(\tau = 0.9)$

4.3.3 Effect of proper handling of infected dead birds on infected human

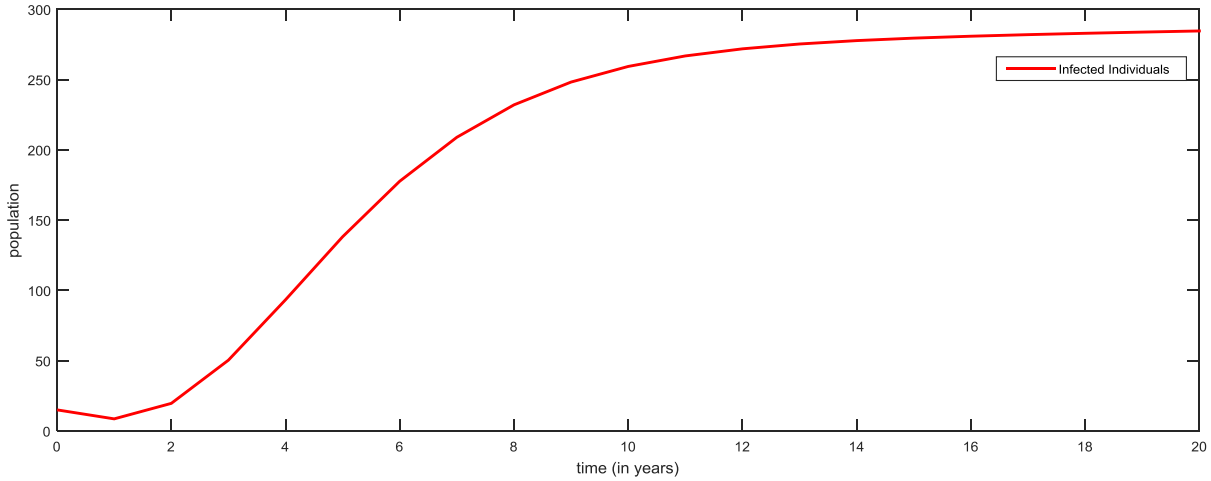

Figure 4.3a: Simulation results for infected human without proper haling of infected dead birds ($\pi = 0$)

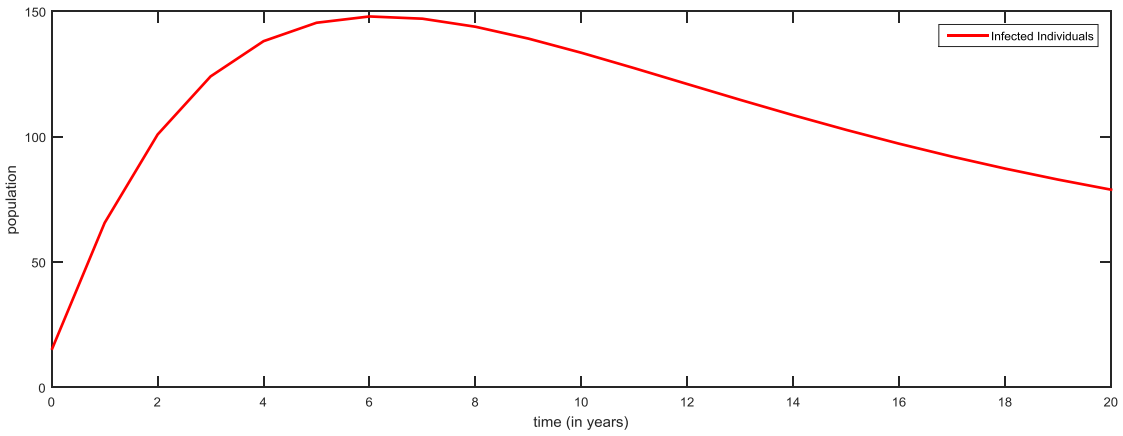

Figure 4.3b: Simulation results for infected human with proper haling of infected dead birds ($\pi = 0.5$)

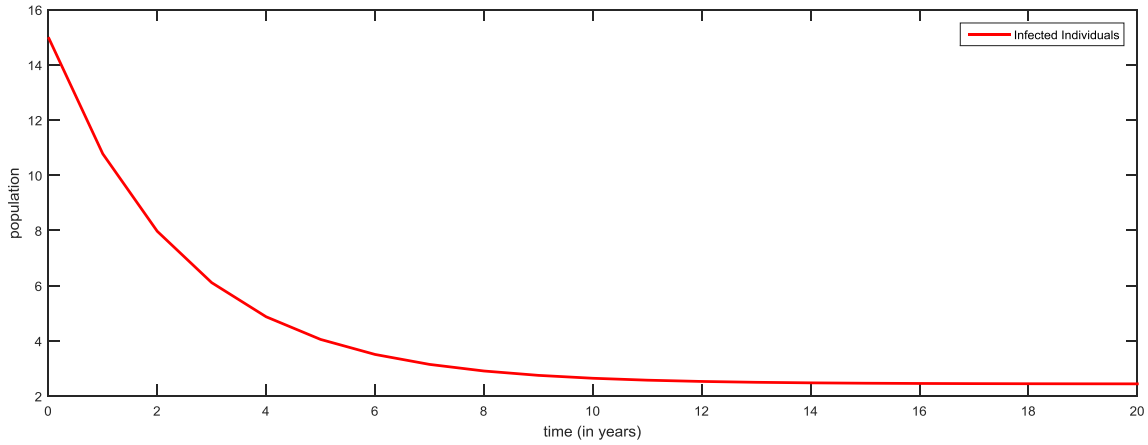

Figure 4.3c: Simulation results for infected human without proper haling of infected dead birds ($\pi = 0.9$)

4.3.4 Effect of proper handling of infected dead birds on infected birds

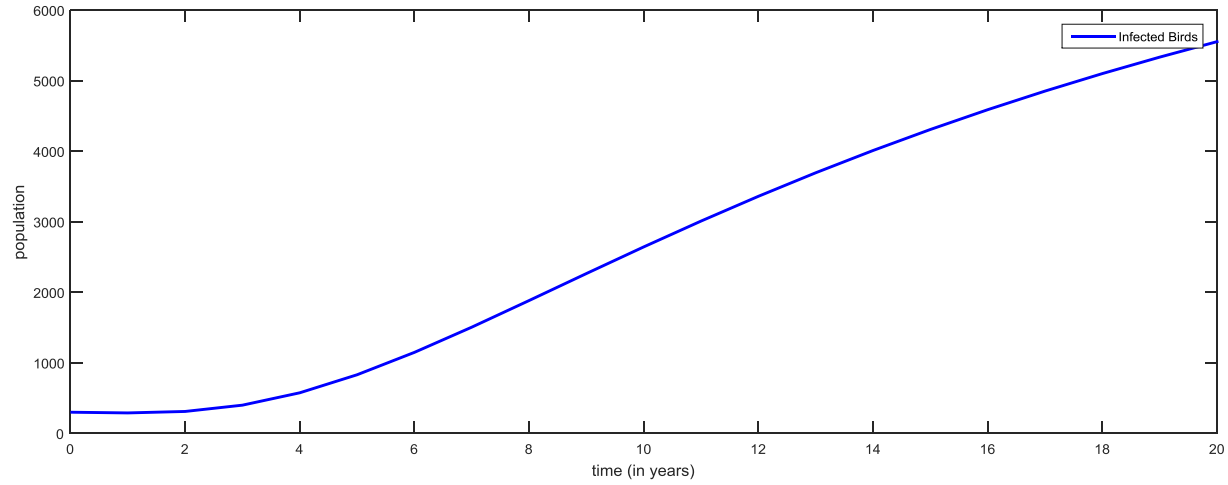

Figure 4.4a: Simulation results for infected birds without proper haling of infected dead birds ($\pi = 0$)

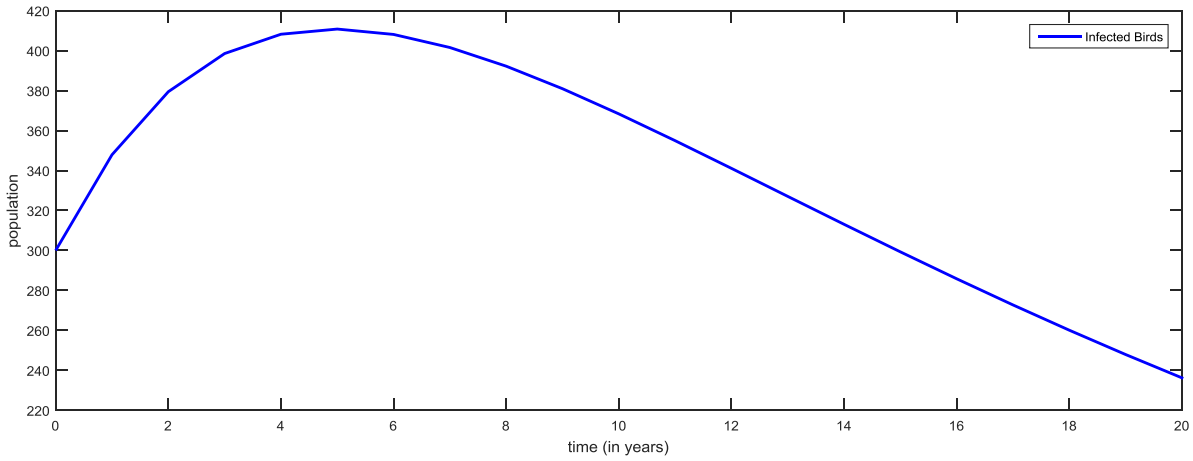

Figure 4.4b: Simulation results for infected birds with proper handling of infected dead birds ($\pi = 0.5$)

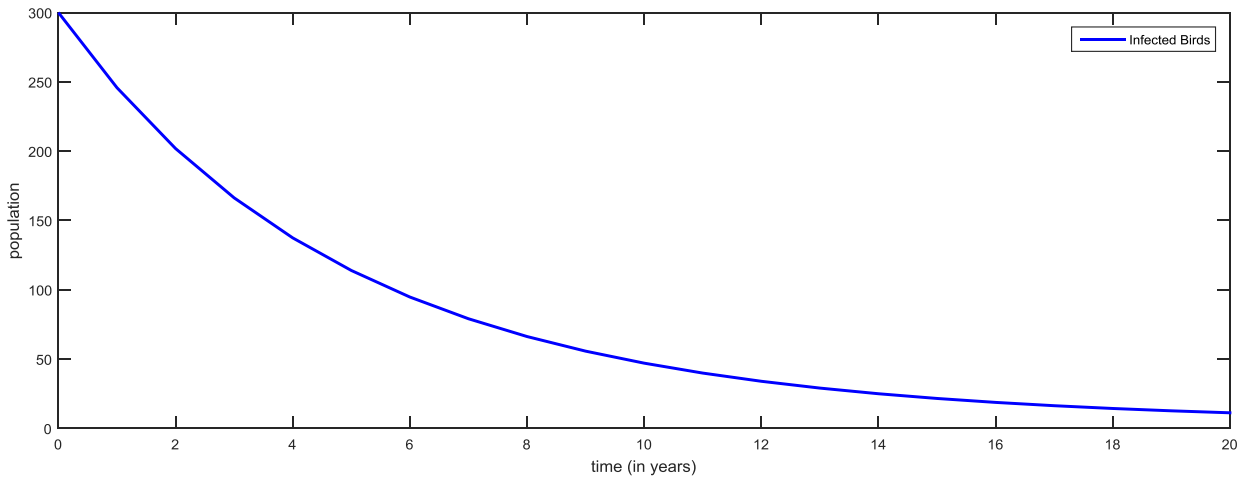

Figure 4.4c: Simulation results for infected birds with proper handling of infected dead birds ($\pi = 0.9$)

4.3.5 Combined effect of vaccination of susceptible birds and proper handling of infected dead birds on infected human

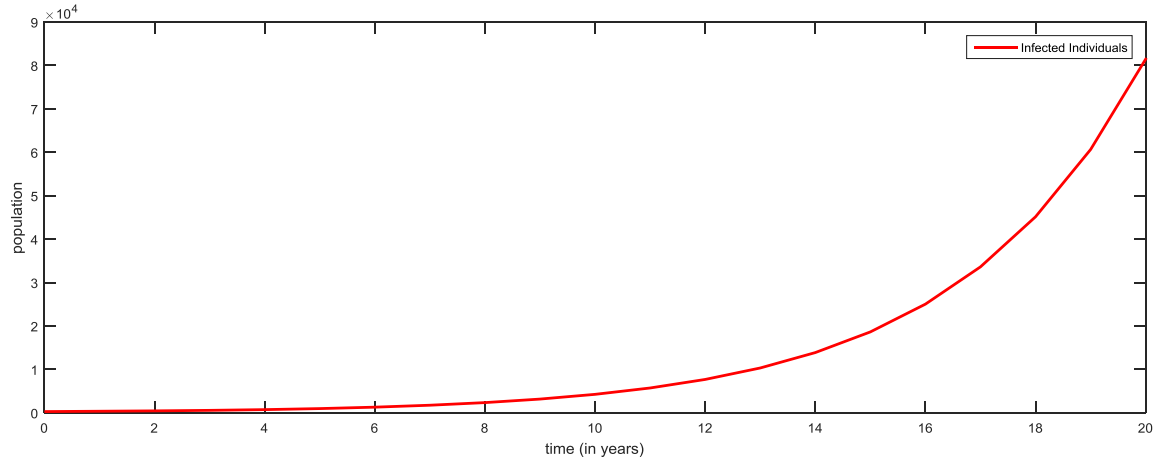

Figure 4.5a: Simulation results for infected human without vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0, \pi = 0)$

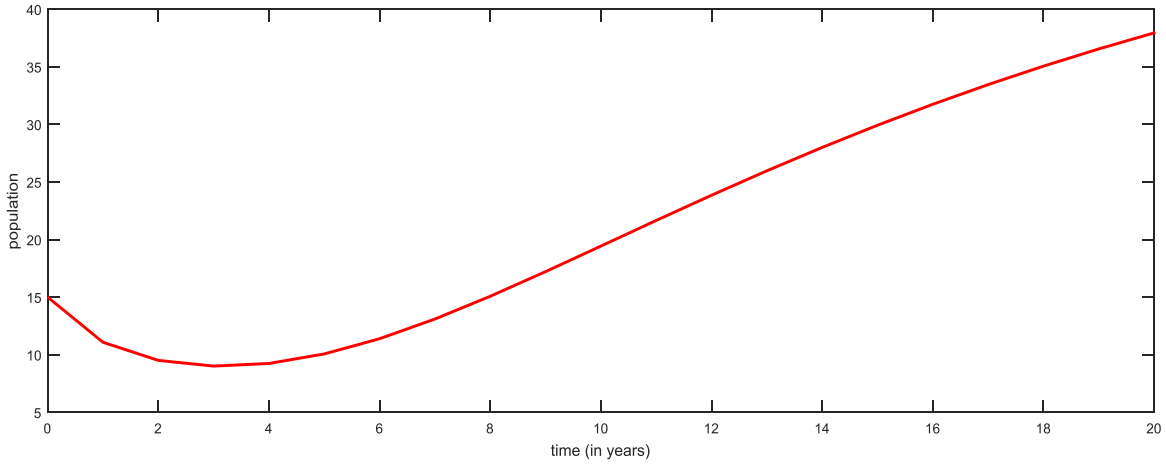

Figure 4.5b: Simulation results for infected human without vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0.5, \pi = 0.5)$

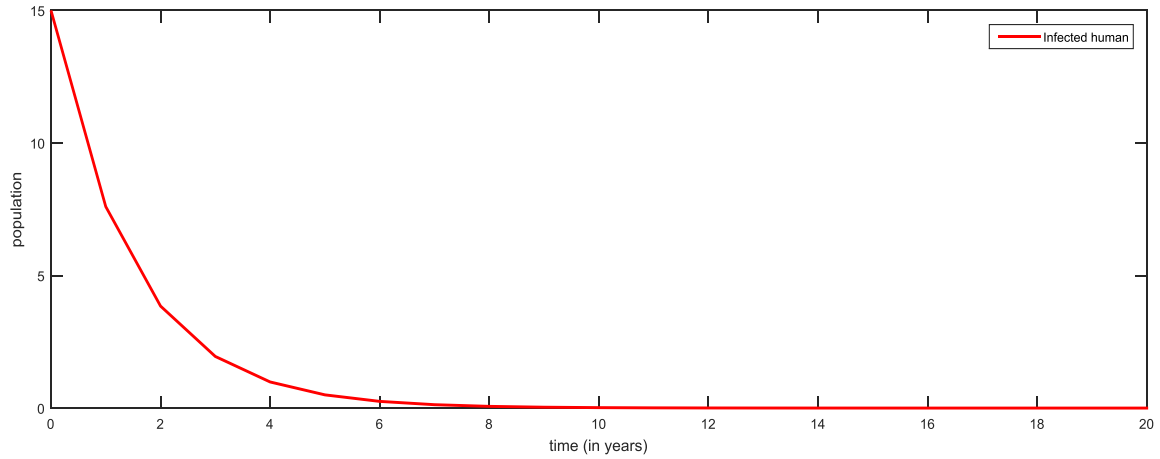

Figure 4.5c: Simulation results for infected human without vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0.95, \pi = 0.95)$

4.3.6 Combined effect of vaccination of susceptible birds and proper handling of infected dead birds on infected birds

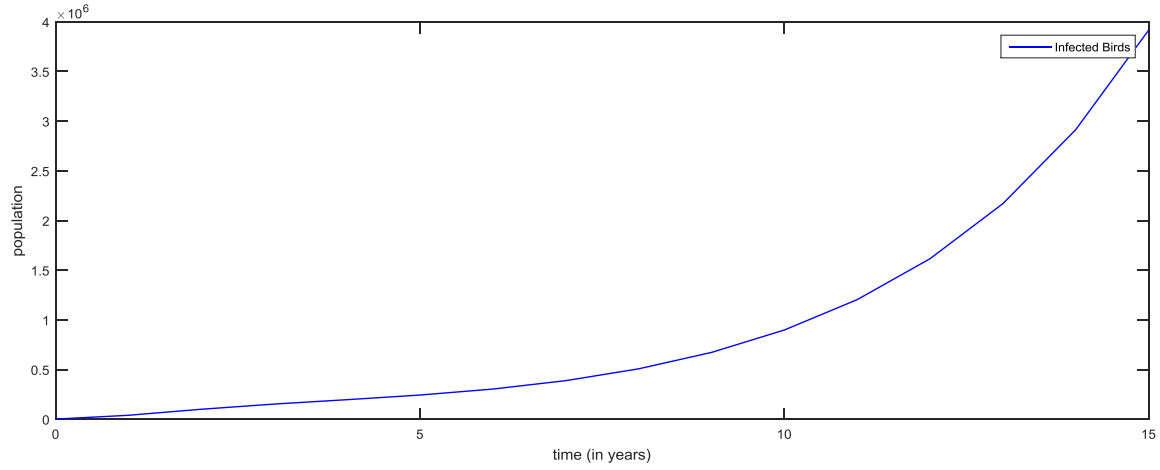

Figure 4.6a: Simulation results for infected birds without vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0, \pi = 0)$

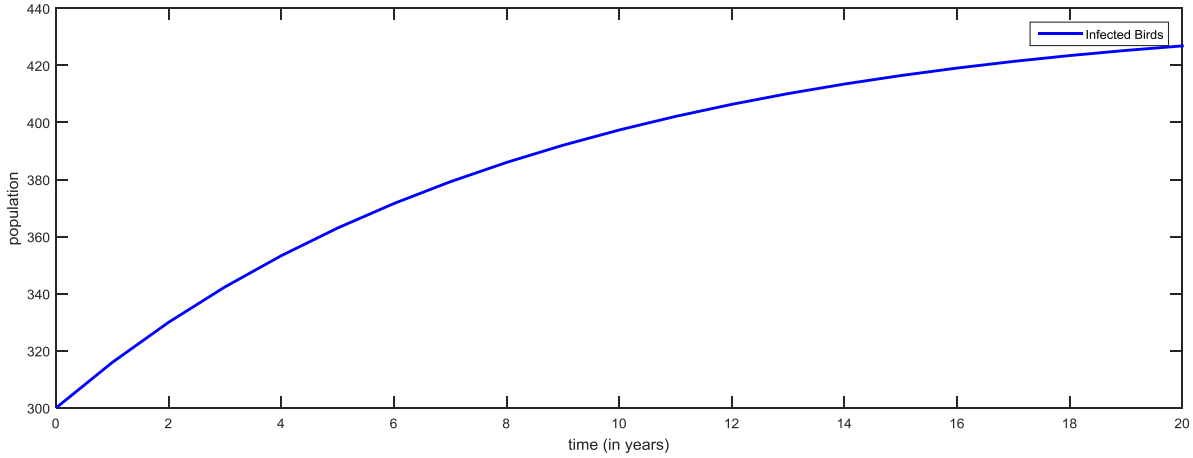

Figure 4.6b: Simulation results for infected birds with vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0.5, \pi = 0.5)$

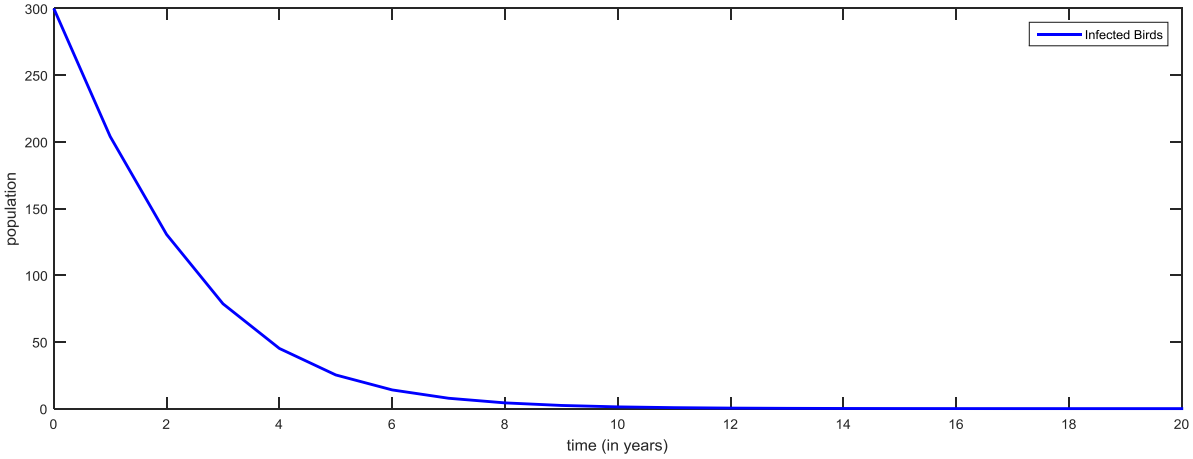

Figure 4.6c: Simulation results for infected birds with vaccination of susceptible birds and proper handling of infected dead birds $(\tau = 0.95, \pi = 0.95)$

4.4 Discussion of Results

In this section, we present the discussion of the results for both analytical and numerical solutions presented in sections (4.2) and (4.3).

4.4.1 Discussion of analytical results

The Avian influenza modified model consists of an 11-dimentional system of ordinary differential equations. We obtained the existence and uniqueness of the solution, positivity of solution, invariant region and disease free equilibrium state of the system. We also computed the basic reproduction number of the model and both the local and global stability analysis of the disease free equilibrium which were found to be stable with some conditions of inequality and 100% efficacy of the bird vaccine respectively.

4.4.2 Discussion *of simulation results*

Figures (4.1a – 4.1c) and (4.2a – 4.2c) show the effects of vaccination of bird on infected human and birds respectively. The population of the infected birds and human rose and latter drop to the minimal. As the simulation results indicate, the bird vaccination plays a vital role in reduction and eradication of Avian influenza in both human and bird population.

Figures (4.3a – 4.3c) and (4.4a – 4.4c) show the effects of proper handling of infected dead birds on infected human and birds respectively. The infected population of both birds and human grows rapidly with no proper handling of infected dead bird. And with proper handling, the population growth reduced by a good number more than half. Proper handling of infected death bird is very important in controlling the spread of Avian influenza. The increase in proper handling rate results to reduction in the number if infections in both populations.

Figures (4.5a – 4.5c) and (4.6a – 4.6c) show the combined effects of both vaccination of birds and proper handling of infected dead birds on infected human and birds. The infected population of both birds and human grows rapidly with no both vaccine for birds and proper handling of infected dead bird. And with these two controls, the infected populations of both human and birds drop rapidly with time as the simulation results indicate. The more greater rate of the two controls, the faster the drop of the number of infected human and birds.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this work, we have modified the mathematical model by Bimal and Durgesh (2016) for transmission dynamics of Avian influenza virus by accounting for accounts for the vaccination for susceptible birds and infection of human from infected dead birds. We carried out analytical studies of the modified model where the population comprises of eleven compartments: Susceptible human, Exposed human, Infected human, Isolated human, Recovered human,and Vaccinated human. The Disease Free Equilibrium (DFE), existence and uniqueness, invariant region, positivity of solution and the reproduction number were obtained. We carried out the stability analysis for the disease free equilibrium using Routh-Hurwitz stability criterion. The numerical solutions were also obtained using MATLAB R2015a inbuilt scheme of ode45

5.2 Conclusion

This research work presents modified model of the transmission dynamics of Avian influenza that account for the vaccination for susceptible bird and the infection of human fron infected dead birds. Analytical study was carried out using Routh-Hurwitz stability criterion and the results shows that the disease free equilibrium (DFE) points are locally asymptotically stable (LAS) whenever $R_0 < 1$ and global asymptotically stable (GAS) whenever $R_0 \leq 1$. The result from the numerical experiments carried out show that significant changes in the rate of vaccination human and birds improves the rate at which Avian influenza is eradicated from the population.

5.3 Recommendations

The cases of Avian influenza can be eradicated completely from the population. This can achieved if the following recommendations are considered.

i. There should be more awareness on the dangers of Avian influenza
ii. Poultry farmers should be sensitized on how to handle poultry farm and be thought on the important of vaccine and how to administer the vaccine on their poultry farm.

iii. People should be educated on the mode of transmission of the virus and how to prevent it especially those that are close to poultry farms.

iv. Government and non-governmental organisation should encourage the use of Avian influenza vaccine among human for it plays a vital in reducing or stopping the spread of virus.

5.4 Contribution to Knowledge

i. Development of mathematical model for the effects of vaccine for birds and infection of human from infected dead birds.

ii. Disease free equilibrium was obtained

iii. Existence and uniqueness of solution, invariant region, positivity of solution and the reproduction number were obtained

iv. Computational result was obtained

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