# DERIVATION OF IMPLICIT RATIONAL RUNGE-KUTTA SCHEMES FOR APPROXIMATION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

**BY**

**SANI USMAN ABDULLAHI**

**(M.Sc/MA/06/0118)**

**SEPTEMBER, 2012**

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## CERTIFICATION

This is to certify that this thesis “Derivation of Implicit Rational Runge – Kutta Schemes for the Approximation of Second Order Ordinary Differential Equations” has been duly presented by SANI, Usman Abdullahi (M.Sc/MA/06/0118) of the department of Mathematics of Modibbo Adama University of Technology, Yola, and been approved by the following having met the stipulated requirement.

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Prof. M. R. Odekunle Date

Supervisor

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Name: Date

Internal Examiner

-------------------------------------- --------------------------

Prof. Y. D. Gulibar Date

External Examiner

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Dr. Samuel Musa Date

Ag. Head of Department

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Prof. M. R. Odekunle Date

Dean SPGS

**DEDICATION**

This work is dedicated to my parent, Alh. Abdullahi Sani and Haj. Zainabu Abu.

## ACKNOWLEDGEMENT

Alhandulillah! To the only wonderful King, who gave me the privilege and life from the beginning of this work up to this stage. Honour and glory be the Almighty Allah.

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Finally, I owe a debt of gratitude to everyone who put up with my down-to-the-wire. The message is I LOVE you all may the Almighty God reward you with the richest of His blessings.

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## ABSTRACT

The development of 3 – stage Implicit Rational Runge – Kutta methods are considered using Taylor and Binomial series expansion for the direct solution of general second order initial value problems of ordinary differential equations with constant step length. The basic properties of the developed method were investigated and found to be consistent and convergent. The efficiency of the method were tested on some numerical examples and found to give better approximations than the existing methods.

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# CHAPTER ONE

# INTRODUCTION

## 1.0 Background of the Study

A numerical method is the study of approximate techniques for solving mathematical problems taking into account the extent of possible errors. Many numerical schemes have been developed for solution of ordinary differential equations which posses no analytic solution. In recent years much research has been carried out to develop derivative free Runge-Kutta methods.

Conventionally, higher order ordinary differential equations are solved by method of reduction to system of first order ordinary differential equations and methods of solving the resultant first order ordinary differential equations can be adopted. The approach has reported to increase the dimension of the problem and therefore result in more computational burden. Awoyemi (2001), Adesanya *et al.* (2009) and Anake *et al.* (2012). Some of the direct methods of solving second order ordinary differential equations are: Nystrom type, self stating Runge – Kutta method which involve several functions evaluation per step and the linear multi-step methods, particularly the Implicit methods which have better stability than the explicit methods and require fewer functions evaluation per step, (Jator (2010), Agular and Ramos (2006)).

## 1.1 Statement of the Problem

Most of the existing Runge – Kutta methods were developed to solve first order initial value problems or special second order ordinary differential equations. This thesis considered the direct solution of general second order initial value problems.

## 1.2 Aim and Objectives of the Study

The aim of this study is to develop a 3 – Stage Implicit Rational Runge – Kutta method for the direct solution of general second order initial value problems, while the objectives are:

i. Use Taylor and Binomial series expansion of the s-stage Rational Runge-Kutta method to derive an implicit rational Runge – Kutta schemes for second order initial value problems.
ii. Analyze the consistency, convergence and the error analysis of the schemes.
iii. Verify the efficiency of the derived scheme using numerical examples.

## 1.3 Significance of the Study

The significance of this study is to derive a better scheme in terms of approximation and time of execution for the direct solution of general second order ordinary differential equations.

## 1.4 Scope of the study

The method developed in this thesis can only solve second order initial value problems.

## 1.5 Definition of terms

**Derivative:**

The rate of change of one variable with respect to another is called derivative.

**Differential equation (D.E):**

An equation involving some of the derivative of functions is called differential equation.

**Type of differential equations:**

An equation that contains one independent variable and its derivative with respect to that independent variable is called Ordinary Differential Equation (ODE). E.g

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = \sin x$$

while on the other hand, if the equation contains more than one independent variable and their derivatives with respect to each independent variables is called Partial Differential Equations (PDE). E.g

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

**Order of Differential equation:**

The order of D.E is the order of the highest derivative of the unknown functions that appear in the equation. Thus, the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 2y = 0 \quad \text{is of order 2.}$$

**Linear and Non – Linear Differential Equation**

Consider the general form of ODE

$$a_n y^n + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \ldots + a_0 y = f(t)$$

If the coefficients $a_n, a_{n-1}, \ldots, a_{0,}$ are either constants or functions of $t$, then the ODE is said to be linear, otherwise is said to be non-linear.

**Consistency**

The Runge-Kutta method is said to be consistent with the initial value condition if

$$\emptyset(x, y, 0) \equiv f(x, y) \qquad 1.10$$

then,

$$y(x+h) - y(x) - h\emptyset(x, y(x), h) = y(x) + hy'(x) + O(h^2) - y(x) - h\emptyset(x, y(x), 0)$$

$$= hy'(x_n) - hy'(x_n) + O(h^2) = O(h^2)$$

Since $y'(x) = f\big(x, y(x)\big) = \emptyset(x, y(x), 0)$ by the above then a consistent method has order at least one, Lambert (1973).

**Order**

A method is said to have order $p$ if $p$ is the largest integer for which

$$y(x+h) - y(x) - h\emptyset(x, y(x), h) = O(h^{p+1})$$

holds, the Taylor algorithm of order $p$ is

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots + \frac{h^p}{p!}y^p(x)$$

**Stability**

A numerical method is stable if small change in the initial condition produces a corresponding small change in the subsequent approximations.

**A – Stability**

A numerical method is said to be "A-Stable" if its region of absolute stability contains the whole of the left hand half plane i.e. $Reh\lambda < 0$.

**Convergent**

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size goes to zero. i.e.

$$\lim_{h \to 0}[y(x_{n+1}) - y_{n+1}] = 0$$

**Stiff Equation**

The initial value problem $y' = f(x, y)$ is considered stiff when its Jacobian matrix has both eigenvalues with very large negative real part and eigenvalues with very small negative real part which constrain the step length to be extremely small to maintain stability requirements in Numerical methods.

**Truncation Error**

Truncation error: an error introduced as a result of ignoring some of the higher terms of the power series expansion during the development of the new formula. Mathematically written as

$$T_{n+1} = |y(x_{n+1}) - y(x_n) - h\phi(x_n, y_n, h)|$$

**Explicit Scheme**

In an explicit numerical scheme, the calculation of the solution at a given step or stage does not depend on the value of the solution at that step or on a later step or stage.

**Implicit Scheme**

In an implicit numerical scheme, the calculation of the solution at a given step or stage does depend on the value of the solution at that step or on a later step or stage. Such methods are usually more expensive than explicit schemes but are better for handling stiff ODEs.

**Embedded Runge-Kutta methods**

Two Runge-Kutta methods that share the same stages. The differences between the solutions give an estimate of the local truncation error.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.0 Introduction

In numerical analysis, Runge-Kutta schemes are important family of implicit and explicit iterative methods for approximation of solution of ordinary differential equations. The method was developed around 1900 by German Mathematician C. Runge and M. W. Kutta.

First order initial value problem is given in the form:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.1}$$

The numerical solution to (2.1) is

$$y_{n+1} = y_n + h\phi(x_n, y_n, h) \tag{2.2}$$

where

$$\phi(x, y, h) = \sum_{i=1}^{s} c_i k_i$$

$$k_1 = f(x, y),$$

$$k_r = f\left( x + ha_i, y + h\sum_{j=1}^{r} b_{ij} k_j \right), \quad r = 1(1)s \tag{2.3}$$

with constraints

$$a_i = \sum_{j=1}^{i} b_{ij}, \quad i = 1(1)s$$

The derivative of suitable parameters $a_{ij}$, $b_i$ and $c_i$ of higher order term involves a large amount of tedious algebraic manipulations and functions evaluations which is both time consuming and error prone, Julyan and Oreste (1992). The derivation of the second order, third order, otherwise known as the Heun's method and the popular fourth order Runge – Kutta methods is extensively discussed by Lambert (1973), Butcher (1987). According to Julyan and Oreste (1992) the minimum number of stages necessary for an explicit method to attain order $p$ is still an open problem. Therefore so many new schemes and approximation formula have been derived this includes the work of Ababneh *et al*. (2009a), Ababneh *et al*. (2009b) Faranak and Ismail (2010).

The Runge-Kutta method is not restricted to solving first order differential equation of the form (2.1). We can do the same by splitting higher order differential equations into an equivalent system of first order equations.

In general, the second order equation can be reduced to an equivalent system of first order of twice the dimension and solved using the standard Runge-Kutta method. However, it is more efficient if the equation can be solved directly using Runge-Kutta-Nystrom method of the special second order differential equations which according to Jain (1984) is given in the form:

$$y'' = f(x, y), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0 \tag{2.4}$$

with numerical solution

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i k_i \tag{2.5a}$$

and

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} b'_i k_i \tag{2.5b}$$

where

$$k_i = f\left( x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{i} a_{ij} k_j \right), \quad i = 1,2,\ldots,s$$

The derivative of suitable parameters $a_{ij}$, $b_i$ and $c_i$ requires extremely lengthy algebraic manipulations, except for small values of $s$. Such method was discussed in Sharp and Fine (1992) and Dormand *et al* (1987).

Runge-Kutta-Nystrom methods are direct extension of Runge-Kutta method to second order differential equations in (2.4) so given by the Finnish Mathematician E. J Nystrom. The best known of these method uses the formula in (2.5) where $h = 0, 1, 2, \ldots, N - 1$ (N, the number of step), Nystrom (1925).

## 2.1: Runge – Kutta Nystrom Methods

Recently, significant advances have been made in the development of general theory of Runge-Kutta-Nystrom methods and in the derivation, error estimation and control, Dormand (1996) which includes many notable ones.

Sharp and Fine (1992) generated some Nystrom pairs for the general second order initial value problem. Dormand *et al.* (1987) obtained some families of Runge-Kutta-Nystrom formula. Fudziah (2003) derived the embedded pairs of Runge-Kutta-Nystrom method which are diagonally implicit and all the diagonal elements are equal. Fudziah (2009) also derived a Sixth Order Singly Diagonally Implicit Runge-Kutta Nystrom Method with Explicit First Stage for Solving Second Order Ordinary Differential Equations. Senu *et al.* (2011) derived a Singly Diagonally Implicit Runge-Kutta - Nystrom Method for Solving Oscillatory Problems. Also recently Okunuga *et al.* (2012) discussed the general techniques for solving equation of the form (2.4) directly without first reducing it to systems of first order ODEs. The above authors observed that the direct solution of second order equations is of greater advantage over reduction to systems of first order equations to increase efficiency and reduced storage requirement.

It should be noted that the methods considered above were Runge-Kutta-Nystrom methods of the special cases for second order ordinary differential equation of the form (2.4). Much attention have not been given to Runge – Kutta method for the solution of general second order ODEs; bearing in mind that many physical systems are modeled using the general second order ODEs, Lambert (1973).

## 2.2: Runge-Kutta Method for Second Order Ordinary Differential Equations.

The general $s-$stage Runge-Kutta scheme for general second order initial value problems of ordinary differential equations of the form

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0 \qquad (2.2.1)$$

is defined by Jain (1984) to be

$$y_{n+1} = y_n + hy'_n + \sum_{r=1}^{s} c_r k_r \qquad (2.2.2)$$

and

$$y'_{n+1} = y'_n + \frac{1}{h}\sum_{r=1}^{s} c'_r k_r \qquad (2.2.3)$$

where

$$K_r = \frac{h^2}{2} f\left( x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij} k_j, y'_n + \frac{1}{h}\sum_{j=1}^{r} b_{ij} k_j \right), \quad i = 1(1)s \qquad (2.2.4)$$

with

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij}, \qquad i = 1(1)r \qquad (2.2.5)$$

where $c_i, a_{ij}, b_{ij}, c_r, c'_r$ are constants to be determined.

The fourth order Runge-Kutta method for the solution of (2.2.1) is:

$$y_{n+1} = \frac{1}{3}(k_1 + k_2 + k_3) \qquad \text{, and}$$

$$y'_{n+1} = \frac{1}{3h}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{2.2.6}$$

where

$$k_1 = h^2 f\left(x, y, y'\right)$$

$$k_2 = \frac{h^2}{2} f\left(x + \frac{1}{2}h, y + \frac{h}{2}y' + \frac{1}{2}k_1, y' + \frac{1}{h}k_1\right)$$

$$k_3 = \frac{h^2}{2} f\left(x + \frac{1}{2}h, y + \frac{h}{2}y' + \frac{1}{4}k_2, y' + \frac{1}{h}k_2\right)$$

$$k_4 = \frac{h^2}{2} f\left(x + h, y + hy' + \frac{1}{4}k_3, y' + \frac{2}{h}k_3\right)$$

## 2.3: Rational Runge – Kutta Methods

Hong (1982) first proposed the use of rational function of Runge – Kutta method then Okunbor (1987) investigated the use of the rational function of the Runge-Kutta scheme:

$$y_{n+1} = \frac{y_n + h\sum_{i=1}^{r} w_i k_i}{1 + h y_n \sum_{i=1}^{r} v_i H_i} \tag{2.3.1}$$

where

$$k_i = f\left(x_n + c_i h, y_n + h\sum_{i=1}^{r} a_{i-1j}\, k_j\right),\ i = 1(1)r \tag{2.3.2}$$

and

$$H_i = g\left(x_n + d_i h, z_n + h\sum_{i=1}^{r} b_{i-1j} H_j\right),\ i = 1(1)r \tag{2.3.3}$$

in which

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \quad \text{and} \quad z_n = \frac{1}{y_n}$$

where $c_{i}, a_{ij}, b_{ij}, d_i$ are arbitrary constants to be determined.

The scheme is said to be

i. Explicit if $b_{ij} = 0, j \geq i$

ii. Semi-implicit if $b_{ij} = 0, j > i$

iii. Implicit if $b_{ij} \neq 0$ for at least one $j > i$ , $h$ is the step size and the constraint

$$d_i = \sum_{j=1}^{i} b_{ij} , \qquad (2.3.4)$$

is imposed to ensure consistency of the method.

The explicit scheme can only solve non-stiff and mildly stiff equations effectively. They performed poorly on stiff equations.

In view of these inadequacies of the explicit schemes and the superior region of absolute stability associated with implicit schemes, Ademuluyi and Babatola (2000) redefined the parameters; $a_{ij}, b_{ij}, v_i, w_i, d_{i,}$ and $c_{i's}$ so that the resulting method to be implicit and generates also the parameters so that the resulting numerical approximation method shall be A-stable and will have low bound for local truncation error. The scheme was used to solve some stiff initial value problems.

Odekunle (2001) generated some semi-implicit rational Runge-Kutta scheme which is also A-stable and can solve mildly and slightly stiff problems and reduces problems in solving coupled simultaneous non linear systems when compared with implicit schemes. Odekunle *et al.* (2004) developed a method that takes into consideration the point of singularities which other methods mentioned above cannot solve.

Since then many new rational Runge –Kutta schemes have been developed for the solution of first order initial value problems and found out to give better estimates. Among these authors are:

Bolarinwa (2005), who derived some class of semi-implicit rational Runge–Kutta schemes for solving ordinary differential equations with derivatives discontinuities, Babatola *et al.* (2007) developed a One-Stage Implicit rational Runge-Kutta schemes for the solution of discontinuous initial value problems and found out to be consistent, A-Stable and convergent.

Bolarinwa *et al.* (2012) developed a Two-Stage Semi-Implicit Rational Runge – Kutta Scheme for solving first order ordinary differential equations. The scheme is absolutely stable, consistent and convergent and was used to approximate a variety of first order differential equations. However, the methods are presently receiving more attention as efficient schemes for the solutions of various types of first order initial value problems.

The rational function of Runge-Kutta method is still open for more approximation schemes, estimation and control.

Soomiyol (2011) investigated the use of rational form of (2.2.2) and (2.2.3), defined by

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{r=1}^{s} w_r K_r}{1 + y'_n \sum_{r=1}^{s} v_r H_r} \qquad (2.3.5)$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \qquad (2.3.6)$$

where

$$K_r = \frac{h^2}{2} f\left( x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{r} b_{ij} K_j \right),\ i = 1(1)s \tag{2.3.7}$$

and

$$H_r = \frac{h^2}{2} g\left( x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{r} \alpha_{ij} H_{ij}, z'_n + \frac{1}{h}\sum_{j=1}^{r} \beta_{ij} H_j \right),\ i = 1(1)s \tag{2.3.8}$$

with constraints

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij}, \qquad i = 1(1)r$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j=1}^{i} \beta_{ij}, \qquad i = 1(1)r$$

in which

$$g(x_n, z_n, z'_n) = -z_n^2 f(x_n, y_n, y'_n) \quad and\ z_n = \frac{1}{y_n}$$

The above scheme is said to be

i. Explicit if $a_{ij}, b_{ij} = 0, j \geq i$
ii. Semi-Implicit if $a_{ij}, b_{ij} = 0, j > i$
iii. Implicit if $a_{ij}, b_{ij} \neq 0$, for at leasst one $j > i$

For direct solution of general second order ordinary differential equation and derived the Explicit Rational Runge-Kutta schemes. The scheme is found to be efficient, and was used to solve second order initial value problems. The methods mentioned above are explicit, hence are unsuitable for stiff systems; then there is the need to develop a method that is implicit in nature which have better stability properties and able to solve stiff problems. In this thesis, we addressed the problem by developing implicit rational Runge-Kutta schemes.

# CHAPTER THREE

# METHODOLOGY

## 3.0 RATIONAL RUNGE-KUTTA SCHEME FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this chapter, the derivation of one stage, two stage and three stage implicit rational Runge-Kutta schemes is considered using the techniques of rational form of Runge-Kutta methods proposed by Okunbor (1987).

The *s*-stage rational Runge-Kutta method as proposed by Soomiyol (2011) is given as follows:

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{r=1}^{s} w_r K_r}{1 + y'_n \sum_{r=1}^{s} v_r H_r} \tag{3.1}$$

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \tag{3.2}$$

where

$$K_r = \frac{h^2}{2} f\left( x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{r} b_{ij} K_j \right),\ i = 1(1)s\ , \tag{3.3}$$

$$H_r = \frac{h^2}{2} g\left( x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{r} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{r} \beta_{ij} H_j \right),\ i = 1(1)s\ , \tag{3.4}$$

with constraints

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j}^{i} b_{ij}, \quad i = 1(1)r \tag{3.5}$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j}^{i} \beta_{ij}, \quad i = 1(1)r \tag{3.6}$$

in which

$$g(x_n, z_n, z'_n) = -z_n^2 f(x_n, y_n, y'_n) \quad and\ z_n = \frac{1}{y_n} \tag{3.7}$$

The constraint equations (3.5) and (3.6) are to ensure consistency of the method, $h$ is the step size and the parameters $a_{ij}, b_{ij}, c_i, d_{i,} \alpha_{ij}, \beta_{ij}$ are constants called the parameters of the method.

In this research work, we shall consider the implicit scheme where all the $a_{ij}$ and $b_{ij} \neq 0$ for at least one $j > i$.

## 3.1 DERIVATION OF ONE STAGE IMPLICIT SCHEME

From equations (3.1), (3.2), (3.3) and (3.4), setting $s = 1$, we have

$$y_{n+1} = \frac{y_n + hy'_n + w_1 K_1}{1 + y'_n v_1 H_1} \tag{3.1.1}$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h} w'_1 K_1}{1 + \frac{1}{h} y'_n v'_1 H_1} \tag{3.1.2}$$

where

$$k_1 = \frac{h^2}{2} f\left(x_n + c_1 h, y_n + hc_1 y'_n + a_{11}K_1, y'_n + \frac{1}{h} b_{11} K_1\right), i = 1(1)s \quad (3.1.3a)$$

and

$$H_1 = \frac{h^2}{2} g\left(x_n + d_1 h, z_n + hd_1 z'_n + \alpha_{11}H_1, z'_n + \frac{1}{h} \beta_{11} H_1\right), i = 1(1)s \quad (3.1.3b)$$

with constraints

$$c_1 = a_{11} = \tfrac{1}{2} b_{11} \qquad and \qquad d_1 = \alpha_{11} = \tfrac{1}{2} \beta_{11} \quad (3.1.4)$$

where $c_1, a_{11}, b_{11}, d_1, \alpha_{11}, \beta_{11}, w_1, w'_1, v_1$ and $v'_1$ are all constants to be determined.

Equation (3.1.1) can be written as

$$y_{n+1} = \left(y_n + hy'_n + w_1 k_1\right)\left(1 + y_n v_1 H_1\right)^{-1} \quad (3.1.5)$$

Expanding the bracket and neglecting 2nd and higher orders gives

$$y_{n+1} = \left(y_n + hy'_n + w_1 k_1\right)\left(1 - y_n v_1 H_1\right) \quad (3.1.6)$$

Expanding (3.1.6) and re-arranging, gives

$$y_{n+1} = y_n + hy'_n - \left(y_n^2 v_1 + hy_n y'_n v_1\right)H_1 + (w_1 - y_n v_1 H_1 w_1)K_1 \quad (3.1.7)$$

Equation (3.1.2) can be written as

$$y'_{n+1} = (y'_n + \tfrac{1}{h} w'_1 K_1)\left(1 + \tfrac{1}{h} y'_n v'_1 H_1\right)^{-1} \quad (3.1.8)$$

$$y'_{n+1} = \left(y'_n + \frac{1}{h} w'_1 k_1\right)\left(1 - \frac{1}{h} y'_n v'_1 H_1\right)$$

Expanding the binomial and re-arranging also gives

$$y'_{n+1} = y'_n + \tfrac{1}{h} w'_1 K_1 - \left(\tfrac{1}{h} y'^2_n v'_1 + \tfrac{1}{h^2} y'_n w'_1 v'_1 K_1\right) H_1 \quad (3.1.9)$$

Now, the Taylor's series expansion of $y_{n+1}$ about $x_n$ is given as

$$y_{n+1} = y_n + hy'_n + \frac{h^2 y_n''}{2!} + \frac{h^3 y_n'''}{3!} + \frac{h^4 y_n^{iv}}{4!} + \ \dots \qquad (3.1.10)$$

and

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2 y_n'''}{2!} + \frac{h^3 y_n^{iv}}{3!} + \ \dots \qquad (3.1.11)$$

where

$$\left.\begin{aligned} y_n'' &= f(x_n, y_n, y_n') = f_n \\ y_n''' &= f_x + y' f_y + f_n f_{y'} = \Delta f_n \\ y_n^{iv} &= f_{xx} + y_n'^2 f_{yy} + f^2 f_{y'y'} + 2y' f_n f_{yy'} + 2 f_n f_{xy'} + f_{y'} \Delta f_n \\ y_n^{iv} &= \Delta^2 f_n + f_{y'} \Delta f_n + f_n f_y \end{aligned}\right\} \qquad (3.1.12)$$

Since $\Delta = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y'}$ (3.1.13)

The Taylor's series of the function of three variables is defined as

$$f(x+h, y+A, y'+B) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left( h\frac{\partial}{\partial x} + A\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \right)^j f(x', y', y'')' \right\}_{x=x', y=y', y'=y'} \qquad (3.1.14)$$

This implies that from equation (3.1.3a), we have

$$K_1 = \frac{h^2}{2} f\left( x_n + c_1 h, y_n + hc_1 y'_n + a_{11} K_1, y'_n + \frac{1}{h} b_{11} K_1 \right) \qquad (3.1.15)$$

Expanding (3.1.15) in Taylor's series gives

$$\frac{2}{h^2}K_1 = f_n + \left(c_1 h f_x + (hc_1 y'_n + a_{11}K_1)f_n + \frac{1}{h}b_{11}f_{y'}\right)$$

$$+\frac{1}{2!}\Big((c_1 h)^2 f_{xx} + 2c_1 h(hc_1 y'_n + a_{11}K_1)f_{xy} + 2c_1 h\left(\frac{1}{h}b_{11}K_1\right)f_{xy'}$$

$$+ (hc_1 y'_n + a_{11}K_1)^2 f_{yy} + 2(hc_1 y'_n + a_{11}K_1)\left(\frac{1}{h}b_{11}K_1\right)f_{yy'}$$

$$+\left(\frac{1}{h}b_{11}K_1\right)^2 f_{y'y'}\Big) + \ \ldots \qquad (3.1.16)$$

this implies that

$$K_1 = \frac{h^2}{2}f_n + \frac{h^2}{2}\left(c_1 h f_x + (hc_1 y'_n + a_{11}K_1)f_n + \frac{1}{h}b_{11}f_{y'}\right)$$

$$+\frac{h^2}{4}\Big((c_1 h)^2 f_{xx} + 2c_1 h(hc_1 y'_n + a_{11}K_1)f_{xy} + 2c_1 h\left(\frac{1}{h}b_{11}K_1\right)f_{xy'}$$

$$+ (hc_1 y'_n + a_{11}K_1)^2 f_{yy} + 2(hc_1 y'_n + a_{11}K_1)\left(\frac{1}{h}b_{11}K_1\right)f_{yy'}$$

$$+\left(\frac{1}{h}b_{11}K_1\right)^2 f_{y'y'}\Big) + \ 0(h^3)$$

Simplifying further and arranging the equation in powers of $h$ gives,

$$K_1 = \frac{h}{2}\left[b_{11}K_1 f_{y'} + a_{11}b_{11}K_1^2 f_{yy'}\right]$$

$$+\frac{h^2}{2}\left[f_n + a_{11}K_1 f_y + c_1 b_{11}K_1 f_{xy'} + a_{11}^2 K_1^2 f_{yy} + c_1 y'_n b_{11}K_1 f_{yy'}\right]$$

$$+\frac{h^3}{2}\left[c_1 f_x + c_1 y'_n f_y + c_1 a_{11}K_1 f_{xy} + c_1 a_{11} y'_n K_1 f_{yy}\right]$$

$$+\frac{h^4}{4}\left[c_1^2 f_{xx} + c_1^2 y'_n f_{xy} + c_1^2 y'^2_n f_{yy}\right] + 0(h^5) \qquad (3.1.17)$$

Equation (3.1.17) is implicit; one cannot proceed by successive substitution. Following *Lambert (1973)*, we can assume that the solution for $K_1$ may be express in the form

$$K_1 = hA_1 + h^2 B_1 + h^3 C_1 + h^4 D_1 + 0(h^5) \qquad (3.1.18)$$

Substituting equation (3.1.18) into (3.1.17) gives

$$K_1 = \frac{h}{2}\left[b_{11}(hA_1 + h^2B_1 + h^3C_1)f_{y'} + a_{11}b_{11}(hA_1 + h^2B_1)^2 f_{yy'}\right]$$
$$+ \frac{h^2}{2}\left[f_n + a_{11}(hA_1 + h^2B_1)f_y + c_1b_{11}(hA_1 + h^2B_1)f_{xy'} + a_{11}^2(hA_1)^2 f_{yy}\right.$$
$$\left. + c_1y_n'b_{11}(hA_1 + h^2B_1)f_{yy'}\right]$$
$$+ \frac{h^3}{2}\left[c_1f_x + c_1y_n'f_y + c_1a_{11}(hA_1)f_{xy} + c_1a_{11}y_n'(hA_1)f_{yy}\right]$$
$$+ \frac{h^4}{4}\left[c_1^2f_{xx} + c_1^2y_n'f_{xy} + c_1^2y'^2_nf_{yy}\right] + 0(h^5) \qquad (3.1.19)$$

On equating powers of $h$ from equation (3.1.17) and (3.1.18), gives

$A_1 = 0,\ B_1 = \frac{1}{2}f_n, \quad C_1 = \frac{1}{2}\left(c_1f_x + c_1y_n'f_y + 1/2b_{11}f_nf_{y'}\right) = \frac{1}{2}c_1\Delta f_n$ , since $c_1 = \frac{1}{2}b_{11}$

$$D_1 = \frac{1}{4}\left(c_1^2\Delta^2 f_n + b_{11}\Delta f_nf_{y'} + a_{11}f_nf_y\right) \qquad (3.1.20)$$

Substituting $A_1, B_1, C_1$ and $D_1$ into (3.1.18) gives

$$K_1 = \frac{h^2}{2}f_n + \frac{h^3}{2}c_1\Delta f_n + \frac{h^4}{2}\left(c_1^2\Delta^2 f_n + b_{11}\Delta f_nf_{y'} + a_{11}f_nf_y\right) \qquad (3.1.21)$$

Similarly, expanding $H_1$ in Taylor's series about $(x_n, z_n, z_n')$, from (3.1.3b), we have

$$\frac{2}{h^2}H_1 = g_n + \left(d_1hg_x + (hd_1z'_n + \alpha_{11}H_1)g_z + \frac{1}{h}\beta_{11}g_{z'}\right)$$
$$+ \frac{1}{2!}\left((d_1h)^2g_{xx} + 2d_1h(hd_1y'_n + \alpha_{11}H_1)g_{xz} + 2c_1h\left(\frac{1}{h}\beta_{11}H_1\right)g_{xz'}\right.$$
$$+ (hdy'_n + \alpha_{11}H_1)^2g_{zz} + 2(hd_1y'_n + \alpha_{11}H_1)\left(\frac{1}{h}\beta_{11}H_1\right)g_{zz'}$$
$$\left. + \left(\frac{1}{h}\beta_{11}H_1\right)^2 g_{z'z'}\right) + \ \dots$$

Expanding further as in the case of $K_1$ above gives

$$H_1 = \frac{h}{2}[\beta_{11}H_1 g_{z\prime} + \alpha_{11}\beta_{11}H_1^2 g_{zz\prime}]$$
$$+\frac{h^2}{2}[g_n + \alpha_{11}H_1 g_z + d_1\beta_{11}H_1 g_{xz\prime} + \alpha_{11}^2 H_1^2 g_{zz} + d_1 z_n'\beta_{11}H_1 g_{zz\prime}]$$
$$+\frac{h^3}{2}[d_1 g_x + d_1 z_n' g_z + d_1\alpha_{11}H_1 g_{xz} + d_1\alpha_{11}z_n' H_1 g_{zz}]$$
$$+\frac{h^4}{4}[c_1^2 g_{xx} + d_1^2 z_n' g_{xz} + d_1^2 z'^2_n g_{zz}] + 0(h^5) \quad (3.1.22)$$

Equation (3.1.22) is also implicit which cannot be proceed by successive substitution. Assuming a solution of the equation is of the form

$$H_1 = hL_1 + h^2 M_1 + h^3 N_1 + h^4 R_1 + 0(h^5) \quad (3.1.23)$$

Substituting the values of $H_1$ in (3.1.23) into equation (3.1.22) and equating powers of $h$ of the equation, we can get the following after substitutions:

$$L_1 = 0, \quad M_1 = \frac{1}{2}g_{n,} \quad N_1 = \frac{1}{2}d_1\Delta g_n \quad and\ R_1 = \frac{1}{4}(d_1^2\Delta^2 g_n + \beta_{11}\Delta g_n g_{z\prime} + \alpha_{11}g_n g_z)$$

(3.1.24)

Substituting equation (3.1.24) into equation (3.1.23) gives

$$H_1 = \frac{h^2}{2}g_n + \frac{h^3}{2}d_1\Delta g_n + \frac{h^4}{2}(d_1^2\Delta^2 g_n + \beta_{11}\Delta g_n g_{z\prime} + \alpha_{11}g_n g_z) \quad (3.1.25)$$

Using equations (3.1.18) and (3.1.23) into equations (3.1.7) and (3.1.9) respectively gives

$$y_{n+1} = y_n + hy_n' - (y_n^2 v_1 + hy_n y_n' v_1)(h^2 M_1 + h^3 N_1 + h^4 R_1)$$
$$+ [w_1 - y_n v_1 w_1(h^2 M_1 + h^3 N_1 + h^4 R_1)](h^2 B_1 + h^3 C_1 + h^4 D_1)$$

Expanding the brackets and re-arranging in powers of $h$ gives

$$y_{n+1} = y_n + hy_n' + h^2(w_1 B_1 - y_n^2 v_1 M_1) + h^3(w_1 C_1 - y_n^2 v_1 N_1 - y_n y_n' v_1 M_1) + 0(h^4)$$

(3.1.26)

Also for $y_{n+1}'$ gives

$$y'_{n+1} = y'_n + \frac{1}{h} w'_1 (h^2 B_1 + h^3 C_1 + h^4 D_1) - \left[\frac{1}{h} y'^2_n v'_n + \frac{1}{h^2} y'_n w'_1 v'_n (h^2 B_1 + h^3 C_1 + h^4 D_1)\right] (h^2 M_1 + h^3 N_1 + h^4 R_1)$$

Expanding the brackets and re-arrange in powers of $h$ gives

$$y'_{n+1} = y'_n + h(w'_1 B_1 - y'^2_n v'_n M_1) + h^2 (w'_1 C_1 - y'^2_n v'_n N_1 - y'_n w'_1 v'_n B_1 M_1) + h^3 (w'_1 D_1 - y'^2_n v'_n R_1 - y'_n w'_1 v'_n B_1 N_1 - y'_n w'_1 v'_n C_1 M_1) + 0(h^4) \quad \ldots (3.1.27)$$

Comparing the corresponding powers in $h$ of equations (3.1.26) and (3.1.27) with equations (3.1.10) and (3.1.11) we obtain

$$\left.\begin{aligned} &\tfrac{1}{2} w_1 f_n - \tfrac{1}{2} y'_n v_1 g_n = \tfrac{1}{2} f_n \\ &\tfrac{1}{2} w_1 w c_1 \Delta f_n - \tfrac{1}{2} y'^2_n v_1 d_1 \Delta g_n - \tfrac{1}{2} y_n y'_n v_1 g_n = \tfrac{1}{6} \Delta f_n \\ &\tfrac{1}{2} w'_1 f_n - \tfrac{1}{2} y'^2_n v'_1 g_n = f_n \\ &\tfrac{1}{2} w'_1 c_1 \Delta f_n - \tfrac{1}{2} y'^2_n v'_1 d_1 \Delta g_n - \tfrac{1}{2} y'_n w'_1 v'_1 f_n (\tfrac{1}{2} g_n) = \tfrac{1}{2} \Delta f_n \end{aligned}\right\} \quad (3.1.28)$$

(By using the equations in (3.1.20) and (3.1.24))

Since from equation (3.7)

$$\left.\begin{aligned} &g_n = -\frac{f_n}{y_n^2}, \quad g_x = -\frac{f_x}{y_n^2}, \quad g_z = -2\frac{f_n}{y_n} + f_y, \quad g_{z'} = -2\frac{f_n}{y_n} + f_{y'}, \quad z'_n = -\frac{y'_n}{y_n^2} \\ &\text{and} \\ &\Delta g_n = g_n + z'_n g_z + g_n g_{z'} \end{aligned}\right\} \quad (3.1.29)$$

Using those equations into equation (3.1.28), we get the following simultaneous equations

$$\left.\begin{aligned} &w_1 + v_1 = 1 \\ &w_1 c_1 + v_1 d_1 = \tfrac{1}{3} \\ &w'_1 + v'_1 = 2 \\ &w'_1 c_1 + v'_1 d_1 = 1 \end{aligned}\right\} \quad (3.1.30)$$

Equation (3.1.30) has (4) equations with (6) unknowns; there will not be a unique solution for (3.1.30). There will be a family of one-stage scheme of order four.

i. Choosing the parameters

$$w_1 = \frac{1}{3}, \quad v_1 = \frac{2}{3}, \quad c_1 = a_{11} = b_{11} = 0, \quad w_1' = 0, \quad v_1' = 2, \quad d_1 = \alpha_{11} = \frac{1}{2}, \quad \beta_{11} = 1$$

arbitrarily the following scheme is obtain.

$$y_{n+1} = \frac{hy_n' + \frac{1}{3}K_1}{1 + \frac{2}{3}y_n H_1} \tag{3.1.32}$$

and

$$y_{n+1}' = \frac{y_n'}{1 + \frac{2}{h}H_1} \tag{3.1.33}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y_n')$$

$$H_1 = \frac{h^2}{2} f\left( x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz_n' + \frac{1}{2}H_1, z_n' + \frac{1}{h}H_1 \right)$$ , since $d_1 = \alpha_{11} = \frac{1}{2}\beta_{11}$

ii. Choosing the parameters

From (3.1.30) setting

$$w_1 = v_1 = \frac{1}{2}, \;\; c_1 = a_{11} = \frac{1}{2}, \;\; d_1 = \alpha_{11} = \frac{1}{6}, \;\; w_1' = 2, \;\; v_1' = 0, \;\; b_{11} = 1, \;\; \beta_{11} = \frac{1}{3}$$

Then,

$$y_{n+1} = \frac{y_n + hy_n' + \frac{1}{2}K_1}{1 + \frac{1}{2}y_n H_1} \tag{3.1.34}$$

and

$$y'_{n+1} = y'_n + \frac{2}{h}K_1 \qquad (3.1.35)$$

where

$$K_1 = \frac{h^2}{2} f\left( x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy'_n + \frac{1}{2}K_1, y'_n + \frac{1}{h}K_1 \right), \text{ since } c_1 = a_{11} = \frac{1}{2}b_{11}$$

and

$$H_1 = \frac{h^2}{2} g\left( x_n + \frac{1}{6}h, z_n + \frac{1}{6}hz'_n + \frac{1}{6}H_1, z'_n + \frac{1}{3h}H_1 \right), \text{ since } d_1 = \alpha_{11} = \frac{1}{2}\beta_{11}$$

## 3.2 DERIVATION OF TWO - STAGE SCHEME

From equations (3.1), (3.2), (3.3) and (3.4), setting $s = 2$, we can have

$$y_{n+1} = \frac{y_n + hy'_n + w_1K_1 + w_2K_2}{1 + y_n(v_1H_1 + v_2H_2)} \qquad (3.2.1)$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h}(w'_1K_1 + w'_2K_2)}{1 + \frac{1}{h}y'_n(v'_1H_1 + v'_2H_2)} \qquad (3.2.2)$$

where

$$k_2 = \frac{h^2}{2} f\left( x_n + c_2h, y_n + hc_2y'_n + a_{21}K_1 + a_{22}K_2, y'_n + \frac{1}{h}(b_{21}K_1 + b_{22}K_2) \right) \qquad (3.2.3a)$$

and

$$H_2 = \frac{h^2}{2} g\left( x_n + d_2h, z_n + hd_2z'_n + \alpha_{21}H_1 + \alpha_{22}H_2, z'_n + \frac{1}{h}(\beta_{21}H_1 + \beta_{22}H_2) \right) \qquad (3.2.3b)$$

with constraints

$$c_2 = a_{21} + a_{22} = \tfrac{1}{2}(b_{21} + b_{22}) \quad \text{and} \quad d_2 = \alpha_{21} + \alpha_{22} = \tfrac{1}{2}(\beta_{21} + \beta_{22}) \qquad (3.2.4)$$

Where $c_2, a_{21}, a_{22}, b_{21}, b_{22}\ d_2, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}, w_1, w'_1, v_1$ and $v'_1$ are all constants to be determine.

Now by adopting a binomial expansion on equations (3.2.1) and (3.2.2), gives

This implies that from equation (3.2.1), we have

$$y_{n+1} = (y_n + hy'_n + w_1K_1 + w_2K_2)(1 + y_n(v_1H_1 + v_2H_2))^{-1}$$

$$y_{n+1} = (y_n + hy'_n + w_1K_1 + w_2K_2)\big(1 - y_n(v_1H_1 + v_2H_2)\big) - \text{Higher order terms}$$

Expanding, re-arranging and ignoring the higher order terms, gives

$$\begin{aligned} y_{n+1} = y_n + hy'_n + w_1K_1 + w_2K_2 - y_n^2(v_1H_1 + v_2H_2) - hy_ny'_n(v_1H_1 + v_2H_2) \\ - y_n(w_1K_1 + w_2K_2)(v_1H_1 + v_2H_2) \end{aligned} \quad (3.2.5)$$

also

$$y'_{n+1} = \left(y'_n + \frac{1}{h}(w'_1K_1 + w'_2K_2)\right)\left(1 - \frac{1}{h}y'_n(v'_1H_1 + v'_2H_2)\right)^{-1}$$

Expanding and re-arranging also gives

$$\begin{aligned} y'_{n+1} = y'_n - \frac{1}{h}y'^2_n(v'_1H_1 + v'_2H_2) + \frac{1}{h}(w'_1K_1 + w'_2K_2) \\ - \frac{1}{h^2}y'_n(w'_1K_1 + w'_2K_2)(v'_1H_1 + v'_2H_2) \end{aligned} \quad (3.2.6)$$

From equations (3.2.3a) and (3.2.3b), the Taylor series expansion of $K_2$ about $(x_n, y_n, y'_n)$ is given as

$$\begin{aligned} \frac{2}{h^2}K_2 = f_n + \left(c_2hf_x + (hc_2y'_n + a_{21}K_1 + a_{22}K_2)f_n + \frac{1}{h}(b_{21}K_1 + b_{22}K_2)f_{y'}\right) \\ + \frac{1}{2!}\Bigg((c_2h)^2f_{xx} + 2c_2h(hc_1y'_n + a_{21}K_1 + a_{22}K_2)f_{xy} \\ + 2c_2(b_{21}K_1 + b_{22}K_2)f_{xy'} + (hc_2y'_n + a_{21}K_1 + a_{22}K_2)^2f_{yy} \\ + \frac{2}{h}(a_{21}K_1 + a_{22}K_2)(b_{21}K_1 + b_{22}K_2)f_{yy'} + \left(\frac{1}{h}(b_{21}K_1 + b_{22}K_2)\right)^2 f_{y'y'}\Bigg) \\ + \dots \end{aligned}$$

Expanding the brackets and re-arranging in powers of $h$ gives

$$\frac{2}{h^2}K_2 = f_n + (a_{21}K_1 + a_{22}K_2)f_y$$
$$+ h\left[c_2f_x + c_2y_n'f_y + c_2(a_{21}K_1 + a_{22}K_2)f_{xy} + 2c_2y_n'\,(a_{21}K_1 + a_{22}K_2)f_{yy}\right]$$
$$+ \frac{h^2}{2}\left[{c_2}^2f_{xx} + 2c_2y_n'f_{xy} + c_2^2y_n'^2f_{yy}\right]$$
$$+ \frac{1}{2}\left[2c_2(b_{21}K_1 + b_{22}K_2)f_{xy'} + (a_{21}K_1 + a_{22}K_2)^2f_{yy}\right.$$
$$\left. + 2c_2y_n'(b_{21}K_1 + b_{22}K_2)f_{yy'}\right] + \frac{2}{h}(a_{21}K_1 + a_{22}K_2)(b_{21}K_1 + b_{22}K_2)f_{yy'}$$
$$+ \frac{1}{h}(b_{21}K_1 + b_{22}K_2)f_{y'} + \frac{1}{h^2}(b_{21}K_1 + b_{22}K_2)^2f_{y'y'} + 0(h^3)$$

This implies that

$$K_2 = \frac{h^2}{2}\left[f_n + (a_{21}K_1 + a_{22}K_2)f_y + c_2(b_{21}K_1 + b_{22}K_2)f_{xy'} + (a_{21}K_1 + a_{22}K_2)^2f_{yy}\right.$$
$$\left. + c_2y_n'(b_{21}K_1 + b_{22}K_2)f_{yy'}\right]$$
$$+ \frac{h^3}{2}\left[c_2f_x + c_2y_n'f_y + c_2(a_{21}K_1 + a_{22}K_2)f_{xy} + c_2y_n'\,(a_{21}K_1 + a_{22}K_2)f_{yy}\right]$$
$$+ \frac{h^4}{4}\left[{c_2}^2f_{xx} + 2c_2y_n'f_{xy'} + c_2^2y_n'^2f_{yy}\right]$$
$$+ h(a_{21}K_1 + a_{22}K_2)(b_{21}K_1 + b_{22}K_2)f_{yy'} + \frac{h}{2}(b_{21}K_1 + b_{22}K_2)f_{y'}$$
$$+ \frac{1}{2}(b_{21}K_1 + b_{22}K_2)^2f_{y'y'} + 0(h^5) \qquad (3.2.7)$$

Equation (3.2.7) is implicit, which cannot be proceed by successive substitutions. We assume a solution for $K_2$ which may be expressed as

$$K_2 = h^2B_2 + h^3C_2 + h^4D_2 + 0(h^5) \qquad (3.2.8)$$

Substituting equation (3.2.8) into (3.2.7) gives

$$K_2 = \frac{h^2}{2}\Big[f_n + \big(a_{21}(h^2B_1 + h^3C_1 + h^4D_1) + a_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_y$$

$$+ c_2\big(b_{21}(h^2B_1 + h^3C_1 + h^4D_1) + b_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{xy'}$$

$$+ \big(a_{21}(h^2B_1 + h^3C_1 + h^4D_1) + a_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)^2 f_{yy}$$

$$+ c_2y_n'\big(b_{21}(h^2B_1 + h^3C_1 + h^4D_1) + b_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{yy'}\Big]$$

$$+ \frac{h^3}{2}\big[c_2f_x + c_2y_n'f_y$$

$$+ c_2\big(a_{21}(h^2B_1 + h^3C_1 + h^4D_1) + a_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{xy}$$

$$+ c_2y_n'\,\big(a_{21}(h^2B_1 + h^3C_1 + h^4D_1) + a_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{yy}\big]$$

$$+ \frac{h^4}{4}\big[{c_2}^2f_{xx} + 2c_2y_n'f_{xy'} + c_2^2y_n'^2f_{yy}\big]$$

$$+ h\big(a_{21}(h^2B_1 + h^3C_1 + h^4D_1)$$

$$+ a_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)\big(b_{21}(h^2B_1 + h^3C_1 + h^4D_1)$$

$$+ b_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{yy'}$$

$$+ \frac{h}{2}\big(b_{21}(h^2B_1 + h^3C_1 + h^4D_1) + b_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)f_{y'}$$

$$+ \frac{1}{2}\big(b_{21}(h^2B_1 + h^3C_1 + h^4D_1) + b_{22}(h^2B_2 + h^3C_2 + h^4D_2)\big)^2 f_{y'y'} + \cdots$$

Expanding and re-arranging gives

$$K_2 = \frac{h^2}{2}\,f_n + \frac{h^3}{2}\big[c_2f_x + c_2y_n'f_y + (b_{21}B_1 + b_{22}B_2)f_{y'}\big]$$
$$+ \frac{h^4}{4}\big[{c_2}^2f_{xx} + 2c_2y_n'f_{xy'} + c_2^2y_n'^2f_{yy} + 2(b_{21}B_1 + b_{22}B_2)^2f_{yy'}$$
$$+ 2c_2(b_{21}B_1 + b_{22}B_2)f_{xy'} + c_2y_n'(b_{21}B_1 + b_{22}B_2)f_{yy'}$$
$$+ 2(a_{21}B_1 + a_{22}B_2)f_y + 2(b_{21}C_1 + b_{22}C_2)f_{y'}\big] + 0(h^5) \qquad (3.2.9)$$

On equating powers of $h$ from equation (3.2.9) and (3.2.8), gives

$$\left.\begin{aligned} B_2 &= \tfrac{1}{2}f_n \\ C_2 &= \tfrac{1}{2}\big(c_2f_x + c_2y_n'f_y + (b_{21}B_1 + b_{22}B_2)f_{y'}\big) = \tfrac{1}{2}c_2\Delta f_n \\ D_2 &= \tfrac{1}{4}\big(c_2^2\Delta^2 f_n + 2c_2\Delta f_nf_{y'} + c_2f_nf_y\big) \end{aligned}\right\} \qquad (3.2.10)$$

since $\left(c_2 = a_{21} + a_{22} = \frac{1}{2}(b_{21} + b_{22})\right)$

then

$$K_2 = \frac{h^2}{2} f_n + \frac{h^3}{2} c_2 \Delta f_n + \frac{h^4}{4}\left(c_2^2 \Delta^2 f_n + 2c_2 \Delta f_n f_{y'} + c_2 f_n f_y\right) + 0(h^5) \qquad (3.2.11)$$

In a similar manner

$$H_2 = h^2 M_2 + h^3 N_2 + h^4 R_2 + 0(h^5) \qquad (3.2.12)$$

Where

$$\left.\begin{aligned} M_2 &= \frac{1}{2} g_n \\ N_2 &= \frac{1}{2} d_2 \Delta g f_n \\ R_2 &= \frac{1}{4}(d_2^2 \Delta^2 g_n + 2d_2 \Delta g_n g_{z'} + d_2 g_n g_z) \end{aligned}\right\} \qquad (3.2.13)$$

And also,

$$H_2 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_2 \Delta g f_n + \frac{h^4}{4}(d_2^2 \Delta^2 g_n + 2d_2 \Delta g_n g_{z'} + d_2 g_n g_z) + 0(h^5) \qquad (3.2.14)$$

Substituting equations (3.1.20), (3.1.23), (3.2.8) and (3.2.12) into equations (3.2.5) and (3.2.6), re – arranging and compare the resulting equation with equations (3.1.7) and (3.1.8) gives the following

$$\left.\begin{aligned} w_1 B_1 + w_2 B_2 - y_n^2(v_1 M_1 + v_2 M_2) &= \frac{1}{2} f_n \\ w_1 C_1 + w_2 C_2 - y_n^2(v_1 M_1 + v_2 M_2) &= \frac{1}{6} \Delta f_n \\ w_1' B_1 + w_2' B_2 - y_n^2(v_1' M_1 + v_2' M_2) &= f_n \\ w_1' C_1 + w_2' C_2 - y_n'^2(v_1' M_1 + v_2' M_2) &= \frac{1}{2} \Delta f_n \end{aligned}\right\} \qquad (3.2.15)$$

Substituting the values of $B_1$, $B_2$, $C_1$, $C_2$, $M_1$, $M_2$, $N_2$, and $N_2$, we have the following non – liner differential equations

$$
\left.\begin{aligned}
&w_1 + w_2 + v_1 + v_2 = 1\\
&w_1 c_1 + w_2 c_2 + v_1 d_1 + v_2 d_2 = \tfrac{1}{3}\\
&w_1' + w_2' + v_1' + v_2' = 2\\
&w_1' c_1 + w_1' c_1 + v_1' d_1 + v_2' d_2 = 1
\end{aligned}\right\} \qquad (3.2.16)
$$

Which are four equations with twelve unknowns from (3.2.16).This implies that there are families of solutions.

1. From (3.2.16) setting $w_1 = w_2 = 0,\ \ v_1 = \frac{1}{4},\ \ v_2 = \frac{3}{4},\ \ w_1' = w_2' = 0,\ \ v_1' = v_2' = 1,\ \ d_1 = \frac{1}{6},\ \ d_2 = \frac{5}{6}$

   Then we have,

$$y_{n+1} = \frac{y_n + h y_n'}{1 + y_n \frac{1}{4}(H_1 + 3H_2)} \qquad (3.2.17)$$

and

$$y'_{n+1} = \frac{y_n'}{1 + \frac{1}{h} y_n' (H_1 + H_2)} \qquad (3.2.18)$$

where

$$H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{6}h, z_n + \frac{1}{6}hz_n' + \frac{1}{6}H_1, z_n' + \frac{1}{3h}H_1\right)$$

$$H_2 = \frac{h^2}{2} g\left(x_n + \frac{5}{6}h, z_n + \frac{5}{6}hz_n' + \frac{1}{2}H_1 + \frac{1}{3}H_2, z_n' + \frac{1}{h}\left(H_1 + \frac{2}{3}H_2\right)\right)$$

2. Also from (3.2.16) setting $w_1 = w_2 = \frac{1}{8},\ \ v_1 = v_2 = \frac{3}{8},\ \ \ c_1 = c_2 = d_1 = d_2 = \frac{1}{3}$ and $w_1' = w_2' = v_1' = v_2' = \frac{1}{2}$

   Then we have,

$$y_{n+1} = \frac{y_n + h y_n' + \frac{1}{8}(K_1 + K_2)}{1 + y_n \frac{3}{8}(H_1 + H_2)} \qquad (3.2.19)$$

and

$$y'_{n+1} = \frac{y'_n + \frac{1}{2h}(K_1 + K_2)}{1 + \frac{1}{2h} y'_n (H_1 + H_2)} \quad (3.2.20)$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hy'_n + \frac{1}{3}K_1, y'_n + \frac{2}{3h}K_1\right)$$

$$K_2 = \frac{h^2}{2} f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hy'_n + \frac{1}{3}(K_1 + K_2), y'_n + \frac{2}{3h}(K_1 + K_2)\right)$$

and

$$H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{3}h, z_n + \frac{1}{3}hz'_n + \frac{1}{3}H_1, z'_n + \frac{2}{3h}H_1\right)$$

$$H_2 = \frac{h^2}{2} g\left(x_n + \frac{1}{3}h, z_n + \frac{1}{3}hz'_n + \frac{1}{3}(H_1 + H_2), z'_n + \frac{2}{3h}(H_1 + H_2)\right)$$

## 3.3 DERIVATION OF THREE - STAGE SCHEME

From equations (3.1), (3.2), (3.3) and (3.4), setting $s = 3$, we can have

$$y_{n+1} = \frac{y_n + hy'_n + w_1K_1 + w_2K_2 + w_3K_3}{1 + y_n(v_1H_1 + v_2H_2 + v_3H_3)} \quad (3.3.1)$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h}(w'_1K_1 + w'_2K_2 + w'_3K_3)}{1 + \frac{1}{h}y'_n(v'_1H_1 + v'_2H_2 + v'_3H_3)} \quad (3.3.2)$$

where

$$K_3 = \frac{h^2}{2} f\left(x_n + c_3h, y_n + hc_3y'_n + a_{31}K_1 + a_{32}K_2 + a_{33}K_3, y'_n + \frac{1}{h}(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)\right), \quad (3.3.3a)$$

and

$$H_3 = \frac{h^2}{2} g\left(x_n + d_3 h, z_n + hd_3 z'_n + \alpha_{31}H_1 + \alpha_{32}H_2 + \alpha_{33}H_3, z'_n + \frac{1}{h}(\beta_{31}H_1 + \beta_{32}H_2 + \beta_{33}H_3)\right) \quad (3.2.3b)$$

with constraints

$$\left.\begin{aligned} c_3 &= a_{31} + a_{32} + a_{33} = \tfrac{1}{2}(b_{31} + b_{32} + b_{33}) \\ d_3 &= \alpha_{31} + \alpha_{32} + \alpha_{33} = \tfrac{1}{2}(\beta_{31} + \beta_{32} + \beta_{33}) \end{aligned}\right\} \quad (3.3.4)$$

where $c_3, a_{31}, a_{32}, a_{33}$ $b_{31}, b_{32}, b_{33}, d_3, \alpha_{31}, \alpha_{23}, \alpha_{23}, \beta_{31}, \beta_{32}, \beta_{33}, w_1, w_2, w_3,$ $w'_1, w'_2, w'_3, v_1, v_2, v_3$ and $v'_1, v'_2, v'_3$ are all constants to be determine.

Now by adopting a binomial expansion on equations (3.3.1) and (3.3.2), gives

This implies that from equation (3.3.1), we have

$$y_{n+1} = (y_n + hy'_n + w_1K_1 + w_2K_2 + w_3K_3)(1 + y_n(v_1H_1 + v_2H_2 + v_3H_3))^{-1}$$

$y_{n+1} = (y_n + hy'_n + w_1K_1 + w_2K_2 + w_3K_3)\big(1 - y_n(v_1H_1 + v_2H_2 + v_3H_3)\big)$ – Higher order terms

Expanding, re-arranging and ignoring the higher order terms, gives

$$\begin{aligned} y_{n+1} = y_n + hy'_n + w_1K_1 + w_2K_2 + w_3K_3 - y_n^2(v_1H_1 + v_2H_2 + v_3H_3) \\ - hy_ny'_n(v_1H_1 + v_2H_2 + v_3H_3) \\ - y_n[(w_1K_1 + w_2K_2 + w_3K_3)(v_1H_1 + v_2H_2 \\ + v_3H_3)] \end{aligned} \quad (3.3.5)$$

also

$$y'_{n+1} = \left(y'_n + \frac{1}{h}(w'_1K_1 + w'_2K_2 + w'_3K_3)\right)\left(1 - \frac{1}{h}y'_n(v'_1H_1 + v'_2H_2 + v'_3K_3)\right)^{-1}$$

Expanding and re-arranging also gives

$$y'_{n+1} = y'_n - \frac{1}{h} y'^2_n (v'_1 H_1 + v'_2 H_2 + v'_3 K_3) + \frac{1}{h}(w'_1 K_1 + w'_2 K_2 + w'_3 K_3)$$
$$- \frac{1}{h^2} y'_n (w'_1 K_1 + w'_2 K_2 + w'_3 K_3)(v'_1 H_1 + v'_2 H_2$$
$$+ v'_3 K_3) \qquad (3.3.6)$$

Now, expanding $K_3$Taylor series gives

$$\frac{2}{h^2} K_3 = f_n + \Big( c_3 h f_x + (hc_3 y'_n + a_{31} K_1 + a_{32} K_2 + a_{33} K_3) f_y$$
$$+ \frac{1}{h}(b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{y'} \Big)$$
$$+ \frac{1}{2!} \Big( (c_3 h)^2 f_{xx} + 2 c_3 h (hc_1 y'_n + a_{31} K_1 + a_{32} K_2 + a_{33} K_3) f_{xy}$$
$$+ 2 c_2 (b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{xy'}$$
$$+ (hc_2 y'_n + a_{31} K_1 + a_{32} K_2 + a_{33} K_3)^2 f_{yy}$$
$$+ \frac{2}{h}(a_{31} K_1 + a_{32} K_2 + a_{33} K_3)(b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{yy'}$$
$$+ \frac{1}{h^2} \big( (b_{31} K_1 + b_{32} K_2 + b_{33} K_3) \big)^2 f_{y'y'} \Big) + \ldots$$

Expanding the brackets and re-arranging in powers of $h$ gives

$$\frac{2}{h^2} K_3 = f_n + (a_{31} K_1 + a_{32} K_2 + a_{33} K_3) f_y$$
$$+ h \big[ c_3 f_x + c_3 y'_n f_y + c_3 (a_{31} K_1 + a_{32} K_2 + a_{33} K_3) f_{xy}$$
$$+ 2 c_3 y'_n \, (a_{31} K_1 + a_{32} K_2 + a_{33} K_3) f_{yy} \big] + \frac{h^2}{2} \big[ {c_3}^2 f_{xx} + 2 c_3 y'_n f_{xy} + c_3^2 y'^2_n f_{yy} \big]$$
$$+ \frac{1}{2} \big[ 2 c_3 (b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{xy'} + (a_{31} K_1 + a_{32} K_2 + a_{33} K_3)^2 f_{yy}$$
$$+ 2 c_3 y'_n (b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{yy'} \big]$$
$$+ \frac{2}{h}(a_{31} K_1 + a_{32} K_2 + a_{33} K_3)(b_{21} K_1 + b_{22} K_2) f_{yy'}$$
$$+ \frac{1}{h}(b_{31} K_1 + b_{32} K_2 + b_{33} K_3) f_{y'} + \frac{1}{h^2}(b_{31} K_1 + b_{32} K_2 + b_{33} K_3)^2 f_{y'y'}$$
$$+ 0(h^3)$$

this implies that

$$K_3 = \frac{h^2}{2}\left[f_n + (a_{31}K_1 + a_{32}K_2 + a_{33}K_3)f_y + 2c_3(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)f_{xy'} + (a_{31}K_1 + a_{32}K_2 + a_{33}K_3)^2 f_{yy} + 2c_3y'_n(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)f_{yy'}\right] + \frac{h^3}{2}\left[c_3f_x + c_3y'_nf_y + c_3(a_{31}K_1 + a_{32}K_2 + a_{33}K_3)f_{xy} + c_3y'_n\,(a_{31}K_1 + a_{32}K_2 + a_{33}K_3)f_{yy}\right] + \frac{h^4}{4}\left[c_3{}^2f_{xx} + 2c_3y'_nf_{xy} + c_3^2y'^2_nf_{yy}\right] + h(a_{31}K_1 + a_{32}K_2 + a_{33}K_3)(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)f_{yy'} + \frac{h}{2}(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)f_{y'} + \frac{1}{2}(b_{31}K_1 + b_{32}K_2 + b_{33}K_3)^2 f_{y'y'} + 0(h^5) \qquad (3.3.7)$$

Equation (3.3.7) is implicit, which cannot be proceed by successive substitutions. We assume a solution for $K_2$ which may be expressed as

$$K_3 = h^2B_3 + h^3C_3 + h^4D_3 + 0(h^5) \qquad (3.3.8)$$

Substituting the values of $K_1$, $K_2$ and $K_3$ into equation (3.2.7) and re – arranging in powers of $h$

Expanding and re-arranging gives

$$K_3 = \frac{h^2}{2}\,f_n + \frac{h^3}{2}\left[c_3f_x + c_3y'_nf_y + \frac{1}{2}(b_{31}B_1 + b_{32}B_2 + b_{33}B_3)f_{y'}\right] + \frac{h^4}{4}\left[c_3^2f_{xx} + 2c_3y'_nf_{xy'} + c_3^2y'^2_nf_{yy} + 2(b_{31}B_1 + b_{32}B_2 + b_{33}B_3)^2f_{yy'} + 2c_3(b_{31}B_1 + b_{32}B_2 + b_{33}B_3)f_{xy'} + c_2y'_n(b_{31}B_1 + b_{32}B_2 + b_{33}B_3)f_{yy'} + 2(b_{31}B_1 + b_{32}B_2 + b_{33}B_3)f_y + 2(b_{31}C_1 + b_{32}C_2 + b_{33}C_3)f_{y'}\right] + 0(h^5) \qquad (3.3.9)$$

On equating powers of $h$ from equation (3.3.9) and (3.3.8), gives

$$\left.\begin{aligned} B_3 &= \tfrac{1}{2}f_n \\ C_3 &= \tfrac{1}{2}\left(c_3f_x + c_3y'_nf_y + (b_{31}B_1 + b_{32}B_2 + b_{33}B_3)f_{y'}\right) = \tfrac{1}{2}c_3\Delta f_n \\ D_2 &= \tfrac{1}{4}\left(c_3^2\Delta^2 f_n + (b_{31}C_1 + b_{32}C_2 + b_{33}C_3)\Delta f_nf_{y'} + c_3f_nf_y\right) \end{aligned}\right\} \qquad (3.3.10)$$

then

$$K_2 = \frac{h^2}{2} f_n + \frac{h^3}{2} c_3 \Delta f_n + \frac{h^4}{4}\left(c_3^2 \Delta^2 f_n + (b_{31}C_1 + b_{32}C_2 + b_{33}C_3)\Delta f_n f_{y'} + c_2 f_n f_y\right) + 0(h^5) \quad (3.3.11)$$

In a similar manner

$$H_3 = h^2 M_3 + h^3 N_3 + h^4 R_3 + 0(h^5) \quad (3.3.12)$$

Where

$$\left.\begin{aligned} M_3 &= \tfrac{1}{2} g_n \\ N_3 &= \tfrac{1}{2} d_3 \Delta g f_n \\ R_3 &= \tfrac{1}{4}(d_3^2 \Delta^2 g_n + (\beta_{31}d_1 + \beta_{32}d_2 + \beta_{33}d_3)\Delta g_n g_{z'} + d_3 g_n g_z) \end{aligned}\right\} \quad (3.3.13)$$

And also,

$$H_3 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_3 \Delta g f_n + \frac{h^4}{4}(d_3^2 \Delta^2 g_n + (\beta_{31}d_1 + \beta_{32}d_2 + \beta_{33}d_3)\Delta g_n g_{z'} + d_3 g_n g_z) + 0(h^5) \quad (3.3.14)$$

Substituting equations (3.1.15), (3.1.21), (3.2.8), (3.3.8), (3.2.12) and (3.3.12) into equations (3.2.5), (3.2.6), (3.3.5) and (3.3.6) re – arranging and compare the resulting equation with equations (3.1.7) and (3.1.8) gives the following

$$w_1B_1 + w_2B_2 + w_3B_3 - y_n^2(v_1M_1 + v_2M_2 + v_3M_3) = \tfrac{1}{2} f_n$$

$$w_1C_1 + w_2C_2 + w_3B_3 - y_n^2(v_1N_1 + v_2N_2 + v_3N_3) = \tfrac{1}{6} \Delta f_n$$

$$w_1D_1 + w_2D_2 + w_3D_3 - y_n^2(v_1R_1 + v_2R_2 + v_3R_3) = \tfrac{1}{24}\left(\Delta^2 f_n + \Delta f_n f_{y'} + f_n f_y\right)$$

$$w_1'B_1 + w_2'B_2 + w_3'B_3 - y_n^2(v_1'M_1 + v_2'M_2 + v_3'M_3) = f_n$$

$$w_1'C_1 + w_2'C_2 + w_3'C_3 - y_n'^2(v_1'N_1 + v_2'N_2 + v_3'N_3) = \tfrac{1}{2} \Delta f_n$$

$w'_1 D_1 + w'_2 D_2 + w'_3 D_3 - y'^2_n (v'_1 R_1 + v'_2 R_2 + v'_3 R_3) = \frac{1}{6}\left(\Delta^2 f_n + \Delta f_n f_{y'} + f_n f_y\right)$

Substituting the values of $B_1$, $B_2$, $B_3$, $C_1$, $C_2$, $C_3$, $M_1$, $M_2$, $M_3$, $N_1$, $N_2$, $N_3$, $R_1$, $R_2$, and $R_3$ in the above equation and re-arranging, we have the following sets of non – liner differential equations

$w_1 + w_2 + w_3 + v_1 + v_2 + v_3 = 1$

$w_1 c_1 + w_2 c_2 + w_3 c_3 + v_1 d_1 + v_2 d_2 + v_3 d_3 = \frac{1}{3}$

$w_1 c_1^2 + w_2 c_2^2 + w_3 c_3^2 + v_1 d_1^2 + v_2 d_2^2 + v_3 d_3^2 = \frac{1}{6}$

$w_1(c_1 b_{11} + c_2 b_{12} + c_3 b_{13}) + w_2(c_1 b_{21} + c_2 b_{22} + c_3 b_{23}) + w_3(c_1 b_{31} + c_2 b_{32} + c_3 b_{33}) + v_1(\beta_{11} d_1 + \beta_{12} d_2 + \beta_{13} d_3) + v_2(\beta_{21} d_1 + \beta_{22} d_2 + \beta_{23} d_3) + v_3(\beta_{31} d_1 + \beta_{32} d_2 + \beta_{33} d_3) = \frac{1}{6}$

$w'_1 + w'_2 + w'_3 + v'_1 + v'_2 + v'_3 = 2$

$w'_1 c_1 + w'_2 c_2 + w'_3 c_3 + v'_1 d_1 + v'_2 d_2 + v'_3 d_3 = 1$

$w'_1 c_1^2 + w'_2 c_2^2 + w'_3 c_3^2 + v'_1 d_1^2 + v'_2 d_2^2 + v'_3 d_3^2 = \frac{2}{3}$

$w'_1(c_1 b_{11} + c_2 b_{12} + c_3 b_{13}) + w'_2(c_1 b_{21} + c_2 b_{22} + c_3 b_{23}) + w'_3(c_1 b_{31} + c_2 b_{32} + c_3 b_{33}) + v'_1(\beta_{11} d_1 + \beta_{12} d_2 + \beta_{13} d_3) + v'_2(\beta_{21} d_1 + \beta_{22} d_2 + \beta_{23} d_3) + v'_3(\beta_{31} d_1 + \beta_{32} d_2 + \beta_{33} d_3) = \frac{2}{3}$

With constraints

$$c_1 = a_{11} + a_{12} + a_{13} = \frac{1}{2}(b_{11} + b_{12} + b_{13})$$

$$c_2 = a_{21} + a_{22} + a_{23} = \frac{1}{2}(b_{21} + b_{22} + b_{23})$$

$$c_3 = a_{31} + a_{32} + a_{33} = \frac{1}{2}(b_{31} + b_{32} + b_{33})$$

$$d_1 = \alpha_{11} + \alpha_{12} + \alpha_{13} = \frac{1}{2}(\beta_{11} + \beta_{12} + \beta_{13})$$

$$d_2 = \alpha_{21} + \alpha_{22} + \alpha_{23} = \frac{1}{2}(\beta_{21} + \beta_{22} + \beta_{23})$$

$d_3 = \alpha_{31} + \alpha_{32} + \alpha_{33} = \frac{1}{2}(\beta_{31} + \beta_{32} + \beta_{33})$

This is fourteen (14) equations with fifty-four (54) unknowns. That means the expected scheme it not unique; we can have a family of 3 – stage schemes.

1. Choosing the Parameters

$w_1 = w_2 = w_3 = 0, \quad v_1 = v_2 = v_3 = \frac{1}{3}, \quad c_1 = d_1 = a_{11} = b_{11} = \alpha_{11} = \beta_{11} = 0$

$w_1' = w_2' = \frac{1}{2}, \quad w_3' = 1, \quad v_1' = v_2' = v_3' = 0, \quad c_2 = c_3 = b_{21} = b_{31} = \frac{2}{3}$

$d_2 = d_3 = a_{12} = b_{12} = \beta_{12} = \beta_{21} = \beta_{31} = \frac{1}{2}, \; a_{22} = a_{23} = a_{32} = a_{33} = \frac{1}{6},$

$a_{21} = a_{31} = b_{22} = b_{23} = b_{32} = b_{33} = \frac{1}{3}, \; \alpha_{12} = \alpha_{21} = \alpha_{31} = \beta_{22} = \beta_{23} = \beta_{32} = \beta_{33} = \frac{1}{4}, \; a_{13} = b_{13} = \beta_{13} = -\frac{1}{2}, \quad \alpha_{13} = -\frac{1}{4}, \quad \alpha_{22} = \alpha_{23} = \alpha_{32} = \alpha_{33} = \frac{1}{8}$

Then equations (3.3.1) and (3.3.2) becomes

$$y_{n+1} = \frac{y_n + hy_n'}{1 + y_n \frac{1}{3}(H_1 + H_2 + H_3)} \tag{3.3.14}$$

and

$$y_{n+1}' = y_n' + \frac{1}{h}\left(\frac{1}{2}K_1 + \frac{1}{2}K_2 + K_3\right) \tag{3.3.15}$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n, y_n + \frac{1}{2}K_2 - \frac{1}{2}K_3, y_n' + \frac{1}{h}\left(\frac{1}{2}K_2 - \frac{1}{2}K_3\right)\right)$$

$$K_2 = K_3 = \frac{h^2}{2} f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hy_n' + \frac{1}{6}(2K_1 + K_2 + K_3), y_n' + \frac{1}{h}\left(\frac{2}{3}K_1 + \frac{1}{3}K_2 + \frac{1}{3}K_3\right)\right)$$

and

$$H_1 = \frac{h^2}{2} g\left(x_n, z_n + \frac{1}{4}H_2 - \frac{1}{4}H_3, z_n' + \frac{1}{h}\left(\frac{1}{4}H_2 - \frac{1}{4}H_3\right)\right)$$

$$H_2 = H_3 = \frac{h^2}{2} g\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz_n' + \frac{1}{4}H_1 + \frac{1}{8}H_2 + \frac{1}{8}H_3, z_n' + \frac{1}{h}\left(\frac{1}{2}H_1 + \frac{1}{4}H_2 + \frac{1}{4}H_3\right)\right)$$

But since $K_2 = K_3$ and $H_2 = H_3$ , equations (3.3.14) and (3.3.15) becomes

$$y_{n+1} = \frac{y_n + hy'_n}{1 + y_n \frac{1}{3}(H_1 + 2H_2)} \tag{3.3.16}$$

and

$$y'_{n+1} = y'_n + \frac{1}{2h}(K_1 + 3K_2) \tag{3.3.17}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y'_n) = \frac{h^2}{2} f_n$$

$$K_2 = \frac{h^2}{2} f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hy'_n + \frac{1}{3}(K_1 + K_2), y'_n + \frac{2}{3h}(K_1 + K_2)\right)$$

and

$$H_1 = \frac{h^2}{2} g(x_n, z_n, z'_n) = \frac{h^2}{2} g_n$$

$$H_2 = \frac{h^2}{2} g\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz'_n + \frac{1}{4}(H_1 + H_2), z'_n + \frac{1}{2h}(H_1 + H_2)\right)$$

2. Choosing the Parameters

$w_1 = w_2 = w_3 = 0,\quad v_1 = \frac{2}{3},\quad v_2 = v_3 = \frac{1}{6},\quad c_1 = d_1 = \frac{1}{3},\quad c_2 = \frac{2-\sqrt{2}}{3},\quad c_3 = \frac{4+\sqrt{2}}{6}$

$a_{11} = a_{22} = a_{33} = b_{11} = \alpha_{11} = \alpha_{22} = \alpha_{33} = \beta_{11} = \beta_{22} = \beta_{33} = \frac{1}{3}\ ,\ b_{22} = b_{33} = 1$

$w'_1 = 1,\quad w'_2 = \frac{1}{3},\quad w'_3 = \frac{2}{3},\quad v'_1 = v'_2 = v'_3 = 0,\quad a_{12} = \frac{1}{3},\quad a_{13} = -\frac{1}{3},\ d_2 = \frac{2-\sqrt{6}}{6}$

$a_{21} = a_{23} = \frac{1-\sqrt{2}}{6}\ ,\quad a_{31} = a_{32} = \frac{2+\sqrt{2}}{12}\ ,\quad b_{12} = \frac{1+2\sqrt{2}}{9}\ ,\quad b_{13} = \frac{2-2\sqrt{2}}{9}\ ,\quad b_{23} = \frac{-2\sqrt{2}}{3}$

$b_{21} = b_{31} = \frac{1}{3}\ ,\quad b_{32} = \frac{\sqrt{2}}{3}\ ,\quad \alpha_{12} = \frac{1}{3}\ ,\quad \alpha_{13} = -\frac{1}{3}\ ,\quad \alpha_{21} = \alpha_{23} = -\frac{\sqrt{6}}{12}\ ,\quad \alpha_{31} = \frac{\sqrt{6}}{12}$

$\alpha_{32} = \frac{\sqrt{6}}{6}\ ,\quad \beta_{12} = \beta_{13} = \frac{1}{6}\ ,\quad \beta_{21} = \beta_{23} = \frac{1-\sqrt{6}}{6}\ ,\quad \beta_{31} = \beta_{32} = \frac{1+\sqrt{6}}{6}\ ,\quad d_3 = \frac{2+\sqrt{6}}{6}$

Then equations (3.3.1) and (3.3.2) becomes

$$y_{n+1} = \frac{y_n + h y_n'}{1 + \frac{1}{6} y_n (4H_1 + H_2 + H_3)} \tag{3.3.18}$$

and

$$y_{n+1}' = y_n' + \frac{1}{3h}(3K_1 + K_2 + 2K_3) \tag{3.3.19}$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hy_n' + \frac{1}{3}(K_1 + K_2 - K_3), y_n' + \frac{1}{3h}\left(K_1 + \left(\frac{1+2\sqrt{2}}{3}\right)K_2 + \left(\frac{2-\sqrt{2}}{3}\right)K_3\right)\right)$$

$$K_2 = \frac{h^2}{2} f\left(x_n + \left(\frac{2-\sqrt{2}}{3}\right)h, y_n + \left(\frac{2-\sqrt{2}}{3}\right)hy_n' + \left(\frac{1-\sqrt{2}}{6}\right)K_1 + \frac{1}{3}K_2 + \left(\frac{1-\sqrt{2}}{6}\right)K_3,\ y_n' + \frac{1}{h}\left(\frac{1}{3}K_1 + K_2 - \frac{2\sqrt{2}}{2}K_3\right)\right)$$

$$K_3 = \frac{h^2}{2} f\left(x_n + \left(\frac{4+\sqrt{2}}{6}\right)h, y_n + \left(\frac{4+\sqrt{2}}{6}\right)hy_n' + \left(\frac{2+\sqrt{2}}{12}\right)K_1 + \left(\frac{2+\sqrt{2}}{12}\right)K_2 + \frac{1}{3}K_3,\ y_n' + \frac{1}{h}\left(\frac{1}{3}K_1 + \frac{2\sqrt{2}}{2}K_2 + K_3\right)\right)$$

and

$$H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{3}h,\ z_n + \frac{1}{3}hz_n' + \frac{1}{3}(H_1 + H_2 + H_3), z_n' + \frac{1}{6h}(2H_1 + H_2 + H_3)\right)$$

$$H_2 = \frac{h^2}{2} g\left(x_n + \left(\frac{2-\sqrt{6}}{6}\right)h,\ z_n + \left(\frac{2-\sqrt{6}}{6}\right)hz_n' - \frac{1}{6}\left(\sqrt{6}H_1 - 2H_2 + \sqrt{6}H_3\right),\ z_n' + \frac{1}{6h}\left(\left(\sqrt{1-6}\right)H_1 + 2H_2 + \left(\sqrt{1-6}\right)H_3\right)\right)$$

$$H_3 = \frac{h^2}{2} g\left(x_n + \left(\frac{2+\sqrt{6}}{6}\right)h,\ z_n + \left(\frac{2+\sqrt{6}}{6}\right)hz_n' + \frac{1}{6}\left(\sqrt{6}H_1 + \sqrt{6}H_2 + 2H_3\right),\ z_n' + \frac{1}{6h}\left(\left(\sqrt{1+6}\right)H_1 + \left(\sqrt{1+6}\right)H_2 + 2H_3\right)\right)$$

# CHAPTER FOUR

# ANALYSIS OF THE METHODS

## 4.0: Introduction

There are two types of error involved in a Runge–Kutta method: *roundoff* error and *truncation* error (also known as *discretization* error). Round-off error occurs usually when the method is implemented on a computer. Normally, round-off error is not considered in the numerical analysis of the algorithm, since it depends on the computer on which the algorithm is implemented, and thus is external to the numerical algorithm, Julyan and Oreste (1992). Truncation error is caused by truncation of the infinite Taylor series during the development of the new formula.

An obvious requirement for a successful numerical scheme is that the truncation error should be as small as is desired by using a sufficiently small step length: this concept is known as *convergence*. A concept closely related to convergence is known as *consistency*; a method is said to be consistent (with the initial value problem) if

$$\emptyset(x_n, y_n, y_n'; 0) = f(x_n, y_n, y_n') \tag{4.1}$$

Where $\emptyset(x_n, y_n, y_n'; h)$ is defined by (3.1)

The two crucial concepts in the analysis of numerical error are *local truncation* error and *global truncation* (*discretization*) error. The Truncation error is defined by

$$T_{n+1} = y(x_{n+1}) - y_{n+1} = |y_{n+1} - y_n - h\emptyset(x_n, y_n, y_n'; h)| \tag{4.2}$$

Where $y(\, x_{n+1})$ is the exact equation of the differential equation (1.1) and $y_{n+1}$ is the approximate solution

## 4.1: THE TRUNCATION ERROR OF THE METHOD

Thus, the Truncation error of the method is given by

$$T_{n+1} = y(x_{n+1}) - \frac{y_n + hy'_n + \sum_{r=1}^{s} w_r k_r}{1 + y_n \sum_{r+1}^{s} v_r H_r} \tag{4.1.1}$$

Using the Taylor's series expansion of $y(x_{n+1})$, $K_r$, and $H_r$ about $(x_n, y_n, y'_n)$. The local truncation error of one – stage scheme (3.1.32) of order three is given as

$$T_{n+1} = \left[\left(\frac{1}{24} - \frac{c_1^2}{4}\right)\Delta^2 f_n + \left(\frac{1}{24} - \frac{c_1}{4}\right)\left(2\Delta f_n f_{y'} + f_n y\right)\right] h^4 \tag{4.1.2}$$

If $c_1 = 0$, then

$$T_{n+1} = \left[\Delta^2 f_n + 2\Delta f_n f_{y'} + f_n f_y\right] h^4 \tag{4.1.3}$$

Where

$$\Delta^2 f_n = f_{xx} + y'^2_n f_{yy} + f^2 f_{y'y'} + 2y' f_n f_{yy'} + 2 f_n f_{xy'}\text{ , and}$$

$$\Delta f_n = f_x + y' f_y + f_n f_{y'}$$

By Lotkin (1957) technique, we can find a bound for the partial derivative of $f_n$ from the inequality defined by

$$f_n < M \text{ and } \left|\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial y'^k} f(x_n, y_n, y'_n)\right| < \frac{L^{i+j+k}}{M^{j+k-1}} \tag{4.1.4}$$

Where $M$ and $L$ are constant and $i + j + k \le M$. This condition is used by Lotkin (1957). Thus using (3.4.6) we have

$$f_x < M,\quad f_y < L,\quad f_{y'} < L,\quad f_{xx} < L^2 M,\quad f_{xy} < L^2,\quad f_{xy'} < L^2$$

$$f_{yy} < \frac{L^2}{M},\quad f_{yy'} < \frac{L^2}{M}$$

So, the error bound for (3.4.5) will be

$$T_{n+1} < \left[\left|\frac{1}{24}\right| |\Delta^2 f_n| + |2|\left|\Delta f_n f_{y'}\right| + \left|f_n f_y\right|\right] h^4 < \frac{1}{24}(10L^2 M + LM)h^4 \tag{4.1.5}$$

Hence the local truncation error is

$$T_{n+1} < \frac{1}{24}(10L^2M + LM)h^4$$

To estimate the step size, we can keep

$$T_{n+1} < \frac{1}{24}(10L^2M + LM)h^4 < TOL \qquad (4.1.6)$$

Where *TOL* is an allowance tolerance to be specified by the user.

## 4.2: CONVERGENCE

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size tends to zero.

$$\text{Convergent} = \lim_{h\to 0}|y(x_{n+1}) - y_{n+1}|$$

In other words, if the *discretiation* error at $x_{n+1}$ tends to zero as $h \to \infty$, i.e if

$$e_{n+1} = |y(x_{n+1}) - y_{n+1}| \to 0 \text{ as } n \to \infty \qquad (4.2.1)$$

From equation (3.1.33),

$$y'_{n+1} = \frac{y'_n}{1+\frac{2}{h}y'_nH_1} \qquad (4.2.2)$$

And the numerical solution $y'(x_{n+1})$ seems to satisfy the equation

$$y'(x_{n+1}) = \frac{y'_n}{1+\frac{2}{h}y'(x_n)H_1} + T_{n+1} \qquad (4.2.3)$$

Where $T_{n+1}$ is a local truncation error.

Subtracting equation (4.2.3) from (4.2.2) gives

$$y'_{n+1} - y'(x_{n+1}) = \frac{y'_n}{1+\frac{2}{h}y'_nH_1} - \frac{y'_n}{1+\frac{2}{h}y'(x_n)H_1} + T_{n+1} \qquad (4.2.4)$$

Adopting equation (4.2.4) gives

$$y'_{n+1} - y'(x_{n+1}) = \frac{y'_n\left(1+\frac{2}{h}y'_nH_1\right) - y'_n\left(1+\frac{2}{h}y'(x_n)H_1\right)}{\left(1+\frac{2}{h}y'_nH_1\right)\left(1+\frac{2}{h}y'(x_n)H_1\right)} + T_{n+1} \qquad (4.2.5)$$

Expanding the brackets and re-arranging gives

$$y'_{n+1} - y'(x_{n+1}) = \frac{\frac{2}{h}y'_nH_1\left(y'_n - y'(x_n)\right)}{\left(1+\frac{2}{h}y'_nH_1\right)\left(1+\frac{2}{h}y'(x_n)H_1\right)} + T_{n+1}$$

This implies that

$$e_{n+1} = \frac{e\left(\frac{2}{h}y'_nH_1\right)}{\left(1+\frac{2}{h}y'_nH_1\right)\left(1+\frac{2}{h}y'(x_n)H_1\right)} + T_{n+1} \qquad (4.2.6)$$

From equations (4.2.6), setting

$A_n = \left(\frac{2}{h}y'_nH_1\right)$, $B_n = \left(1+\frac{2}{h}y'_nH_1\right)$, $C_n = \left(1+\frac{2}{h}y'(x_n)H_1\right)$ and $T_{n+1} = T$

Then

$$e_{n+1} = \frac{A_n}{B_nC_n}e_n + T \qquad (4.2.7)$$

Let $B = \max B_n > 0$, $C = \max C_n > 0$ and $A = \max A_n < 0$ then (4.2.7) becomes,

$$e_{n+1} \leq \frac{A}{BC}e_n + T$$

Set $\frac{A}{BC} = K < 1$, then

$$e_{n+1} \leq Ke_n + T \qquad (4.2.8)$$

If $n = 0$, then from (4.2.8),

$$e_1 = Ke_0 + T$$

$e_2 = Ke_1 + T = K^2e_0 + KT + T$ by substituting the value of $e_1$

$$e_3 = Ke_2 + T = K^3e_0 + K^2T + T$$

Continuing in this manner, we get the following

$$e_{n+1} = K^{n+1}e_0 + \sum_{t=0}^{n+1} K^t T \tag{4.2.9}$$

Since $\frac{A}{BC} = K < 1$, then one can see that as $n \to \infty$, $e_{n+1} \to 0$. This proves that the scheme converges.

## 4.3: CONSISTENCY

A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve as the step size ends to zero. To prove if equation (3.1.32) is consistent, subtract $y_n$ from both side of (3.1.32), then

$$y_{n+1} - y_n = \frac{y_n + hy_n'}{1 + y_n H_1} - y_n \tag{4.3.1}$$

Therefore

$$y_{n+1} - y_n = \frac{y_n + hy_n' - y_n(1 + y_n H_1)}{1 + y_n H_1}$$

Simplifying gives

$$y_{n+1} - y_n = \frac{hy_n' - y_n^2 H_1}{1 + y_n H_1} \tag{4.3.2}$$

But since

$$H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{3}h, z_n + \frac{1}{3}hz_n' + \frac{1}{3}H_1, z_n' + \frac{2}{3h}H_1\right)$$

Then equation (4.3.2) becomes

$$y_{n+1} - y_n = \frac{hy_n' - y_n^2\left[\frac{h^2}{2} g\left(x_n + \frac{1}{3}h, z_n + \frac{1}{3}hz_n' + \frac{1}{3}H_1, z_n' + \frac{2}{3h}H_1\right)\right]}{1 + y_n\left[\frac{h^2}{2} g\left(x_n + \frac{1}{3}h, z_n + \frac{1}{3}hz_n' + \frac{1}{3}H_1, z_n' + \frac{2}{3h}H_1\right)\right]}$$

Dividing all through by $h$ and taking the limit as $h$ tends to zero on both sides gives

$$\lim_{h \to \infty} \frac{y_{n+1} - y_n}{h} = y_n'$$

Hence, the method is consistent.

# CHAPTER FIVE

# NUMERICAL EXAMPLES

## 5.0 INTRODUCTION

Having gotten our schemes, part of the objectives is to apply the scheme derived to approximate real life problems. Second order linear differential equations have a variety of applications in science and engineering. In this research explore one of them, the spring problems.

## 5.1 Numerical Problems

**Example 1.**

A spring of mass 2kg has a damping constant of 8, and a force of 5N is required to keep the spring stretched 0.5m beyond its natural length. The spring is stretched 1m beyond its natural length and then release with zero velocity. Find the position of mass at time t.

<u>Solution</u>

The problem above can be modeled by the initial value problem

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+5x=0, \qquad x(0)=1, \quad x'(0)=0$$

The exact solution is

$$x(t)=e^{-2t}(2\sin t+\cos t)$$

**Example 2.**

A spring of mass 2kg has a damping constant of 10, and a force of 6N is required to keep the spring stretched 0.5m beyond its natural length. The spring is stretched 1m beyond its natural length and then release with zero velocity. Find the position of mass at time t.

Solution

The above spring problem is modeled by the initial value problem

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0, \qquad x(0)=1, \;\; x'(0)=0$$

The exact solution is

$$x(t)=-2e^{-3t}+3e^{-2t}$$

**Example 3. (Variable Coefficient Equation)**

Consider the equation $y''=(1+x^2)y, \quad y(0)=1, \; y'(0)=0, \; x\in[0,1]$

The exact solution is

$$y(x)=e^{x^2/2}$$

[Sources: Jain 1984]

**Example 4. (Non-linear Differential Equation)**

Consider a non-linear ordinary differential equation

$$y''-x(y')^2=0, \qquad y(0)=1, \;\; y'(0)=\frac{1}{2}$$

The exact solution is

$$y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

[Sources: Jacob (2010), Taiwo and Osilagun (2011) and Anake et-al (2012)]

## 5.2 PRESENTATION OF THE RESULTS

The results below are the exact and the approximate solution to each problem.

(**Note that.** All results is in 10 decimal places)

Table 5.2.1 results of example 1 at $h$ = 0.1

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y_n'$ | $y_n$ | $y_n'$ |
| 0.1 | 0.978113886 | -0.408683442 | 0.971852523 | -0.406934307 | 6.26E-03 | -1.75E-03 |
| 0.2 | 0.923302344 | -0.665860175 | 0.901166794 | -0.661027045 | 2.21E-02 | -4.83E-03 |
| 0.3 | 0.848669638 | -0.810924641 | 0.801778305 | -0.798072019 | 4.69E-02 | 1.29E-02 |
| 0.4 | 0.763813263 | -0.874884702 | 0.682786227 | -0.847056781 | 8.10E-02 | -2.78E-02 |
| 0.5 | 0.675586181 | -0.881853996 | 0.548234367 | -0.830600061 | 1.27E-01 | -5.13E-02 |
| 0.6 | 0.588720400 | -0.850335224 | 0.393287231 | -0.765107383 | 1.95E-01 | -8.52E-02 |
| 0.7 | 0.506332013 | -0.794310629 | 0.172338281 | -0.659316519 | 3.34E-01 | -1.35E-01 |
| 0.8 | 0.430326053 | -0.724158485 | 0.636264391 | -0.500370482 | -2.06E-01 | -2.24E-01 |
| 0.9 | 0.361717571 | -0.647415336 | 0.535303951 | -0.585436941 | -1.74E-01 | -6.20E-02 |
| 1.0 | 0.300883394 | -0.569403570 | 0.406425080 | -0.599863195 | -1.06E-01 | 3.05E-02 |

Table 5.2.2 results of example 1 at $h$ = 0.01

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y_n'$ | $y_n$ | $y_n'$ |
| 0.01 | 0.999753311 | -0.049009117 | 0.999746676 | -0.049007072 | 6.63E-06 | -2.05E-06 |
| 0.02 | 0.999026303 | -0.096072539 | 0.999000136 | -0.096068285 | 2.62E-05 | -4.25E-06 |
| 0.03 | 0.997838168 | -0.141243491 | 0.997779902 | -0.141236237 | 5.83E-05 | -7.25E-06 |
| 0.04 | 0.996207568 | -0.18457404 | 0.996104951 | -0.184562407 | 1.03E-04 | -1.16E-05 |
| 0.05 | 0.994152652 | -0.226115112 | 0.993993728 | -0.226097174 | 1.59E-04 | -1.79E-05 |
| 0.06 | 0.991691064 | -0.265916514 | 0.991464159 | -0.265889831 | 2.27E-04 | -1.79E-05 |
| 0.07 | 0.988839958 | -0.304026952 | 0.988533663 | -0.303988608 | 3.06E-04 | -3.83E-05 |
| 0.08 | 0.985616003 | -0.340494051 | 0.985219165 | -0.340440684 | 3.97E-04 | -5.34E-05 |
| 0.09 | 0.982035399 | -0.375364374 | 0.981537104 | -0.375292206 | 4.98E-04 | -7.22E-05 |
| 0.1 | 0.978113886 | -0.408683442 | 0.977503449 | -0.408588307 | 6.10E-04 | -9.51E-05 |

Table 5.2.3 results of example 1 at $h$ = 0.001

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.001 | 0.999997503 | -0.004990009 | 0.999997497 | -0.004990007 | 6.66E-09 | -2.08E-09 |
| 0.002 | 0.999990027 | -0.009960073 | 0.999990000 | -0.009960069 | 2.66E-08 | -4.18E-09 |
| 0.003 | 0.999977590 | -0.014910247 | 0.999977530 | -0.014910241 | 5.98E-08 | -6.35E-09 |
| 0.004 | 0.999960213 | -0.019840585 | 0.999960107 | -0.019840577 | 1.06E-07 | -8.68E-09 |
| 0.005 | 0.999937915 | -0.024751143 | 0.999937749 | -0.024751131 | 1.66E-07 | -1.12E-08 |
| 0.006 | 0.999910717 | -0.029641974 | 0.999910478 | -0.029641959 | 2.39E-07 | -1.40E-08 |
| 0.007 | 0.999878638 | -0.034513132 | 0.999878313 | -0.034513115 | 3.24E-07 | -1.72E-08 |
| 0.008 | 0.999841697 | -0.039364673 | 0.999841274 | -0.039364652 | 4.23E-07 | -2.08E-08 |
| 0.009 | 0.999799915 | -0.044196650 | 0.999799380 | -0.044196625 | 5.35E-07 | -2.48E-08 |
| 0.01 | 0.999753311 | -0.049009117 | 0.999752650 | -0.049009087 | 6.60E-07 | -2.94E-08 |

Table 5.2.4 results of example 2 at $h$ = 0.1

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.1 | 0.974555818 | -0.467475194 | 0.965148944 | -0.490368272 | 9.41E-03 | 2.29E-02 |
| 0.2 | 0.913336866 | -0.729050460 | 0.878800789 | -0.782015951 | 3.45E-02 | 5.30E-02 |
| 0.3 | 0.833295589 | -0.853451858 | 0.762787276 | -0.923295925 | 7.05E-02 | 6.98E-02 |
| 0.4 | 0.745598469 | -0.888808513 | 0.632219564 | -0.955356931 | 1.13E-01 | 6.65E-02 |
| 0.5 | 0.657378003 | -0.868495686 | 0.494344330 | -0.911516387 | 1.63E-01 | 4.30E-02 |
| 0.6 | 0.572984859 | -0.815371942 | 0.342244979 | 0.816304591 | 2.31E-01 | 9.33E-04 |
| 0.7 | 0.494878035 | -0.744843214 | -1.45945457 | -0.681774224 | 1.95E+00 | -6.31E-02 |
| 0.8 | 0.424253647 | -0.667071388 | -1.51700728 | 0.286422849 | 1.94E+00 | -9.53E-01 |
| 0.9 | 0.361485639 | -0.588560253 | -1.44948531 | 0.924225035 | 1.81E+00 | -1.51E+00 |
| 1.0 | 0.306431713 | -0.513289289 | -1.30394335 | 1.292676829 | 1.61E+00 | -1.81E+00 |

Table 5.2.5 results of example 2 at $h$ = 0.01

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.01 | 0.999704953 | -0.058518839 | 0.999694999 | -0.058810937 | 9.95E-06 | 2.92E-04 |
| 0.02 | 0.998839250 | -0.114149433 | 0.998797086 | -0.115138120 | 4.22E-05 | 9.89E-04 |
| 0.03 | 0.997431230 | -0.167000090 | 0.997331275 | -0.169050819 | 1.00E-04 | 2.05E-03 |
| 0.04 | 0.995508166 | -0.217175458 | 0.995321915 | -0.220616869 | 1.86E-04 | 3.44E-03 |
| 0.05 | 0.993096301 | -0.264776650 | 0.992792696 | -0.269902694 | 3.04E-04 | 5.13E-03 |
| 0.06 | 0.990220887 | -0.309901352 | 0.989766661 | -0.316973325 | 4.54E-04 | 7.07E-03 |
| 0.07 | 0.986906214 | -0.352643937 | 0.986266216 | -0.361892421 | 6.40E-04 | 9.25E-03 |
| 0.08 | 0.983175645 | -0.393095567 | 0.982313139 | -0.404722292 | 8.63E-04 | 1.16E-02 |
| 0.09 | 0.979051646 | -0.431344302 | 0.977928587 | -0.445523915 | 1.12E-03 | 1.42E-02 |
| 0.1 | 0.974555818 | -0.467475194 | 0.973133113 | -0.484356955 | 1.42E-03 | 1.69E-02 |

Table 5.2.6 results of example 2 at $h$ = 0.001

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.001 | 0.999997005 | -0.005985019 | 0.999996995 | -0.005988011 | 1.00E-08 | 2.99E-06 |
| 0.002 | 0.999988040 | -0.011940152 | 0.999987997 | -0.011950589 | 4.29E-08 | 1.04E-05 |
| 0.003 | 0.999973135 | -0.017865512 | 0.999973031 | -0.017887807 | 1.03E-07 | 2.23E-05 |
| 0.004 | 0.999952319 | -0.023761212 | 0.999952124 | -0.023799736 | 1.95E-07 | 3.85E-05 |
| 0.005 | 0.999925622 | -0.029627365 | 0.999925299 | -0.029686448 | 3.23E-07 | 5.91E-05 |
| 0.006 | 0.999893074 | -0.035464083 | 0.999892583 | -0.035548015 | 4.91E-07 | 8.39E-05 |
| 0.007 | 0.999854704 | -0.041271478 | 0.999854000 | -0.041384508 | 7.03E-07 | 1.13E-04 |
| 0.008 | 0.999810541 | -0.047049662 | 0.999809576 | -0.047195999 | 9.64E-07 | 1.46E-04 |
| 0.009 | 0.999760614 | -0.052798745 | 0.999759336 | -0.052982558 | 1.28E-06 | 1.84E-04 |
| 0.01 | 0.999704953 | -0.058518839 | 0.999703304 | -0.058744256 | 1.65E-06 | 2.25E-04 |

Table 5.2.7 results of example 3 at $h$ = 0.1

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.1 | 1.005012521 | 0.100501252 | 1.005025126 | 0.100250417 | -1.26E-05 | 2.51E-04 |
| 0.2 | 1.020201340 | 0.204040268 | 1.020129605 | 0.201733389 | 7.17E-05 | 2.31E-03 |
| 0.3 | 1.046027860 | 0.313808358 | 1.045542449 | 0.306464227 | 4.85E-04 | 7.34E-03 |
| 0.4 | 1.083287068 | 0.433314827 | 1.081707487 | 0.416660651 | 1.58E-03 | 1.67E-02 |
| 0.5 | 1.133148453 | 0.566574227 | 1.129290062 | 0.534778113 | 3.86E-03 | 3.18E-02 |
| 0.6 | 1.197217363 | 0.718330418 | 1.189200895 | 0.663611165 | 8.02E-03 | 5.47E-02 |
| 0.7 | 1.277621313 | 0.894334919 | 1.262631449 | 0.806411362 | 1.50E-02 | 8.79E-02 |
| 0.8 | 1.377127764 | 1.101702211 | 1.351102688 | 0.967028171 | 2.60E-02 | 1.35E-01 |
| 0.9 | 1.499302500 | 1.349372250 | 1.456529654 | 1.150081082 | 4.28E-02 | 1.99E-01 |
| 1.0 | 1.648721271 | 1.648721271 | 1.581305030 | 1.361173403 | 6.74E-02 | 2.88E-01 |

Table 5.2.8 results of example 3 at $h$ = 0.01

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.01 | 1.000050001 | 0.010000500 | 1.000050003 | 0.01000025 | -1.25E-09 | 2.50E-07 |
| 0.02 | 1.000200020 | 0.020004000 | 1.000200013 | 0.020001722 | 7.08E-09 | 2.28E-06 |
| 0.03 | 1.000450101 | 0.030013503 | 1.000450054 | 0.030013503 | 4.74E-08 | 7.14E-06 |
| 0.04 | 1.000800320 | 0.040032013 | 1.000800169 | 0.040016172 | 1.51E-07 | 1.58E-05 |
| 0.05 | 1.001250782 | 0.050062539 | 1.001250421 | 0.050033155 | 3.61E-07 | 2.94E-05 |
| 0.06 | 1.001801621 | 0.060108097 | 1.001800894 | 0.060059320 | 7.27E-07 | 4.88E-05 |
| 0.07 | 1.002453004 | 0.070171710 | 1.002451691 | 0.070096677 | 1.31E-06 | 7.50E-05 |
| 0.08 | 1.003205125 | 0.080256410 | 1.003202935 | 0.080147243 | 2.19E-06 | 1.09E-04 |
| 0.09 | 1.004058212 | 0.090365239 | 1.004054772 | 0.090213038 | 3.44E-06 | 1.52E-04 |
| 0.1 | 1.005012521 | 0.100501252 | 1.005007363 | 0.100296091 | 5.16E-06 | 2.05E-04 |

Table 5.2.9 results of example 3 at $h$ = 0.001

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.001 | 1.000000500 | 0.001000001 | 1.000000500 | 0.001000000 | 0.00E-00 | 0.00E-00 |
| 0.002 | 1.000002000 | 0.002000004 | 1.000002000 | 0.002000002 | 0.00E-00 | 2.28E-09 |
| 0.003 | 1.000004500 | 0.003000014 | 1.000004500 | 0.003000006 | 0.00E-00 | 7.14E-09 |
| 0.004 | 1.000008000 | 0.004000032 | 1.000008000 | 0.004000016 | 0.00E-00 | 1.58E-08 |
| 0.005 | 1.000012500 | 0.005000063 | 1.000012500 | 0.005000033 | 0.00E-00 | 2.94E-08 |
| 0.006 | 1.000018000 | 0.006000108 | 1.000018000 | 0.006000059 | 0.00E-00 | 4.87E-08 |
| 0.007 | 1.000024500 | 0.007000172 | 1.000024500 | 0.007000097 | 0.00E-00 | 7.49E-08 |
| 0.008 | 1.000032001 | 0.008000256 | 1.000032000 | 0.008000147 | 0.00E-00 | 1.09E-07 |
| 0.009 | 1.000040501 | 0.009000365 | 1.000040500 | 0.009000213 | 0.00E-00 | 1.52E-07 |
| 0.01 | 1.000050001 | 0.010000500 | 1.000050001 | 0.010000296 | 0.00E-00 | 2.05E-07 |

Table 5.2.10 results of example 4 at $h$ = 0.1

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.1 | 1.050041729 | 0.501253133 | 1.050073940 | 0.503168650 | -3.22E-05 | -1.92E-03 |
| 0.2 | 1.100335348 | 0.505050505 | 1.100577720 | 0.509928057 | -2.42E-04 | -4.88E-03 |
| 0.3 | 1.151140436 | 0.511508951 | 1.151877265 | 0.520516997 | -7.37E-04 | -9.01E-03 |
| 0.4 | 1.202732554 | 0.520833333 | 1.204368009 | 0.535350750 | -1.64E-03 | -1.45E-02 |
| 0.5 | 1.255412812 | 0.533333333 | 1.258494045 | 0.555072830 | -3.08E-03 | -2.17E-02 |
| 0.6 | 1.309519604 | 0.549450549 | 1.314773677 | 0.580648013 | -5.25E-03 | -3.12E-02 |
| 0.7 | 1.365443754 | 0.569800570 | 1.373836442 | 0.613531812 | -8.39E-03 | -4.37E-02 |
| 0.8 | 1.423648930 | 0.595238095 | 1.436481545 | 0.656001612 | -1.28E-02 | -6.08E-02 |
| 0.9 | 1.484700279 | 0.626959248 | 1.503780137 | 0.711893892 | -1.91E-02 | -8.49E-02 |
| 1.0 | 1.549306144 | 0.666666667 | 1.577285372 | 0.788697022 | -2.80E-02 | -1.22E-01 |

Table 5.2.11 results of example 4 at $h$ = 0.01

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| $t$ | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.01 | 1.005000042 | 0.500012500 | 1.005000057 | 0.500025667 | -1.57E-08 | -1.32E-05 |
| 0.02 | 1.010000333 | 0.500050005 | 1.010000457 | 0.500077340 | -1.24E-07 | -2.73E-05 |
| 0.03 | 1.015001125 | 0.500112525 | 1.015001460 | 0.500155033 | -3.35E-07 | -4.25E-05 |
| 0.04 | 1.020002667 | 0.500200080 | 1.020003326 | 0.500258771 | -6.59E-07 | -5.87E-05 |
| 0.05 | 1.025005210 | 0.500312695 | 1.025006315 | 0.500388585 | -1.10E-06 | -7.59E-05 |
| 0.06 | 1.030009005 | 0.500450405 | 1.030010689 | 0.500544515 | -1.68E-06 | -9.41E-05 |
| 0.07 | 1.035014302 | 0.500613251 | 1.035016709 | 0.500726612 | -2.41E-06 | -1.13E-04 |
| 0.08 | 1.040021354 | 0.500801282 | 1.040024636 | 0.500801282 | -3.28E-06 | -1.34E-04 |
| 0.09 | 1.045030412 | 0.501014554 | 1.045034733 | 0.501169545 | -4.32E-06 | -1.55E-04 |
| 0.1 | 1.050041729 | 0.501253133 | 1.050047264 | 0.501430525 | -5.53E-06 | -1.77E-04 |

Table 5.2.12 results of example 4 at $h = 0.001$

| | Exact Solutions | | Approximate Solutions | | Errors | |
|---|---|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y'(x_n)$ | $y_n$ | $y'_n$ | $y_n$ | $y'_n$ |
| 0.001 | 1.000500000 | 0.500000125 | 1.000500000 | 0.500000251 | 0.00E-00 | -1.26E-07 |
| 0.002 | 1.001000000 | 0.500000500 | 1.001000000 | 0.500000752 | 0.00E-00 | -2.52E-07 |
| 0.003 | 1.001500001 | 0.500001125 | 1.001500001 | 0.500001505 | 0.00E-00 | -3.80E-07 |
| 0.004 | 1.002000003 | 0.500002000 | 1.002000003 | 0.500002509 | 0.00E-00 | -5.09E-07 |
| 0.005 | 1.002500005 | 0.500003125 | 1.002500006 | 0.500003763 | 0.00E-00 | -6.38E-07 |
| 0.006 | 1.003000009 | 0.500004500 | 1.003000010 | 0.500005269 | -1.37E-09 | -7.69E-07 |
| 0.007 | 1.003500014 | 0.500006125 | 1.003500016 | 0.500007026 | -1.90E-09 | -9.01E-07 |
| 0.008 | 1.004000021 | 0.500008000 | 1.004000024 | 0.500009033 | -2.53E-09 | -1.03E-06 |
| 0.009 | 1.004500030 | 0.500010125 | 1.004500034 | 0.500011292 | -3.24E-09 | -1.17E-06 |
| 0.01 | 1.005000042 | 0.500012500 | 1.005000046 | 0.500013802 | -4.05E-09 | -1.30E-06 |

Table 5.2.13 Comparing errors of example 4, Jacob (2010) and new scheme at $h = 3.125 \times 10^{-3}$

| | **Exact Solutions** | **Computed with new scheme** | **Absolute Errors in Jacob (2010)** | **Absolute error in new Scheme** |
|---|---|---|---|---|
| ***t*** | $y(x_n)$ | $y_n$ | $y_n$ | $y_n$ |
| 0.1 | 1.001562501 | 1.001562502 | 6.125E-08 | 4.41E-10 |
| 0.2 | 1.003125010 | 1.003125014 | 1.211E-07 | 3.51E-09 |
| 0.3 | 1.004687534 | 1.004687544 | 1.874E-07 | 9.31E-09 |
| 0.4 | 1.006250081 | 1.006250099 | 2.616E-07 | 1.79E-08 |
| 0.5 | 1.007812659 | 1.007812688 | 3.534E-07 | 2.95E-08 |

## 5.3 Discussions

Example 1 and 2 are second order linear ordinary differential equations, the schemes favorably approximate the exact solution of the differential equations. The results are highlighted in tables (5.2.1) through (5.2.6) with various step lengths. Thus, the results are better with smaller step length.

In table (5.2.7), (5.2.8) and (5.2.9) are results obtained by applying the schemes to example 3, the variable coefficient differential equation. The result performed well and approximate the exact solution better as the step size goes to *h = 0.001*.

Example 4 is a non-linear ordinary differential equation which is also well approximated with results in tables (5.2.10), (5.2.11) and (5.2.12). The results show superiority over the results of Optimal Order Method of Jacob (2010).

# CHAPTER SIX

# SUMMARY, RECOMMENDATION AND CONCLUSION

## 6.0 Summary

The 3 – State Implicit Rational Runge-Kutta Scheme was derived and used for the solution of problems leading to second order ordinary differential equation and was found out to be effective. The scheme was found to be consistent, convergent and give better approximation.

## 6.1 Conclusion

The new numerical schemes derived follows the techniques of rational form of Runge – Kutta methods proposed by Hong (1982) which was adopted by Okunbor (1987) and Ademiluyi and Babatola (2000) by using Taylor and Binomial expansion in stages evaluation. The order condition obtained in this research is up to five (5) and the stage is up to three (3). This is an improvement on the work of earlier authors.

The new schemes are of high accuracy for direct numerical solution of general second order ordinary differential equations. The steps to the derivation of the new schemes are presented in the methodology while the analysis of the schemes proved to be consistent, convergent, the results proves to be good estimate of the exact equations. Thus, the scheme is effective and efficient, these suggest a wider application of the schemes for even more complicated physical problems; since the methods is used to solve equations of the form $y'' = (x, y, y')$ and $y'' = (x, y)$ favorably. Equations of constant and variable coefficients are also considered and the fact that is also used to solve non-linear problem.

## 6.2 Recommendation

Since the scheme is capable of approximating various problems, we recommend further research in the area to explore and derive more schemes from the rational function of Runge – Kutta methods defined in (3.1) for solving second order ordinary differential equations as in the case of first order equations where many authors worked in that direction.

## REFERENCES


Ababneh O. Y., Ahmad R, and Ismail E. S. (2009a): *Design of New Diagonally Implicit Runge–Kutta Methods for Stiff Problems.* Journal of Applied Mathematical Sciences, 3(45): 2241 – 2253.

Ababneh O. Y., Ahmad R, and Ismail E. S. (2009b): *New Multi-step Runge-Kutta Method.* Journal of Applied Mathematical Sciences, 3(4):2255 – 2262.

Ademiluyi, R. A, Babatola, P. O. (2000): *Implicit Rational Runge-Kutta scheme for integration of stiffs ODE's.* Nigerian Mathematical Society, (NMS) Journal.

Adesanya, A. O, Anake, T. A, and Oghonyon, G. J, (2009): *Continuous implicit method for the solution of general second order ordinary differential equations.* Journal of Nigerian Association of Mathematics and Physics. 15:71 – 78.

Agular-Vigo J. and Ramos H., (2006): *Variable Step-Size Implementation of Multistep Methods for* $y'' = f(x, y, y')$. Journal of Computer and Applied Mathematics, 192:114 – 131.

Anake T. O, Awoyemi D. O, Adesanya A. A and Fameyo M. M. (2012): *Solving General Second Order Ordinary Differential Equations by One – Step Hybrid Collocation Method.* International Journal of Science and Technology 2(4): 164 – 168.

Awoyemi, D.O. (2001): *A New Sixth-Order Algorithm for General Second Order Ordinary Differential Equations*. International Journal of Computer and Mathematics, 77:117 – 124.

Babatola P. O., Ademiluyi R. A and Areo E. A., (2007): *One-Stage Implicit Rational Runge-Kutta Schemes for treatment of Discontinuous Initial value problems,* Journal of Engineering and Applied Sciences 2(1): 96 – 104.

Bolarinwa, B. (2005): *A Class of Semi – Implicit Rational Runge – Kutta scheme for solving ordinary differential equations with derivative discontinuities.* M. Tech thesis, Federal University of Technology, Akure; Nigeria. Unpublished.

Bolarinwa B, Ademiluyi R. A, Oluwagunwa A. P. and Awomuse B. O. (2012): *A Class of Two-Stage Semi- Implicit Rational Runge - Kutta Scheme for Solving Ordinary Differential Equations.* Canadian Journal of Science and Engineering Mathematics, 3(3):99 – 111.

Butcher, J. C. (1987) *Numerical Analysis of Ordinary Differential Equations: Runge–Kutta and General Linear Methods* (Wiley).

Butcher, J. C. (2003): *Numerical methods for ordinary differential equations.* John wiley and sons.

Dormand, R. J. (1996): *Numerical methods for differential equations. A computational approach,* CRC Press London.

Dormand, J. R, El-Mikkawy, M. E. A, and Prince, P. J. (1987): *Families of Runge-Kutta-Nystrom formula,*IMA journal of Numerical Analysis,7:235-250.

Faranak R. and Ismail F., (2010): *Fifth Order Improved Runge-Kutta Method for Solving Ordinary Differential Equations*. Journal of applied informatics and remote sensing, 129 – 133.

Fudziah I. (2003): *Embedded singly diagonal implicit Runge-Kutta-Nystrom method order 5(4) for the integration of special second order ordinary differential equations.* International journal of mathematics science. 2(2):70-74.

Fudziah I. (2009): *Sixth Order Singly Diagonally Implicit Runge-Kutta Nystrom Method with Explicit First Stage for Solving Second Order Ordinary Differential Equations.* European Journal of Scientific Research. 26(4):470-479.

Hong, Y. F, (1982): *A Class of A – Stable Explicit Scheme, Computational and Asymptotic Method for Boundary and Interior Layer*. Proceeding of ball II conference, trinity college Dublin, 236 – 241.

Jacob K. S. (2010): *A Zero-Stable Optimal Method for Direct Solution of Second Order Differential Equations.* Journal of Mathematics and Statistics. 6(3): 367 – 371.

Jain, M. K. (1984): *Numerical Solutions of Differential Equations,* (Second Edition), Wiley Eastern Limited.

Jator S. N., (2010): *On a class of hybrid methods for* $y'' = f(x, y, y')$. Internal Journal of Pure and Applied Mathematics, 59(4):381-395.

Julyan H. E. and Oreste P, (1992): *The Dynamics of Runge–Kutta Methods*. Internal Journal of Bifurcation and Chaos, **2**, 427–449.

Lambert, J. D. (1973): *Computational methods in ordinary differential equations.* New York: John wiley and sons.

Lotkin M. (1951): *On the accuracy of Runge – Kutta Methods.* MTAC 5 128 – 132

Nystrom, E. J. (1925): *Verber, Die Numericche Integration von Differential-gleichagen, Acta sos., Sc. Fenn, 5013,1-55.*

Odekunle M. R (2001): *Some Semi-Implicit Rational Runge-Kutta Schemes.* Bagale Journal of Pure and Applied Sciences, 1(1):11-14.

Odekunle, M. R, Oye, N. D, Adee, S. O. (2004): *A class of inverse Runge-Kutta schemes for the numerical integration of singular problems.* Journal of Applied Mathematics and Computing. Elsevier, 158:149-158.

Okunuga, S. A, Sofoluwa, A. B, Ehigie, J.O and Akambi, M. A. (2012): *Fifth Order Two – Stage Explicit Runge – Kutta – Nystrom Method for direct integration of Second Order Ordinary Differential Equations.* Journal of Scientific Research and Essay, 7(2):134-144.

Okunbor, D. I. (1987): *Explicit rational Runge-Kutta schemes for stiff system of ordinary differential equations.* M.Sc Thesis, University of Benin, Benin city. (Unpublished).

Osama, Y. A, Rokiah, R. A. and Eddie, S. I. (2009): *On cases of Fourth Order Runge – Kutta Methods.* European Journal of Scientific Research, 31(4):605-615.

Senu, N, Sulaiman M, Ismail, F and Othman M. (2011): *A Singly Diagonally Implicit Runge-Kutta - Nystrom Method for Solving Oscillatory Problems*. International Journal of Applied Mathematics. 41(2):

Sharp P. W. and Fine J. M., (1992): *Nystrom pairs for the general second order initial-value problems,* Journal of Computational and Applied Mathematics 42: 279-291.

Soomiyol M. C (2011): *Rational Explicit Runge – Kutta Method for the Solution of Second Order Ordiary Differential Equations.* M.Sc Thesis, Modibbo Adama University Yola.

Taiwo O. A and Osilagun J. A, (2011): *On approximate Solution of Second Order Differentia Equations by Iterative Decomposition Method.* Asian Journal of Mathematics and Statistics 4(1): 1 – 7.