EINSTEIN'S FIELD EQUATION FOR HOMOGENOUS SPHERICAL MASSIVE BODIES WHOSE TENSOR FIELD VARIES WITH TIME, RADIAL DISTANCE AND POLAR ANGLE

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DECLARATION

I hereby declare that this dissertation has been written by me and it is a report of my research work. It has not been presented in any previous application for state diploma or degree. All quotations are indicated and sources of information specifically acknowledge by means of references.

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CERTIFICATION

The dissertation Einstein's Field Equation For Homogenous Spherical Massive Bodies Whose Tensor Field Varies With Time, Radial Distance And Polar Angle, meets the regulations governing the award of Masters, of the School of Postgraduate Studies, Nasarawa State University, Keffi, and is approved for its contribution to knowledge.

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DEDICATION

I dedicated this research work to Almighty Allah for giving me the privilege and ability to undertake this research work. Also to my late dad in person of Umar Musa Maisalati and my beloved mum in person of Adama Salihu.

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ABSTRACT

In this research work, we constructed and solved Einstein's geometrical field equations exterior to astrophysically real or hypothetical distribution of mass within a spherical geometry using golden metric tensor whose tensor field varies with time, radial distance and polar angle, and the solution was used in the study of motion of particles and photons within this field. Out of the sixteen (16) components of the Ricci tensors calculated, it is most interesting and instructive to note that only R_{00} in the limit of c^0 reduces to the well know Laplacian operator. In Minkwoski coordinate and in the limit of linear term, R_{00} reduces to the well-knownD'Alembertian operator. In general and in Minkowski coordinates, the well known D'Alembertian operator acting on f contained hitherto unknown perfectly consistent additional terms of all order of c^{-2} which could be use to generalized the Maxwell's electric and magnetic wave field equations in gravitational field. Our metric tensors and the solution of the Einstein's field equations have only one arbitrary function f , and this put the Einstein's geometrical theory of gravitation on the same bases with the Newton's dynamical theory of gravitation. The result in this research work is of much importance as it can be applied to the study of rotating bodies such as stars.

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Gravitation is the study of nature, which gives better understanding of the universe. The gravitational force governs most of the bodies in the universe. (Birkhoff & Langer, 1923). Gravity, or gravitation, is a natural phenomenon by which all things with mass are brought toward (or *gravitate* toward) one another, including planets, stars and galaxies. Since energy and mass are equivalent, all forms of energy, including light, also cause gravitation and are under the influence of it. On earth, gravity gives weight to physical objects and causes the ocean tides. The gravitational attraction of the original gaseous matter present in the Universe caused it to begin coalescing, forming stars and the stars to group together into galaxies so gravity is responsible for many of the large-scale structures in the Universe (Green, 2004). There are two major theories of gravitation namely:

i. The Newton's dynamical gravitational theory;

ii. The Einstein's geometrical gravitational theory

According to Newton's dynamical gravitational theory, a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Newton's theory enjoyed its greatest success when it was used to predict the existence of Neptune based on motions of Uranus that could not be accounted for by the actions of the other planets (Jiri, 1999).

A discrepancy in Mercury’s orbit pointed out flaws in Newton's theory. By the end of the 19th century, it was known that its orbit showed slight perturbations that could not be accounted for entirely under Newton's theory, but all searches for another perturbing body (such as a planet orbiting the Sun even closer than Mercury) had been fruitless (Weinberg, 1972; Wudka, 1998).

According to Einstein’s theory of gravitation published in 1915/1916, which unifies Special Relativity and Sir Isaac Newton’s law of universal gravitation, that gravitation is not due to a force but rather a manifestation of curved space and time, with the curvature being produced by the mass-energy and momentum content of the space-time. This is termed General Relativity and is the most widely accepted theory of gravitation (Bragmann, 1987; Howusu, 2010; Chifu, 2012). With the general theory of relativity, in which Einstein managed to reconcile relativity and gravitation, he had to discard the traditional physics worldview, which saw space as merely a stage on which the events of the world unfold. Instead, space-time is a dynamic entity, which is distorted by any matter that is contained in it, and which in turn tells that matter how to move and evolve. This interaction between space-time and matter is described by Einstein's geometric, relativistic theory of gravity. Einstein proposed that objects such as the sun and the Earth change this geometry. In the presence of matter and energy, it can evolve, stretch and warp, forming ridges, mountains and valleys that cause bodies moving through it to zigzag and curve. Therefore, although Earth appears to be pulled towards the sun by gravity, there is no such force. It is simply the geometry of space-time around the sun telling Earth how to move (Green, 2004; Chifu, 2012).

This theory was able to offer the resolution of the anomalous orbital precession of the orbits of the planets as well as the gravitational shift by the sun, in which the Newton's dynamical theory could not account for (Anderson, 1967; Wudka, 1998).

1.2 Statement of the Problem

After the publication of Einstein's geometrical gravitational field equations in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began (Howusu, 2010; Chifu & Howusu, 2009; Chifu, 2012). Schwarzchild first constructed the exact solution to this field equation in static and pure radial spherical polar coordinates in 1916 by considering astrophysical bodies such as the sun and the stars (Lumbi *et al.,* 2014). In Schwarzchild's metric, the tensor field varies with radial distance only.

Due to the fact that some of the astrophysical bodies such as the sun and stars are not perfectly spherical, their field cannot depend on radial distance only as assumed by Schwarzschild (Howusu, 2003).

Based on this fact, in this research work, we introduced an astrophysical distribution of mass within a region of spherical geometry, whose tensor field varies with time, radial distance and polar angle. The Riemann-Christoffel tensor, Ricci tensor, the Einstein's exterior field equations as well as the equations of motion for test particles and photons in the equatorial plane to this astrophysical body were constructed.

1.3 Aim of the Study

The aim of this research work is to construct the Einstein's field equation model for a spherical massive body whose tensor field varies with time, radial distance and polar angle only.

1.4 Objectives of the Study

The objectives of this research work are:

i. To construct the Riemann-Christoffel tensors for a spherical distribution of mass whose tensor field varies with time, radial distance and polar angle

ii. To construct the Ricci tensors for a spherical distribution of mass whose tensor field varies with time, radial distance and polar angle

iii. To construct Einstein's field equations for a spherical distribution of mass whose tensor field varies with time, radial distance and polar angle.

iv. To study the motion of particles and photons in the equatorial plane of a spherical distribution of mass whose tensor field varies with time, radial distance and polar angle.

1.5 Significance of the Study

This research work will gives us a clue to the nature of gravitation and pave way to the advance of the gravitational field to exact shape of the bodies in the universe. The Einstein's field equation whose tensor field varies not only with radial distance but also with time and polar angle will provide a frame work in which it will be possible to write down equations of physics such that they are formally identical for all observers, including accelerating ones within this gravitational field.

It will also help in studying astrophysical bodies whose tensor field varies with time, radial distance and polar angle in order to ascertain the actual behavior of a spherical body.

1.6 Scope of the Study

This study only deals with the Einstein's exterior field equations for rotating astrophysical massive bodies within a spherical geometry whose tensor field varies with time, radial distance and polar angle. However, it can be extended to include azimuthal angle and can be modify to study the interior field equations.

The equations obtained are only valid within a spherical geometry.

CHAPTER TWO

LITERATURE REVIEW

2.1 General Relativity

General relativity (GR, also known as the general theory of relativity or GTR) is the geometrictheory of gravitation published by Albert Einstein in 1915 (Penrose, 1965) and the current description of gravitation in modern physics. General relativity is considered probably the most beautiful of all existing physical theories (Kumar, 1959). General relativity generalizes special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime. In particular, the *curvature of spacetime* is directly related to the energy and momentum of whatever matter and radiation are present. The relation is specified by the Einstein field equations, a system of partial differential equations.

Some predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include gravitational time dilation, gravitational lensing, the gravitational redshift of light, and the gravitational time delay. The predictions of general relativity have been confirmed in all observations and experiments to date. Although general relativity is not the only relativistic theory of gravity, it is the simplest theory that is consistent with experimental data. However, unanswered questions remain, the most fundamental being how *general relativity* can be reconciled with the laws of quantum physics to produce a complete and self-consistent theory of quantum gravity (Cheng, 2005).

Einstein's theory has important astrophysical implications. For example, it implies the existence of black holes—regions of space in which space and time are distorted in such a way that nothing, not even light, can escape—as an end-state for massive stars. There is ample evidence that the intense radiation emitted by certain kinds of astronomical objects is due to black holes; for example, microquasars and active galactic nuclei result from the presence of stellar black holes and supermassive black holes, respectively. The bending of light by gravity can lead to the phenomenon of gravitational lensing, in which multiple images of the same distant astronomical object are visible in the sky. General relativity also predicts the existence of gravitational waves, which have since been observed directly by physics collaboration LIGO. In addition, general relativity is the basis of current cosmological models of a consistently expanding universe.

Soon after publishing the special theory of relativity in 1905, Einstein started thinking about how to incorporate gravity into his new relativistic framework. In 1907, beginning with a simple thought experiment involving an observer in free fall, he embarked on what would be an eight-year search for a relativistic theory of gravity. After numerous detours and false starts, his work culminated in the presentation to the Prussian Academy of Science in November 1915 of what are now known as the

Einstein field equations. These equations specify how the geometry of space and time is influenced by whatever matter and radiation are present, and form the core of Einstein's general theory of relativity (Pais, 1982).

2.2 Einstein’s Field Equations

The field equations for gravitation are inevitably going to be more complicated than those of electromagnetism. Maxwell’s equations are linear because the electromagnetic field does not itself carry charge, whereas gravitational field do carry energy and momentum and must therefore contribute to their own source. That is, the gravitational field equations will have to be nonlinear partial differential equations, the nonlinearity representing the effect of gravitation itself (Schwarzschild, 1916; Weinberg, 1972).

Einstein used approximation methods in working out initial predictions of the theory. However, as early as 1916, the astrophysicist Karl Schwarzschild found the first non-trivial exact solution to the Einstein field equations, the Schwarzschild metric. This solution laid the groundwork for the description of the final stages of gravitational collapse, and the objects known today as black holes. In the same year, the first steps towards generalizing Schwarzschild's solution to electrically charged objects were taken, which eventually resulted in the Reissner–Nordström solution, now associated with electrically charged black holes (Schwarzschild, 1916). In 1917, Einstein applied his theory to the universe as a whole, initiating the field of relativistic cosmology. In line with contemporary thinking, he assumed a static universe, adding a new parameter to his original field equations—the cosmological constant—to match that observational presumption (Einstein, 1917). By 1929, however, the work of Hubble and others had shown that our universe is expanding. This is, readily described by the expanding cosmological solutions found by Friedmann in 1922, which do not require a cosmological constant. Lemaître used these solutions to formulate the earliest version of the Big Bang models, in which our universe has evolved from an extremely hot and dense earlier state (Hubble, 1929). Einstein later declared the cosmological constant the biggest blunder of his life (Green, 2004; Gamow, 1970).

Having formulated the relativistic, geometric version of the effects of gravity, the question of gravity's source remains. In Newtonian gravity, the source is mass. In special relativity, mass turns out to be part of a more general quantity called the energy–momentum tensor, which includes both energy and momentum densities as well as stress (that is, pressure and shear) (Ehlers, 1973). Using the equivalence principle, this tensor is readily generalized to curved spacetime. Drawing further upon the analogy with geometric Newtonian gravity, it is natural to assume that the field equation for gravity relates this tensor and the Ricci tensor, which describes a particular class of tidal effects: the change in volume for a small cloud of test particles that are initially at rest, and then fall freely. In special relativity, conservation of energy–momentum corresponds to the statement that the energy–momentum tensor is divergence-free. This formula, too, is readily generalized to curved spacetime by replacing partial derivatives with their curved-manifold counterparts, covariant derivatives studied in differential geometry. With this additional condition—the covariant divergence of the energy–momentum tensor, and hence of whatever is on the other side of the equation, is zero— the simplest set of equations are what are called Einstein's (field) equations given by (Weinberg, 1972; Arfken, 1985; Bergmann, 1987; Howusu, 2009; Lumbi *et al.*, 2014)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{-8\pi G}{c^4} T_{\mu\nu} \quad : \tag{2.1}$$

On the left-hand side is the Einsteintensor, a specific divergence-free combination of the Ricci tensor $R_{\mu\nu}$ and the metric. Where $G_{\mu\nu}$ is symmetric. In particular, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the curvature scalar. The Ricci tensor itself is related to the more general Riemann curvature tensor as $R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$

On the right-hand side, $T_{\mu\nu}$ is the energy–momentum tensor. All tensors are written in abstract index notation (Ehlers, 1973). Matching the theory's prediction to observational results for planetaryorbits (or, equivalently, assuring that the weak-gravity, low-speed limit is Newtonian mechanics), the proportionality constant can be fixed as $\kappa = 8\pi G/c^4$, with G the gravitational constant and c the speed of light (Kenyon, 1990). When there is no matter present, so that the energy–momentum tensor vanishes, the results are the vacuum Einstein equations, $R_{\mu\nu} = 0$

2.3 Consequences of Einstein's theory

General relativity has a number of physical consequences. Some follow directly from the theory's axioms, whereas others have become clear only in the course of many years of research that followed Einstein's initial publication. Some of the consequences are as follows:

2.3.1 Gravitational time dilation and frequency shift

Assuming that the equivalence principle holds (Rindler, 2001), gravity influences the passage of time. Light sent down into a gravity well is blueshifted, whereas light sent in the opposite direction (i.e., climbing out of the gravity well) is redshifted; collectively, these two effects are known as the gravitational frequency shift. More generally, processes close to a massive body run more slowly when compared with processes taking place farther away; this effect is known as gravitational time dilation (Rindler, 2001).

Gravitational redshift has been measured in the laboratory (Pound & Rebka, 1959; Pound & Rebka, 1960; Pound & Snider, 1964; Ohanian & Ruffini, 1994) and using astronomical observations. Gravitational time dilation in the Earth's gravitational field has been measured numerous times using atomic clocks (Hafele & Keating, 1972a; Hafele & Keating, 1972b), while ongoing validation is provided as a side effect of the operation of the Global Positioning System (GPS) (Ashby, 2002; Ashby, 2003). Tests in stronger gravitational fields are provided by the observation of binary pulsars (Stairs, 2003; Kramer, 2004). All results are in agreement with general relativity (Ohanian & Ruffini, 1994). However, at the current level of accuracy, these observations cannot distinguish between general relativity and other theories in which the equivalence principle is valid (Ohanian & Ruffini, 1994).

2.3.2 Light deflection and gravitational time delay

General relativity predicts that the path of light is bent in a gravitational field; light passing a massive body is deflected towards that body. This effect has been confirmed by observing the light of stars or distant quasars being deflected as it passes the Sun (Kennefick, 2005; Ohanian & Ruffini, 1994; Shapiro *et al.*, 2004).

This and related predictions follow from the fact that light follows what is called a light-like or null geodesic—a generalization of the straight lines along which light travels in classical physics. Such geodesics are the generalization of the invariance of lightspeed in special relativity (Ehlers, 1973). As one examines suitable model spacetimes (either the exterior Schwarzschild solution or, for more than a single mass, the post-Newtonian expansion) (Blanchet, 2006), several effects of gravity on light propagation emerge. Although the bending of light can also be derived by extending the universality of free fall to light (Einstein, 1907; Israel, 1987; Ehlers & Rindler, 1997; Rindler, 2001), the angle of deflection resulting from such calculations is only half the value given by general relativity.

Closely related to light deflection is the gravitational time delay (or Shapiro delay), the phenomenon that light signals take longer to move through a gravitational field than they would in the absence of that field. There have been numerous successful tests of this prediction (Shapiro, 1964; Weinberg, 1972; Bertotti *et al.*, 2003, Stairs, 2003). In the parameterized post-Newtonian formalism (PPN), measurements of both the deflection of light and the gravitational time delay determine a parameter called γ, which encodes the influence of gravity on the geometry of space (Will, 1993).

2.3.3 Gravitational waves

Predicted in 1916 (Einstein, 1916; Einstein, 1918) by Albert Einstein, there are gravitational waves: ripples in the metric of spacetime that propagate at the speed of light. These are one of several analogies between weak-field gravity and electromagnetism in that; they are analogous to electromagnetic waves. On February 11, 2016, the Advanced LIGO team announced that they had directly detected gravitational waves from a pair of black holes merging (Castelvecchi & Witze 2016; Abbot *et al.,* 2016).

The simplest type of such a wave can be visualized by its action on a ring of freely floating particles. A sine wave propagating through such a ring towards the reader distorts the ring in a characteristic, rhythmic fashion (animated image to the right) (Schutz, 1985). Since Einstein's equations are non-linear, arbitrarily strong gravitational waves do not obey linear superposition, making their description difficult. However, for weak fields, a linear approximation can be made. Such linearized gravitational waves are sufficiently accurate to describe the exceedingly weak waves that are expected to arrive here on Earth from far-off cosmic events, which typically result in relative distances increasing and decreasing by 10^{-21} or less. Data analysis methods routinely make use of the fact that these linearized waves can be Fourier decomposed (Jaranowski & Krolak, 2005).

Some exact solutions describe gravitational waves without any approximation, e.g., a wave train traveling through empty space (Rindler, 2001) or Gowdy universes, varieties of an expanding cosmos filled with gravitational waves (Gowdy, 1971;

Gowdy, 1974). But for gravitational waves produced in astrophysically relevant situations, such as the merger of two black holes, numerical methods are presently the only way to construct appropriate models (Seidal, 1998; Lehner, 2002).

2.4 Solutions of Einstein's Field Equations

2.4.1 The Reissner–Nordstr¨om Solution

This spherically symmetric solution of the Einstein–Maxwell equations was derived independently (Bicak, 1989) by H. Reissner in 1916, H. Weyl in 1917, and G. Nordstr¨om in 1918. It represents a spacetime with no matter sources exceptfor a radial electric field, the energy of which has to be included on the righthandside of the Einstein equations. The Reissner–Nordstr¨om solution is the uniquespherical electrovacuum solution. Similarly to the Schwarzschild solution, itthus describes the exterior gravitational and electromagnetic fields of an arbitrary– static, oscillating, collapsing or expanding – spherically symmetric,charged body of mass M and charge Q. The metric reads

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (2.2)$$

The electromagnetic field in these spherical coordinates is described by the"classical" expressions for the time component of the electromagnetic potentialand the (only non-zero) component of the electromagnetic field tensor:

$$A_t = -\frac{Q}{r},\ F_{tr} = -F_{rt} = -\frac{Q}{r^2} \quad (2.3)$$

2.4.1.1 Reissner–Nordstr¨om Black Holesand the Question of Cosmic Censorship

The analytic extensions have qualitatively different character in three cases, depending on the relationship between the mass M and the charge Q. In the case $Q^2 > M^2$ (corresponding, for example, to the field outside an electron),the complete electrovacuum spacetime is covered by the coordinates(t, r, θ, ϕ), $0 < r < \infty$. There is a *naked singularity* (visible from infinity) at$r = 0$ in which the curvature invariants diverge. If $Q^2 < M^2$, the metric equation (2.2) describes a (generic) *Reissner–Nordstr¨om black hole*; it becomes singular at two radii:

$$r = r_{\pm} = M \pm \left(M^2 - Q^2\right)^{\frac{1}{2}} \qquad (2.4)$$

Similar to the Schwarzschild case, these are only coordinate singularities. Graves and Brill (Graves & Brill, 1960) discovered, however, that the analytic extension and the causal structure of the Reissner–Nordstr¨om spacetime with $M^2 > Q^2$ is fundamentally different from that of the Schwarzschild spacetime. There are two null hypersurfaces, at $r = r_+$ and $r = r_-$, which are known as the *outer(event) horizon* and the *inner horizon*; the Killing vector $\partial/\partial t$ is null at the horizons, time-like at $r > r_+$ and $r < r_-$, but space-like at $r_- < r < r_+$ (Misner *et al.*, 1973; Hawking & Ellis,1973; d'Inverno, 1992).

A number of authors have discussed spherically symmetric, static charged dust configurations producing a Reissner–Nordstr¨om metric outside, some of them with a hope to construct a "classical model" of a charged elementary particle (Kramer, 1980). The main influence the metric has exerted on the developments of general relativity, and more recently in supersymmetric and superstring theories is however in its analytically extended electrovacuum form when it represents charged, spherical black holes.

2.4.2 The Kerr Metric

The discovery of the Kerr metric in 1963 and the proof of its unique role in the physics of black holes have made an immense impact on the development of general relativity and astrophysics. This can hardly be more eloquently demonstrated than by an emotional text from Chandrasekhar (Chandrasekhar, 1986): "In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the Universe...". In Boyer–Lindquist coordinates the Kerr metric (Kerr, 1963) looks as follows (Misner *et al.,* 1973; d'Inverno, 1992):

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - 2\frac{2aMr\sin^2\theta}{\Sigma}dtd\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\mathrm{A}}{\Sigma}\sin^2\theta d\varphi^2 \quad (2.5)$$

Where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad \mathrm{A} = \Sigma\left(r^2 + a^2\right) + 2Mra^2\sin^2\theta, \quad (2.6)$$

M and a are constants

2.4.2.1 Basic Features

The Boyer–Lindquist coordinates follow naturally from the symmetries of the Kerr spacetime. The scalars t and ϕ can be fixed uniquely (up to additive constants) as parameters varying along the integral curves of (unique) stationary and axial Killing vector fields $\boldsymbol{k}$ and $\boldsymbol{m}$; and the scalars r and θ can be fixed (up to constant factors) as parameters related as closely as possible to the (geometrically preferred) principal null congruences, which in the Kerr spacetime exist (Misner *et al.*, 1973; d'Inverno, 1992), and their projections on to the two dimensional space-like submanifolds orthogonal to both ***k*** and ***m*** (Stewart & Walker, 1973). The Boyer–Lindquist coordinates represent the natural generalization of Schwarzschild coordinates. With a = 0 the metric (2.5) reduces to the Schwarzschild metric. By examining the Kerr metric in the asymptotic region $r\to\infty$, one finds that M represents the mass and $J = Ma$ the angular momentum pointing in the z-direction, so that a is the angular momentum per unit mass. One can arrive at these results by considering, for example, the weak field and slow motion limit, $M/r \ll 1$ and $a/r \ll 1$. The Kerr metric (2.5) can then be written in the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) - \frac{4aM}{r}\sin^2\theta d\varphi dt, \tag{2.7}$$

which is the weak field metric generated by a central body with mass M and angular momentum $J = Ma$. A general, rigorous way of interpreting the parameters entering the Kerr metric starts from the *definition of multipole moments* of asymptotically flat, stationary vacuum spacetimes. This is given in physical space by Thorne (Thorne, 1980), using his "asymptotically Cartesian and mass centered" coordinate systems, and by Hansen (Hansen, 1974), who, generalizing the definition of Geroch for the static case, gives the coordinate independent definition based on the conformal completion of the 3-dimensional manifold of trajectories of a timelike Killing vector ***k***. The exact Kerr solution has served as a convenient "test-bed" for such definitions.

The mass monopole moment – the mass – is *M*, the mass dipole moment vanishes in the "mass centered"coordinates, the quadrupole moment components are $\frac{1}{3}Ma^2$ and $-\frac{2}{3}Ma^2$. The current dipole moment – the angular momentum – is nonvanishing only along the axis of symmetry and is equal to $J = Ma$, while the current quadrupole moment vanishes. All other nonvanishing *l*-pole moments are proportional to Ma^l (Thorne, 1980; Hansen, 1974). Because these specific values of the multipolemoments depend on only two parameters, the Kerr solution clearly cannot represent the gravitational field outside a general rotating body. The fundamental significance of the Kerr spacetime, however, lies in its role as the *only vacuum rotating blackhole solution*. Many texts give excellent and thorough discussions of properties of Kerr black holes from various viewpoints (Misner *et al.*, 1973; Wald, 1984; Shapiro & Teukolsky, 1983; de Felice & Clarke, 1990). The Kerr metric entered the new edition of "Landau and Lifshitz" (Landau & Lifshitz, 1962). A few years ago, a book devoted entirely to the Kerr geometry appeared (O'Neill, 1994). Here we can list only a few basic points. As with the Reissne Nordstr¨om spacetime, one can make the maximal analytic extension of the Kerr geometry. This, in fact, has much in common with the Reissner–Nordstr¨om case. Loosely speaking, the "repulsive" characters of both charge and rotation have somewhat similar manifestations.

When $a^2 < M^2$, the metric (2.5) has coordinate singularities at $\Delta = 0$, i.e. at (equation (2.6))

$$r = r_{\pm} = M \pm \left(M^2 - a^2\right)^{\frac{1}{2}} \tag{2.8}$$

A crucial difference between the Reissner–Nordstr¨om and Kerr geometry is the existence of the *ergosphere* (or, more precisely, ergoregion) in the Kerr case. The dragging of inertial frames due to a nonvanishing angular momentum causes this. The timelike Killing vector ***k***, given in the Boyer–Lindquist coordinates by *∂/∂t*, becomes null "sooner", at $r = r_0$, than at the event horizon, $r_0 > r_+$ at $\theta \neq 0, \pi$, as a consequence of this dragging:

$$k^\alpha k_\alpha = -g_{tt} = 1 - 2Mr/\Sigma = 0, \quad r = r_0 = M + \left(M^2 - a^2\cos^2\theta\right)^{\frac{1}{2}}, \qquad (2.9)$$

This is the location of the *ergosurface*, the ergoregion being between this surface and the (outer) horizon. In the ergosphere, the "rotating geometry" drags the particles and light so strongly that all physical observers must corotate with the hole, and so rotate with respect to distant observers – "fixed stars" –at rest in the Boyer–Lindquist coordinates.15 Static observers, whose worldliness (*r, θ, ϕ*) = constant would have ***k*** as tangent vectors, cannot exist since ***k*** is spacelike in the ergosphere. Indeed, a non-spacelike worldline with *r, θ* fixed must satisfy the condition

$$g_{tt}dt^2 + g_{\varphi\varphi}d\varphi^2 + 2g_{\varphi t}d\varphi dt \leq 0, \qquad (2.10)$$

in which $g_{tt} = -k^\alpha k_\alpha$, $g_{\varphi\varphi} = m^\alpha m_\alpha$, $g_{\varphi t} = k^\alpha m_\alpha$, are invariants. In the ergosphere, the metric equations (2.5), (2.6) yields $g_{tt} > 0$, $g_{\varphi\varphi} > 0$, $g_{\varphi t} < 0$, so that $d\varphi / dt > 0$, an observer moving along a non-spacelike worldline must corotate with the hole. The effect of dragging on the forms of photon escape cones in a general Kerr field (without restriction $a^2 < M^2$) has been numerically studied and carefully illustrated in a number of figures only recently (Semer´ak, 1996).In order to "compensate" the dragging, the congruence of *"locally nonrotating frames"* (LNRFs), or *"zero-*

angular momentum observers" (ZAMOS),has been introduced. These frames have also commonly been used outside relativistic, rapidly rotating stars constructed numerically, but the Kerr metric played an inspiring role (as, after all, in several other issues, such as in understanding the ergoregions, etc.). The four-velocity of these (not freely falling!) observers, given in Boyer–Lindquist coordinates by

$$e_{(t)}{}^{\alpha} = [\left(\mathrm{A}/\Sigma\Delta\right)^{\frac{1}{2}}, 0, 0, 2Ma/\left(\mathrm{A}\Sigma\Delta\right)^{\frac{1}{2}}], \tag{2.11}$$

is orthogonal to the hypersurfaces t = constant. The particles falling from rest at infinity with zero total angular momentum fall exactly in the radial direction in the locally nonrotating frames with an orthogonal triad tied to the r, θ, ϕ coordinate directions (for the study of the shell of suchparticles falling on to a Kerr black hole) (Bicak & Stuchl´ık, 1976).

2.4.3 The Schwarzschild Solution

2.4.3.1 Spherically Symmetric Spacetimes

In the early days of general relativity, spherical symmetry was introduced in an intuitive manner. It is because of the existence of exact solutions which are singular at their centers (such as the Schwarzschild or the Reissner– Nordstr¨om solutions), and a realization that spherically symmetric, topologically non-trivial smooth spacetimes without any centre may exist (K¨unzle, 1967), that today the group-theoretical definition of spherical symmetry is preferred. Following Ehlers (Ehlers, 1973), we define a spacetime $(\mathcal{M}, g_{\alpha\beta})$ to be spherically symmetric if the rotation group $SO3$ acts on $(\mathcal{M}, g_{\alpha\beta})$ as an isometry group with simply connected, complete, spacelike, 2-

dimensional orbits. One can then prove the theorem (Ehlers, 1973; Schmidt, 1967) that a spherically symmetric spacetime is the direct product ($\mathcal{M}, S^2$) where S^2 is the 2-sphere manifold with the standard metric gSon the unit sphere; and *N* is a 2-dimensional manifold with a Lorentzian (indefinite) metric *gN*, and with a scalar *r* such that the complete spacetime metric $g_{\alpha\beta}$is "conformally decomposable", i.e. *r*−2$g_{\alpha\beta}$ is the direct sum of the 2-dimensional parts *gN* and *gS*. Leaving further technicalities aside (Hawking & Ellis, 1973; Ehlers, 1973; Schmidt, 1967) we write down the final spherically symmetric line element in the form

$$ds^2 = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (2.12)$$

where we permit *φ*(*r, t*) and *λ*(*r, t*) to have an imaginary part $\frac{i\pi}{2}$ so that the signs of dt^2 and dr^2 in (2.13), and thus the role of *r* and *t* as space- and time- coordinates may interchange. The "curvature coordinate" *r* is defined invariantly by the area, $4\pi r^2$, of the 2-spheres *r* = constant, *t* = constant. There is no a priori relation between *r* and the proper distance from the centre (if there is one) to the spherical surface.

2.4.3.2 The Schwarzschild Metric and Its Role in the Solar System

Starting from the line element (2.13) and imposing Einstein's *vacuum* field equations, but allowing spacetime to be in general dynamical, we are led uniquely (Misner, *et al.*, 1973; Hawking & Ellis, 1973) to the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (2.13)$$

where *M* = constant has to be interpreted as a mass, as test particle orbits show. The resulting spacetime is static at *r* >2*M* (no spherically symmetric gravitational waves exist), and asymptotically flat at *r*→∞. Undoubtedly, the Schwarzschild solution, describing the exterior gravitational field of an arbitrary – static, oscillating, collapsing or expanding – spherically symmetric body of (Schwarzschild) mass *M*, is among the most influential solutions of the gravitational field equations, if not of any type of field equations invented in the 20th century. It is the first exact solution of Einstein's equations obtained – by K. Schwarzschild in December 1915, still before Einstein's theory reached its definitive form and, independently, in May 1916, by J. Droste, a Dutch student of Lorentz. However, in its exact form (involving regions near $r \approx 2M$) the metric (2.13) has not yet been experimentally tested. When in 1915 Einstein explained the perihelion advance of Mercury, he found and used only an approximate (to second order in the gravitational potential) spherically symmetric solution. In order to find the value of the deflection of light passing close to the surface of the Sun, in his famous 1911 Prague paper, Einstein used just the equivalence principle within his "Prague gravity theory", based on the variable velocity of light. Then, in 1915, he obtained this value to be twice as big in general relativity, when, in addition to the equivalence principle, the curvature of space (determined from (2.13) to first order in M/r) was taken into account.

One limitation of the Scwartzchild's solution is that it describe a static space-time in which all metric components are independent of the time coordinate t, (so that $\frac{\partial}{\partial t} g_{\mu\nu} = 0$) and the geometry of the space-time is unchanged under a time-reversal $t \rightarrow -t$.

2.4.4 Tests of General Relativity

In 1915, the general theory of relativity was introduced but did not have a solid empirical foundation. It was known that it correctly accounted for the "anomalous" precession of the perihelion of Mercury and on philosophical grounds it was considered satisfying that it was able to unify Newton's law of universal gravitation with special relativity. That light appeared to bend in gravitational fields in line with the predictions of general relativity was found in 1919 but it was not until a program of precision tests was started in 1959 that the various predictions of general relativity were tested to any further degree of accuracy in the weak gravitational field limit, severely limiting possible deviations from the theory (Davide & Witze, 2016).

2.4.4.1Classical tests

Albert Einstein proposed (Einstein, 1916; Davide & Witze, 2016*)*, three tests of general relativity, subsequently called the classical tests of general relativity, in 1916:

1. the perihelion precession of Mercury's orbit
2. the deflection of light by the Sun
3. the gravitational redshift of light

(1) Perihelion Precession of Mercury

Under Newtonian physics, a two-body system consisting of a lone object orbiting a spherical mass would trace out an ellipse with the spherical mass at a focus. The point of closest approach, called the periapsis (or, because the central body in the Solar System is the Sun, perihelion), is fixed. A number of effects in the Solar System cause the perihelia of planets to precess (rotate) around the Sun. The principal cause is the presence of other planets which perturb one another's orbit. Another (much less significant) effect is solar oblateness.

Mercury deviates from the precession predicted from these Newtonian effects. This anomalous rate of precession of the perihelion of Mercury's orbit was first recognized in 1859 as a problem in celestial mechanics, by Urbain Le Verrier. His reanalysis of available timed observations of transits of Mercury over the Sun's disk from 1697 to 1848 showed that the actual rate of the precession disagreed from that predicted from Newton's theory by 38" (arc seconds) per tropical century (later re-estimated at 43") (Le Verrier, 1859). A number of ad-hoc and ultimately unsuccessful solutions were proposed, but they tended to introduce more problems. In general relativity, this remaining precession, or change of orientation of the orbital ellipse within its orbital plane, is explained by gravitation being mediated by the curvature of spacetime. Einstein showed that general relativity (Davide & Witze, 2016), agrees closely with the observed amount of perihelion shift. This was a powerful factor motivating the adoption of general relativity.

Although earlier measurements of planetary orbits were made using conventional telescopes, more accurate measurements are now made with radar. The total observed precession of Mercury is 574.10±0.65 arc-seconds per century(Clemence, 1947) relative to the inertial ICRF. This precession can be attributed to the following causes as shown in table 2.1

Table 2.1: Sources of the Precession of Perihelion for Mercury

Amount (arcsec/Julian century)	Cause
531.63 ± 0.69	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun (quadrapole moment)
42.98 ± 0.04	General Relativity
574.64 ± 0.69	Total
574.10 ± 0.65	Observed

(Clemence, 1947)

The correction by 42.98" is 3/2 multiple of classical prediction with PPN parameter $\gamma = \beta = 0$relativity. More recent calculations based on measurements that are more precise have not materially changed the situation.

The other planets experience perihelion shifts as well, but, since they are farther from the Sun and have longer periods, their shifts are lower, and could not be observed accurately until long after Mercury's. For example, the perihelion shift of Earth's orbit due to general relativity is of 3.84 seconds of arc per century, and Venus's is 8.62". Both values are in good agreement with observation (Abhijit and Krishnan, 2008). The periapsis shift of binary pulsar systems have been measured, with PSR 1913+16 amounting to 4.2º per year (Alfred, 2001). These observations are consistent with general relativity (Weisberg and Taylor, 2005). It is also possible to measure periapsis shift in binary star systems which do not contain ultra-dense stars, but it is more difficult to model the classical effects precisely - for example, the alignment of the stars' spin to their orbital plane needs to be known and is hard to measure directly - so a few systems such as DI Herculis have been considered as problematic cases for general relativity.

In general relativity the perihelion shift σ, expressed in radians per revolution, is approximately given by (Adrian-Horia *et al*, 2015):

$$\sigma = \frac{24\pi^3 L^2}{T^2 c^2 (1-e^2)} \tag{2.13}$$

where L is the semi-major axis, T is the orbital period, c is the speed of light, and e is the orbital eccentricity.

(2) Deflection of light by the Sun

Henry Cavendish in 1784 (in an unpublished manuscript) and Johann Georg von Soldner in 1801 (published in 1804) had pointed out that Newtonian gravity predicts that starlight will bend around a massive object (Soldner, 1804; Soares, 2009). The same value as Soldner's was calculated by Einstein in 1911 based on the equivalence principle alone. However, Einstein noted in 1915 in the process of completing general relativity, that his (and thus Soldner's) 1911 result is only half of the correct value. Einstein became the first to calculate the correct value for light bending (Will, 2006).

The first observation of light deflection was performed by noting the change in position of stars as they passed near the Sun on the celestial sphere. The observations were performed by Arthur Eddington and his collaborators during the total solar eclips of May 29, 1919, (Dyson *et at*, 1920) when the stars near the Sun (at that time in the constellation Taurus) could be observed (Dyson *et at*, 1920). Observations were made simultaneously in the cities of Sobral, Ceara, Brazil and in Sao Tome and Principe on the west coast of Africa (Stanley, 2003). The result was considered spectacular news and made the front page of most major newspapers. It made Einstein and his theory of general relativity world-famous. When asked by his assistant what his reaction would have been if general relativity had not been confirmed by Eddington and Dyson in 1919, Einstein famously made the quip: "Then I would feel sorry for the dear Lord. The theory is correct anyway" (Rosenthal, 1980).

The early accuracy, however, was poor. The results were argued by Harry and Trevor, to have been plagued by systematic error and possibly confirmation bias, although modern reanalysis of the dataset (Daniel, 2007) suggests that Eddington's analysis was accurate (Ball, 2007; Kennefick, 2009). The measurement was repeated by a team from the Lick Observatory in the 1922 eclipse, with results that agreed with the 1919 results (Kennefick, 2009)and has been repeated several times since, most notably in 1953 by Yerkes Observatory astronomers (van Biesbroeck, 1952)and in 1973 by a team from the University of Texas (Texas Mauritanian Eclipse Team, 1973). Considerable uncertainty remained in these measurements for almost fifty years, until observations started being made at radio frequencies. It was not until the 1960s that it was definitively accepted that the amount of deflection was the full value predicted by general relativity, and not half that number. The Einstein ring is an example of the deflection of light from distant galaxies by more nearby objects.

(3) Gravitational Redshift of Light

Einstein predicted the gravitational redshift of light from the equivalence principle in 1907, but it is very difficult to measure astrophysically (see the discussion under *Equivalence Principle* below). Although it was measured by Walter Sydney Adams in 1925, it was only conclusively tested when the Pound-

Rebka experiment in 1959 measured the relative redshift of two sources situated at the top and bottom of Harvard University's Jefferson tower using an extremely sensitive phenomenon called the Mossbauer Effect (Pound & Rebka, 1959; Pound & Rebka, 1960). The result was in excellent agreement with general relativity. This was one of the first precision experiments testing general relativity.

2.4.4.2 Modern Tests

The modern era of testing general relativity was ushered in largely at the impetus of Dicke and Schiff who laid out a framework for testing general relativity (Dicke, 1959; Dicke, 1962; Schiff, 1960). They emphasized the importance not only of the classical tests, but of null experiments, testing for effects which in principle could occur in a theory of gravitation, but do not occur in general relativity. Other important theoretical developments included the inception of alternative theories to general relativity, in particular, scalar-tensor theories such as the Brans-Dicke theory (Brans and Dicke, 1961); the parametrized post-Newtonian formalism in which deviations from general relativity can be quantified; and the framework of the equivalence principle.

Experimentally, new developments in space exploration, electronics and condense matter physics have made additional precise experiments possible, such as the Pound–Rebka experiment, laser interferometry and lunar range finding.

2.4.4.3 Post-Newtonian Tests of Gravity

Early tests of general relativity were hampered by the lack of viable competitors to the theory: it was not clear what sorts of tests would distinguish it from its competitors. General relativity was the only known relativistic theory of gravity compatible with special relativity and observations. Moreover, it is an extremely simple and elegant theory. This changed with the introduction of Brans-Dicke theory in 1960. This theory is arguably simpler, as it contains no dimensionful constants, and is compatible with a version of Mach's principle and Dirac's large number hypothesis, two philosophical ideas which have been influential in the history of relativity. Ultimately, this led to the development of the parameterized post-Newtonian formalism by Nordtvedt and Will, which parameterizes, in terms of ten adjustable parameters, all the possible departures from Newton's law of universal gravitation to first order in the velocity of moving objects (*i.e.* to first order in where v is the velocity of an object and c is the speed of light). This approximation allows the possible deviations from general relativity, for slowly moving objects in weak gravitational fields, to be systematically analyzed. Much effort has been put into constraining the post-Newtonian parameters, and deviations from general relativity are at present severely limited (Dicke, 1962).

The experiments testing gravitational lensing and light time delay limits the same post-Newtonian parameter, the so-called Eddington parameter γ, which is a straightforward parametrization of the amount of deflection of light by a gravitational source. It is equal to one for general relativity, and takes different values in other theories (such as Brans–Dicke theory). It is the best constrained of the ten post-

Newtonian parameters, but there are other experiments designed to constrain the others. Precise observations of the perihelion shift of Mercury constrain other parameters, as do tests of the strong equivalence principle (Dicke, 1962).

2.4.4.4 Gravitational Lensing

One of the most important tests is gravitational lensing. It has been observed in distant astrophysical sources, but these are poorly controlled and it is uncertain how they constrain general relativity. The most precise tests are analogous to Eddington's 1919 experiment: they measure the deflection of radiation from a distant source by the Sun. The sources that can be most precisely analyzed are distant radio sources. In particular, some quasars are very strong radio sources. The directional resolution of any telescope is in principle limited by diffraction; for radio telescopes this is also the practical limit. An important improvement in obtaining positional high accuracies (from milli-arcsecond to micro-arcsecond) was obtained by combining radio telescopes across Earth. The technique is called very long baseline interferometer (VLBI). With this technique radio, observation couple the phase information of the radio signal observed in telescopes separated over large distances. Recently, these telescopes have measured the deflection of radio waves by the Sun to extremely high precision, confirming the amount of deflection predicted by general relativity aspect to the 0.03% level (Fomalont *et al.*, 2009). At this level of precision systematic effects has to be carefully taken into account to determine the precise location of the telescopes on Earth. Some important effects are Earth's nutation, rotation, atmospheric refraction, tectonic displacement and tidal waves. Another important effect is refraction of the radio waves by the solar coronation. Fortunately, this effect has a characteristic spectrum, whereas gravitational distortion is independent of wavelength. Thus, careful analysis, using measurements at several frequencies, can subtract this source of error.

The entire sky is slightly distorted due to the gravitational deflection of light caused by the Sun (the anti-Sun direction excepted). The European Space Agency astrometric satellite Hipparcos has observed this effect. It measured the positions of about 10^5 stars. During the full mission about 3.5×10^6 relative positions have been determined, each to an accuracy of typically 3 milliarcseconds (the accuracy for an 8–9 magnitude star). Since the gravitation deflection perpendicular to the Earth–Sun direction is already 4.07 milliarcseconds, corrections are needed for practically all stars. Without systematic effects, the error in an individual observation of 3 milliarcseconds, could be reduced by the square root of the number of positions, leading to a precision of 0.0016 milliarcseconds. Systematic effects, however, limit the accuracy of the determination to 0.3% (Froeschlé *et al.*, 1997).

2.4.4.5 Light Travel Time Delay Testing

Irwin I. Shapiro proposed another test, beyond the classical tests, which could be performed within the Solar System. It is sometimes called the fourth "classical" test of general relativity. He predicted a relativistic time delay (Shapiro delay) in the round-trip travel time for radar signals reflecting off other planets (Shapiro, 1964). The mere curvature of the path of a photon passing near the Sun is too small to have an observable delaying effect (when the round-trip time is compared to the time taken if the photon had followed a straight path), but general relativity predicts a time delay that becomes progressively larger when the photon passes nearer to the Sun due to the time dilation in the gravitational potential of the Sun. Observing radar reflections from Mercury and Venus just before and after it is eclipsed by the Sun agrees with general relativity theory at the 5% level (Shapiro *et al*., 1971). More recently, the Cassini probe has undertaken a similar experiment which gave agreement with general relativity at the 0.002% level (Bertotti *et al.*, 2003). Very Long Baseline Interferometry has measured velocity-dependent (gravitomagnetic) corrections to the Shapiro time delay in the field of moving Jupiter (Fomalont & Kopeikin, 2003; Kopeikin & Fomalont, 2007) and Saturn (Fomalont *et al.,* 2010).

2.4.4.6 The Equivalence Principle

The equivalence principle, in its simplest form, asserts that the trajectories of falling bodies in a gravitational field should be independent of their mass and internal structure, provided they are small enough not to disturb the environment or be affected by tidal forces. This idea has been tested to extremely high precision by Eotvos torsion balance experiments, which look for a differential acceleration between two test masses. Constraints on this, and on the existence of a composition-dependent fifth force or gravitational Yukawa interaction are very strong, and are discussed under fifth forces and weak equivalence principle (Nordtvedt, 1968).

A version of the equivalence principle, called the strong equivalence principle, asserts that self-gravitation falling bodies, such as stars, planets or black holes (which are all held together by their gravitational attraction) should follow the same trajectories in a gravitational field, provided the same conditions be satisfied. This is called the

Nordtvedt effect and is most precisely tested by the Lunar Laser Ranging Experiments (Nordtvedt, 1968; Nordtvedt, 1968). Since 1969, it has continuously measured the distance from several range finding stations on Earth to reflectors on the Moon to approximately centimeter accuracy (Williams *et al*, 2004). These have provided a strong constraint on several of the other post-Newtonian parameters.

Another part of the strong equivalence principle is the requirement that Newton's gravitational constant be constant in time, and have the same value everywhere in the universe. There are many independent observations limiting the possible variation of Newton's gravitational constant, (Uzan, 2003) but one of the best comes from lunar range finding which suggests that the gravitational constant does not change by more than one part in 10^{11} per year. The constancy of the other constants is discussed in the Einstein's equivalence principle section of the equivalence principle article.

2.4.4.7 Gravitational Redshift

The first of the classical tests discussed above, the gravitational redshift, is a simple consequence of the Einstein's equivalence principle and was predicted by Einstein in 1907. As such, it is not a test of general relativity in the same way as the post-Newtonian tests, because any theory of gravity obeying the equivalence principle should also incorporate the gravitational redshift. Nonetheless, confirming the existence of the effect was an important substantiation of relativistic gravity, since the absence of gravitational redshift would have strongly contradicted relativity. The first observation of the gravitational redshift was the measurement of the shift in the spectral lines from the white dwarf star Sirius B by Adams in 1925. Although this measurement, as well as later measurements of the spectral shift on other white dwarf stars, agreed with the prediction of relativity, it could be argued that the shift could possibly stem from some other cause, and hence experimental verification using a known terrestrial source was preferable.

Experimental verification of gravitational redshift using terrestrial sources took several decades, because it is difficult to find clocks (to measure time dilation) or sources of electromagnetic radiation (to measure redshift) with a frequency that is known well enough that the effect can be accurately measured. It was confirmed experimentally for the first time in 1960 using measurements of the change in wavelength of gamma-ray photons generated with the Mossbauer effect, which generates radiation with a very narrow line width. The experiment, performed by Pound and Rebka and later improved by Pound and Snyder, is called the Pound-Rebka experiment. The accuracy of the gamma-ray measurements was typically 1%. The blueshift of a falling photon can be found by assuming it has an equivalent mass based on its frequency$E = hf$(where h is Plank's constant) along with $= mc^2$, a result of special relativity. Such simple derivations ignore the fact that in general relativity the experiment compares clock rates, rather than energies. In other words, the "higher energy" of the photon after it falls can be equivalently ascribed to the slower running of clocks deeper in the gravitational potential well. To fully validate general relativity, it is important to also show that the rate of arrival of the photons is greater than the rate at which they are emitted. A very accurate gravitational redshift experiment, which deals with this issue, was performed in 1976 (Vessot *et al*, 1980), where a hydrogen maser clock on a rocket was launched to a height of 10,000 km, and its rate compared with an identical clock on the ground. It tested the gravitational redshift to 0.007%.

Although the Global Positioning System (GPS) is not designed as a test of fundamental physics, it must account for the gravitational redshift in its timing system, and physicists have analyzed timing data from the GPS to confirm other tests. When the first satellite was launched, some engineers resisted the prediction that a noticeable gravitational time dilation would occur, so the first satellite was launched without the clock adjustment that was later built into subsequent satellites. It showed the predicted shift of 38 microseconds per day. This rate of discrepancy is sufficient to substantially impair function of GPS within hours if not accounted for (Will, 2010).

Other precision tests of general relativity (Schiller, 2007), not discussed here, are the Gravity Probe A satellite, launched in 1976, which showed gravity and velocity affect the ability to synchronize the rates of clocks orbiting a central mass; the Hafele-Keating experiment, which used atomic clocks in circumnavigating aircraft to test general relativity and special relativity together (Hafele and Keating, 1972; Hafele and Keating, 1972); and the forthcoming Satellite Test of the Equivalence Principle.

2.4.4.8 Frame-Dragging Tests

Tests of the Lense-Thirring precession, consisting of small secular precession of the orbit of a test particle in motion around a central rotating mass, for example, a planet or a star, have been performed with the LAGEOS satellites, (Ciufolini & Pavlis, 2004) but many aspects of them remain controversial. The same effect may have been detected in the data of the Mars Global Surveyor (MGS) spacecraft, a former probe in orbit around Mars; also such a test raised a debate (Krogh, 2007). First attempts to detect the Sun's Lense–Thirring effect on the perihelia of the inner planet have been recently reported as well. Frame dragging would cause the orbital plane of stars orbiting near a supermassive black hole to precess about the black hole spin axis. This effect should be detectable within the next few years via astrometric monitoring of stars at the center of the Milky Way galaxy (Merritt *et al*, 2010). By comparing the rate of orbital precession of two stars on different orbits, it is possible in principle to test the no-hair theorem of general relativity.

The Gravity Probe B satellite, launched in 2004 and operated until 2005, detected frame-dragging and the geodetic effect. The experiment used four quartz spheres the size of ping-pong balls coated with a superconductor. Data analysis continued through 2011 due to high noise levels and difficulties in modelling the noise accurately so that a useful signal could be found. Principal investigators at Stanford University reported on May 4, 2011, that they had accurately measured the frame dragging effect relative to the distant star IM pegasi, and the calculations proved to be in line with the prediction of Einstein's theory. The results, published in Physical Review Journal measured the geodetic effect with an error of about 0.2 percent. The results reported the frame dragging effect (caused by Earth's rotation) added up to 37 milliarcseconds with an error of about 19 percent (Everitt, 2011). Investigator Francis Everitt explained that a milliarcsecond "is the width of a human hair seen at the distance of 10 miles" (Ker, 2011).

In January 2012, LARES satellite was launched on a VEGA rocket to measure Lense-Thirring effect with an accuracy of about 1%, according to its proponents (Ciufolini, *et al*.,2009). This evaluation of the actual accuracy obtainable is a subject of debate (Ciufolini, *et al*., 2009; Ciufolini, *et al*., 2010; Paolozzi, *et al.*, 2011).

2.4.4.9 Tests of the Gravitational Potential at Small Distances

It is possible to test whether the gravitational potential continues with the inverse square law at very small distances. Tests so far have focused on a divergence from GR in the form of a Yukawa potential $V(r) = V_0(1 + \alpha e^{-r/\lambda})$, but no evidence for a potential of this kind has been found. The Yukawa potential with $\alpha = 1$ has been ruled out down to $\lambda = 5.6 \times 10^{-5}$m (Kapner, 2007).

2.4.4.10 Strong Field Tests: Binary Pulsars

Pulsars are rapidly rotating neutron stars, which emit regular radio pulses as they rotate. As such, they act as clocks, which allow very precise monitoring of their orbital motions. Observations of pulsars in orbit around other stars have all demonstrated substantial periapsis precessions that cannot be accounted for classically but can be accounted for by using general relativity. For example, the Hulse–Taylor binary pulsar PSR B1913+16 (a pair of neutron stars in which one is detected as a pulsar) has an observed precession of over 4° of arc per year (periastron shift per orbit only about 10^{-6}). This precession has been used to compute the masses of the components.

Similarly, to the way in which atoms and molecules emit electromagnetic radiation, a gravitating mass that is in quadrupole type or higher order vibration, or is asymmetric and in rotation, can emit gravitational waves (Deser & Franklin, 2005). This gravitational wave is predicted to travel at the speed of light. For example, planets orbiting the Sun constantly lose energy via gravitational radiation, but this effect is so small that it is unlikely it will be observed in the near future (Earth radiates about 200 watts (see gravitational wave of gravitational radiation).

The radiation of gravitational waves has been inferred from the Huslse-Taylor binary (and other binary pulsars) (Stairs, 2003). Precise timing of the pulses shows that the stars orbit only approximately according to Keplars law: over time, they gradually spiral towards each other, demonstrating an energy loss in close agreement with the predicted energy radiated by gravitational waves (Weisberg, *et al.,* 1981; Weisberg, *et al.,* 2010).For their discovery of the first binary pulsar and measuring its orbital decay due to gravitational-wave emission, Hulse and Taylor won the 1993 Nobel Prize in Physics (Novel Price, 1993).

A "double pulsar" discovered in 2003, PSR J0737-3039, has a periastron precession of 16.90° per year; unlike the Hulse–Taylor binary, both neutron star are detected as pulsars, allowing precision timing of both members of the system. Due to this, the tight orbit, the fact that the system is almost edge-on, and the very low transverse velocity of the system as seen from Earth, J0737−3039 provides by far the best system for strong-field tests of general relativity known so far. Several distinct relativistic effects are observed, including orbital decay as in the Hulse–Taylor system. After observing the system for two and a half years, four independent tests of general relativity were possible, the most precise (the Shapiro delay) confirming the general relativity prediction within 0.05% (Kramer, *et al*., 2006), (nevertheless the periastron shift per orbit is only about 0.0013% of a circle and thus it is not a higher-order relativity test).

In 2013, an international team of astronomers reported new data from observing a pulsar-white dwarf system PSR J0348+0432, in which they have been able to measure a change in the orbital period of 8 millionths of a second per year, and confirmed GR predictions in a regime of extreme gravitational fields never probed before (Antoniadi, 2013); but there are still some competing theories that would agree with these data (Ron, 2013).

2.4.4.11 Direct Detection of Gravitational Waves

A number of gravitational-wave detectors have been built with the intent of directly detecting the gravitational waves emanating from such astronomical events as the merger of two neutron stars or black holes. In February 2016, the Advanced LIGO team announced that they had directly detected gravitational waves from a stellar binary black hole merger (Castelvecchi & Witze, 2016; Abbott *et al*., 2016).

General relativity predicts gravitational waves, as does any theory of gravitation that obeys special relativity and so has changes in the gravitational field propagate at a finite speed (Schutz, 1984). Since gravitational waves can be directly detected (Castelvecchi & Witze, 2016), it is possible to use them to learn about the Universe. This is gravitational-wave astronomy. Gravitational-wave astronomy can test general relativity by verifying that the observed waves are of the form predicted (for example, that they only have two transverse polarizations), and by checking that black holes are the objects described by solutions of the Einstein's field equation (Gair, *et al.,* 2013; Yunes & Siemens, 2013; Abbott,*et al.,* 2016).

"These amazing observations are the confirmation of a lot of theoretical work, including Einstein's general theory of relativity, which predicts gravitational waves," says physicist Stephen Hawking (Castelvecchi & Witze,2016*)*.

2.4.4.12 Cosmological Tests

Tests of general relativity on the largest scales are not nearly as stringent as Solar System tests (Peebles, 2004). The earliest such test was prediction and discovery of the expansion of the universe (Rudnicki, 1991). In 1922, Alexander Friedman found that Einstein equations have non-stationary solutions (even in the presence of the cosmological constant) (Pauli, 1958; Krangh, 2003). In 1927 Georges Lemaitre showed that static solutions of the Einstein equations, which are possible in the presence of the cosmological constant, are unstable, and therefore the static universe envisioned by Einstein could not exist (it must either expand or contract) (Pauli, 1958). Lemaître made an explicit prediction that the universe should expand (Krangh, 2003). He also derived a redshift-distance relationship, which is now known as the Hubble Law (Krangh, 2003). Later, in 1931, Einstein himself agreed with the results of Friedmann and Lemaître (Pauli, 1958). The expansion of the universe discovered by Edwin Hubble in 1929 (Pauli, 1958), was then considered by many (and continues to be considered by some now) as a direct confirmation of general relativity (Rudnicki, 1991). In the 1930s, largely due to the work of E.A Milni, it was realised that the linear relationship between redshift and distance derives from the general assumption of uniformity and isotropy rather than specifically from general relativity (Rudnicki, 1991). However the prediction of a non-static universe was non-trivial, indeed dramatic, and primarily motivated by general relativity (Chandrasekhar, 1980).

Some other cosmological tests include searches for primordial gravitational waves generated during cosmic inflation, which may be detected in the cosmic microwave background polarization (Hand, 2009) or by a proposed space-based gravitational wave interferometer called the Big Bang Observer. Other tests at high redshift are constraints on other theories of gravity (Reyes, 2010; Guzzo, 2008), and the variation of the gravitational constant since big bang nucleosynthesis (it varied by no more than 40% since then).

2.5 Review of Related Work

2.5.1 Analytical Solution of Einstein's Geometrical Field Equation Interior/Exterior to Homogeneous Distribution of Masses

Chifu, 2009, published a work on the exact analytical solutions of Einstein's geometrical field equations to the order of c^{-2} using a new approach. In his work, the field equations exterior and interior to the mass distribution have only one unknown function determined by the mass or pressure distribution. The obtained solutions yield the unknown function as generalizations of Newton's gravitational scalar potential. Thus, his solution puts Einstein's geometrical theory of gravity on same footing with Newton's dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newton's gravitational scalar potential. The results in this article are of much significance as the Sun and planets in the solar system are known to be more precisely oblate spheroidal in geometry. The oblate spheroidal geometries of these bodies have effects on their gravitational fields and the motions of test particles and photons in these fields (Chifu, 2009).

Chifu and Howusu, 2009, published a work on the complete analyticalsolution of Einstein's geometrical gravitational field equations exterior to astrophysically real or hypothetical time varying distributions of mass within regions of spherical geometry. The obtained solution of Einstein's gravitational field equations tends out to be a generalization of Newton's gravitational scalar potential exterior to the spherical mass or pressure distribution under consideration (Chifu & Howusu, 2009).

2.5.2 Exact Solution for Gravitational Field Equations

In 2008, Biswas et al., obtained an exact solution of Einstein's equation to the gravitational field of mass point in polar coordinates and the singularity analysis of the solution. This solution is of great importance on account of the fact that it provides a treatment of gravitational field surrounding the sun (Biswas *et al.,* 2008).

Lenk, 2010 in USA, conducted a work on the general, exact solution for the gravitational field equations for diagonal, vacuum, separable metrics. These are metrics each of whose terms can be separated into functions of each space-time variable separately. Other than this, the functions are completely arbitrary; no symmetries are assumed; no limitations are placed on the coordinates. There are 16 functions, which with specific selection of coordinates reduce to 12. Since there are 10 field equations, two functions in the solution are completely arbitrary. The field equations were solved exactly. The solution for each function was presented analytically, with a total of three parameters and ten constants in addition to the two arbitrary functions.It was observe that this metric cannot be reduced to the form of the Schwarzschild solution, nor to that of flat space-time. Because of the high degree of symmetry of these solutions, this was to be expected. They correspond to some of the 'special' cases that were not explicitly presented here (Lenk, 2010).

2.5.3 The Einstein's Equation of Motion

Chifu, 2012, published a work on the general relativistic mechanics in gravitational fields exterior to a static homogeneous spheroidal masses using a new approach. Einstein's field equations in the gravitational field exterior to a static homogeneous prolate spheroid were derived and a solution for the first field equations constructed.

The derived field equations exterior to the mass distribution have only one unknown function determined by the mass or pressure distribution. The obtained solutions yield the unknown function as generalizations of Newton's gravitational scalar potential. Remarkably, the solution puts Einstein's geometrical theory of gravity on same footing with Newton's dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newton's gravitational scalar potential. The consequences of the homogeneous spheroidal gravitational field on the motion of test particles have been theoretically investigated. The effect of the oblate nature of the Sun and planets on some gravitational phenomena were examined. These are gravitational time dilation, gravitational length contraction and gravitational spectral shift of light. The obtained theoretical value for the Pound-Rebka experiment on gravitational spectra shift ($2.578\times10-15$) agrees satisfactorily with the experimental value of $2.45\times10-15$ (Chifu, 2012).

Izam and Jwanbot, 2013, conducted a work on the series solution of the complete Golden Dynamical Equation of motion for a photon in a gravitational field of a massive body and compared the solution to the solutions of Einstein Equation for a photon in the same gravitational field. A value of 1.875" was found as the total deflection angle (Izam & Jwanbot, 2013).

In the year 2013 Chifu, conducted a work on the gravitational field of conical mass distributions using the general theory of relativity. The gravitational metric tensor was constructed and applied to the motion of test particles and photons in this gravitational field. The expression for gravitational time dilation was found to have the same form as that in spherical, oblate spheroidal, and prolate spheroidal gravitational fields and hence confirms an earlier assertion that this gravitational phenomenon is invariant in form with various mass distributions (Chifu, 2013).

In his work, the gravitational field depends on radial distance, polar angle and azimutal angle, which were later, transform to the corresponding canonical coordinate system.

Lumbi *et al.*, 2014, conducted a research on Einstein's equations of motion for test particles exterior to spherical distributions of mass whose tensor field varies with time, radial distance and polar angle. In his article, the metric tensor exterior to hypothetical spherical distributions of mass whose tensor field varies with time, radial distance and polar angle was extended to derive equations of motion for test particles in the gravitational field. The time equation was used to derive the expression for the variation of the time on a clock moving in this gravitational field. For pure polar motion, test particles move with velocity that has an inverse dependence on the radial distance. The results show that the introduction of θin this field does not alter the inverse dependence of velocity on the radial distance (Lumbi *et al.*, 2014).

Lumbi et al., based my thesis on the recommendation giving in the above research work.

CHAPTERTHREE

METHDOLOGY

3.1 Construction of the Riemann-Christoffel tensor

3.1.1 Construction of metric tensors and affine connections

Schwartzchild's metric is the solution of Einstein's field equations exterior to a static homogeneous spherical body (Schwarzschild, 1916; Howusu, 2009; Chifu & Howusu, 2008; Lumbi *et al.*, 2014) given by

$$g_{00} = 1 - 2f(r) \quad (3.1)$$

$$g_{11} = -\left(1 - 2f(r)\right) \quad (3.2)$$

$$g_{22} = -r^2 \quad (3.3)$$

$$g_{33} = -r^2 \sin^2\theta \quad (3.4)$$

$$g_{\mu\nu} = 0 \text{, otherwise} \quad (3.5)$$

Where

c is the speed of light in vacuum

From the condition that this metric component should reduced to the field of a point mass located at the origin (Wikipedia, 2008; Lumbi *et al.*, 2014) and contains Newton's equation of motion in the gravitational field of the static homogenous spherical body, it follows that $f(r)$ is the Newtonian gravitational scalar potential in the exterior region of the body defined in this field as

$$f(r) = \frac{GM}{r} \quad r > R_0 \quad (3.6)$$

where

G is the universal gravitational constant,

M is the mass of the spherical body,

R_0 is the radius of the spherical body.

Let us consider an astrophysical mass distribution within spherical geometry in which the tensor field varies with time, radial distance and polar angle. The covariant metric tensors for this distribution of mass or pressure is given as (Howusu, 2007; Howusu, 2009; Lumbi *et al.,* 2014)

$$g_{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \qquad (3.7)$$

$$g_{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \qquad (3.8)$$

$$g_{22} = -r^2 \qquad (3.9)$$

$$g_{33} = -r^2 \sin^2\theta \qquad (3.10)$$

$$g_{\mu\nu} = 0 \text{, otherwise} \qquad (3.11)$$

where

$f(t,r,\theta)$ is an arbitrary function, determined by the mass or pressure and possess symmetries of the latter. In approximate gravitational field, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

To obtained the corresponding contravariant metric tensors for this gravitational field, we impose the Quotient Theorem (Dimitri & Larisa, 2008; Lumbi *et al.,* 2014) of the tensor analysis to obtain the components of the contravariant tensor as

$$g^{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \tag{3.12}$$

$$g^{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \tag{3.13}$$

$$g^{22} = -\frac{1}{r^2} \tag{3.14}$$

$$g^{33} = -\frac{1}{r^2 \sin^2\theta} \tag{3.15}$$

$$g^{\mu\nu} = 0, \text{ otherwise} \tag{3.16}$$

The coefficients of affine connections, defined by the metric tensors of space-time are determined (Weinberg, 1972; Arfken, 1985; Bergmann, 1987; Howusu, 2009; Lumbi *et al.*, 2014) using the tensor equation

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\xi}\left(g_{\alpha\xi,\beta} + g_{\beta\xi,\alpha} - g_{\alpha\beta,\xi}\right) \tag{3.17}$$

They are found to be given in terms of (ct, r, θ) as

$$\Gamma^{0}{}_{00} = \frac{1}{2} g^{00} g_{00,0} \tag{3.18}$$

$$\Gamma^{0}{}_{01} = \Gamma^{0}{}_{10} = \frac{1}{2} g^{00} g_{00,1} \tag{3.19}$$

$$\Gamma^{0}{}_{11} = -\frac{1}{2} g^{00} g_{11,0} \tag{3.20}$$

$$\Gamma^{0}{}_{02} = \Gamma^{0}{}_{20} = \frac{1}{2} g^{00} g_{00,2} \tag{3.21}$$

$$\Gamma^1{}_{00} = -\frac{1}{2} g^{11} g_{00,1} \tag{3.22}$$

$$\Gamma^1{}_{01} = \Gamma^1{}_{10} = \frac{1}{2} g^{11} g_{11,0} \tag{3.23}$$

$$\Gamma^1{}_{11} = \frac{1}{2} g^{11} g_{11,1} \tag{3.24}$$

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = \frac{1}{2} g^{11} g_{11,2} \tag{3.25}$$

$$\Gamma^1{}_{22} = -\frac{1}{2} g^{11} g_{22,1} \tag{3.26}$$

$$\Gamma^1{}_{33} = -\frac{1}{2} g^{11} g_{33,1} \tag{3.27}$$

$$\Gamma^2{}_{00} = -\frac{1}{2} g^{22} g_{00,2} \tag{3.28}$$

$$\Gamma^2{}_{11} = -\frac{1}{2} g^{22} g_{11,2} \tag{3.29}$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{2} g^{22} g_{22,1} \tag{3.30}$$

$$\Gamma^2{}_{33} = -\frac{1}{2} g^{22} g_{33,2} \tag{3.31}$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{2} g^{33} g_{33,1} \tag{3.32}$$

$$\Gamma^3{}_{23} = \Gamma^3{}_{32} = \frac{1}{2} g^{33} g_{33,2} \tag{3.33}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = 0 \text{; otherwise} \tag{3.34}$$

where the comma denotes partial differentiation w.r.t (0,1,2)=(ct, r, θ). Equation (3.18) to (3.34) can be written explicitly in terms of (ct, r, θ,) as

$$\Gamma^0{}_{00} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t} \tag{3.35}$$

$$\Gamma^0{}_{01} = \Gamma^0{}_{10} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r} \tag{3.36}$$

$$\Gamma^0{}_{11} = -\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\frac{\partial f(t,r,\theta)}{\partial t} \tag{3.37}$$

$$\Gamma^0{}_{02} = \Gamma^0{}_{20} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta} \tag{3.38}$$

$$\Gamma^1{}_{00} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial f(t,r,\theta)}{\partial r} \tag{3.39}$$

$$\Gamma^1{}_{01} = \Gamma^1{}_{10} = -\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t} \tag{3.40}$$

$$\Gamma^1{}_{11} = -\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r} \tag{3.41}$$

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = -\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta} \tag{3.42}$$

$$\Gamma^1{}_{22} = -r\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \tag{3.43}$$

$$\Gamma^1{}_{33} = -r\sin^2\theta\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \tag{3.44}$$

$$\Gamma^2{}_{00} = \frac{1}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial\theta} \tag{3.45}$$

$$\Gamma^2{}_{11} = \frac{1}{c^2 r^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial\theta} \tag{3.46}$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r} \tag{3.47}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \tag{3.48}$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r} \tag{3.49}$$

$$\Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta \tag{3.50}$$

$$\Gamma^\mu{}_{\alpha\beta} = 0;\ \text{otherwise} \tag{3.51}$$

3.1.2 Construction of the Riemann-Christofell Tensors

The well-known Riemann-Christofell tensor for a spherical distribution of mass is given as (Weinberg, 1972; Arfken, 1985;Bergmann, 1987; Howusu, 2009),

$$R^\lambda{}_{\alpha\beta\gamma} = \Gamma^\lambda{}_{\alpha\gamma,\beta} - \Gamma^\lambda{}_{\alpha\beta,\lambda} + \Gamma^\xi{}_{\alpha\gamma}\Gamma^\lambda{}_{\xi\beta} - \Gamma^\xi{}_{\alpha\beta}\Gamma^\lambda{}_{\xi\gamma} \tag{3.52}$$

where a comma denotes differentiation with respect to $\beta and \gamma$.

$R^{\lambda}_{\alpha\beta\gamma}$ is the Riemann curvature tensor or Riemann-Christoffel tensor

$\Gamma^{\lambda}_{\alpha\gamma,\beta}$ is the coefficient of affine connections for this field

This objective was achieved using the affine connections for this field, which was constructed by Lumbi et al, in 2014 (Lumbi *et al*, 2014). The Riemann-Christoffel tensors are found to be given in terms of $(0,1,2,3)$ as shown below:

$$R^{0}_{000} = \Gamma^{0}_{00,0} - \Gamma^{0}_{00,0} + \Gamma^{0}_{00}\Gamma^{0}_{00} - \Gamma^{0}_{00}\Gamma^{0}_{00} + \Gamma^{1}_{00}\Gamma^{0}_{10}$$

$$-\Gamma^{1}_{00}\Gamma^{0}_{10} - \Gamma^{2}_{00}\Gamma^{0}_{20} - \Gamma^{2}_{00}\Gamma^{0}_{20} + \Gamma^{3}_{00}\Gamma^{0}_{30} - \Gamma^{3}_{00}\Gamma^{0}_{30} \quad (3.53)$$

$$R^{1}{}_{001} = \Gamma^{1}{}_{01,0} - \Gamma^{1}{}_{00,1} + \Gamma^{0}{}_{01}\Gamma^{1}{}_{00} - \Gamma^{0}{}_{00}\Gamma^{1}{}_{01} + \Gamma^{1}{}_{01}\Gamma^{1}{}_{10} - \Gamma^{1}{}_{00}\Gamma^{1}{}_{11} - \Gamma^{2}{}_{00}\Gamma^{1}{}_{21} \quad (3.54)$$

$$R^{2}{}_{002} = -\Gamma^{2}{}_{00,2} + \Gamma^{0}{}_{02}\Gamma^{2}{}_{00} - \Gamma^{1}{}_{00}\Gamma^{2}{}_{12} \quad (3.55)$$

$$R^{3}{}_{003} = -\Gamma^{1}{}_{00}\Gamma^{3}{}_{13} - \Gamma^{2}{}_{00}\Gamma^{3}{}_{23} \quad (3.56)$$

$$R^{0}{}_{110} = \Gamma^{0}{}_{10,1} - \Gamma^{0}{}_{11,0} + \Gamma^{0}{}_{10}\Gamma^{0}{}_{01} - \Gamma^{0}{}_{11}\Gamma^{0}{}_{00} + \Gamma^{1}{}_{10}\Gamma^{0}{}_{11} - \Gamma^{1}{}_{11}\Gamma^{0}{}_{10} - \Gamma^{2}{}_{11}\Gamma^{0}{}_{20} \quad (3.57)$$

$$R^{1}_{111} = \Gamma^{1}_{11,1} - \Gamma^{1}_{11,1} + \Gamma^{0}_{11}\Gamma^{1}_{01} - \Gamma^{0}_{11}\Gamma^{1}_{01} + \Gamma^{1}_{11}\Gamma^{1}_{11}$$

$$-\Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{11}\Gamma^{1}_{21} - \Gamma^{2}_{11}\Gamma^{1}_{21} + \Gamma^{3}_{11}\Gamma^{1}_{31} - \Gamma^{3}_{11}\Gamma^{1}_{31} \quad (3.58)$$

$$R^{2}{}_{112} = \Gamma^{2}{}_{12,1} - \Gamma^{2}{}_{11,2} + \Gamma^{1}{}_{12}\Gamma^{2}{}_{11} - \Gamma^{1}{}_{11}\Gamma^{2}{}_{12} + \Gamma^{2}{}_{12}\Gamma^{2}{}_{21} \quad (3.59)$$

$$R^{3}{}_{113} = \Gamma^{3}{}_{13,1} - \Gamma^{1}{}_{11}\Gamma^{3}{}_{13} - \Gamma^{2}{}_{11}\Gamma^{3}{}_{23} + \Gamma^{3}{}_{13}\Gamma^{3}{}_{31} \quad (3.60)$$

$$R^{0}{}_{220} = \Gamma^{0}{}_{20,2} + \Gamma^{0}{}_{20}\Gamma^{0}{}_{02} - \Gamma^{1}{}_{22}\Gamma^{0}{}_{10} \quad (3.61)$$

$$R^{1}{}_{221} = \Gamma^{1}{}_{21,2} - \Gamma^{1}{}_{22,1} + \Gamma^{1}{}_{21}\Gamma^{1}{}_{12} - \Gamma^{1}{}_{22}\Gamma^{1}{}_{11} - \Gamma^{2}{}_{12}\Gamma^{1}{}_{22} \quad (3.62)$$

$$R^{2}_{222} = \Gamma^{2}_{22,2} - \Gamma^{2}_{22,2} + \Gamma^{0}_{22}\Gamma^{2}_{02} - \Gamma^{0}_{22}\Gamma^{2}_{02} + \Gamma^{1}_{22}\Gamma^{2}_{12}$$

$$-\Gamma^{1}_{22}\Gamma^{2}_{12} + \Gamma^{2}_{22}\Gamma^{2}_{22} - \Gamma^{2}_{22}\Gamma^{2}_{22} + \Gamma^{3}_{22}\Gamma^{2}_{32} - \Gamma^{3}_{22}\Gamma^{2}_{32} \quad (3.63)$$

$$R^{3}{}_{223} = \Gamma^{3}{}_{23,2} - \Gamma^{1}{}_{22}\Gamma^{3}{}_{13} + \Gamma^{3}{}_{23}\Gamma^{3}{}_{32} \quad (3.64)$$

$$R^{0}{}_{330} = -\Gamma^{1}{}_{00}\Gamma^{3}{}_{13} - \Gamma^{2}{}_{00}\Gamma^{3}{}_{23} \quad (3.65)$$

$$R^{1}{}_{331} = -\Gamma^{1}{}_{33,1} - \Gamma^{1}{}_{33}\Gamma^{1}{}_{11} - \Gamma^{2}{}_{33}\Gamma^{1}{}_{21} + \Gamma^{3}{}_{31}\Gamma^{1}{}_{33} \quad (3.66)$$

$$R^{2}{}_{332} = -\Gamma^{2}{}_{33,2} - \Gamma^{1}{}_{33}\Gamma^{2}{}_{12} + \Gamma^{3}{}_{32}\Gamma^{2}{}_{33} \quad (3.67)$$

$$R^{3}_{333} = \Gamma^{3}_{33,3} - \Gamma^{3}_{33,3} + \Gamma^{0}_{33}\Gamma^{3}_{03} - \Gamma^{0}_{33}\Gamma^{3}_{03} + \Gamma^{1}_{33}\Gamma^{3}_{13}$$

$$-\Gamma^{1}_{33}\Gamma^{3}_{13} + \Gamma^{2}_{33}\Gamma^{3}_{23} - \Gamma^{2}_{33}\Gamma^{3}_{23} + \Gamma^{3}_{33}\Gamma^{3}_{33} - \Gamma^{3}_{33}\Gamma^{3}_{33} \quad (3.68)$$

$$R^{\lambda}{}_{\alpha\beta\gamma} = 0\text{; otherwise} \quad (3.69)$$

3.2 Construction of the Ricci Tensors for $f(t, r, \theta)$ Field

The well-known Ricci tensors for spherical massive bodies is giving as (Weinberg, 1972; Arfken, 1985; Bergmann, 1987; Howusu, 2009),

$$R_{\mu\nu} = R^{0}{}_{\mu\nu 0} + R^{1}{}_{\mu\nu 1} + R^{2}{}_{\mu\nu 2} + R^{3}{}_{\mu\nu 3} \quad (3.70)$$

where,

$R_{\mu\nu}$ is the Ricci curvature tensor

$R^0_{\mu\nu 0}, R^1_{\mu\nu 1}, R^2_{\mu\nu 2}$ *and* $R^3_{\mu\nu 3}$ are the Riemann-Christoffel tensors.

This objective, that is, the Ricci tensors for spherical massive bodies whose tensor field varies with time, radial distance and polar angle, was achieved using the Riemann-Christoffel tensors (3.53)-(3.3.68). The Ricci tensors are found to be given in terms of$(0,1,2) = (ct, r, \theta)$ as

$$R_{00} = R^{0}_{000} + R^{1}_{001} + R^{2}_{002} + R^{3}_{003} \quad (3.71)$$

$$R_{00} = \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^0_{00}\Gamma^0_{00} - \Gamma^0_{00}\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{10}\Gamma^1{}_{01,0} - \Gamma^1_{00}\Gamma^0_{10} - \Gamma^2_{00}\Gamma^0_{20} - \Gamma^2_{00}\Gamma^0_{20}$$

$$+\Gamma^3_{00}\Gamma^0_{30} - \Gamma^3_{00}\Gamma^0_{30} - \Gamma^1{}_{00,1} + \Gamma^0{}_{01}\Gamma^1{}_{00} - \Gamma^0{}_{00}\Gamma^1{}_{01} + \Gamma^1{}_{01}\Gamma^1{}_{10} - \Gamma^1{}_{00}\Gamma^1{}_{11} - \Gamma^2{}_{00}\Gamma^1{}_{21}$$

$$-\Gamma^2{}_{00,2} + \Gamma^0{}_{02}\Gamma^2{}_{00} - \Gamma^1{}_{00}\Gamma^2{}_{12} - \Gamma^1{}_{00}\Gamma^3{}_{13} - \Gamma^2{}_{00}\Gamma^3{}_{23} \tag{3.72}$$

$$R_{11} = R^0_{110} + R^1_{111} + R^2_{112} + R^3_{113} \tag{3.73}$$

$$R_{11} = \Gamma^0{}_{10,1} - \Gamma^0{}_{11,0} + \Gamma^0{}_{10}\Gamma^0{}_{01} - \Gamma^0{}_{11}\Gamma^0{}_{00} + \Gamma^1{}_{10}\Gamma^0{}_{11} - \Gamma^1{}_{11}\Gamma^0{}_{10} - \Gamma^2{}_{11}\Gamma^0{}_{20}$$

$$\Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^0_{11}\Gamma^1_{01} - \Gamma^0_{11}\Gamma^1_{01} + \Gamma^1_{11}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{21} + \Gamma^3_{11}\Gamma^1_{31} - \Gamma^3_{11}\Gamma^1_{31}$$

$$+\Gamma^2{}_{12,1} - \Gamma^2{}_{11,2} + \Gamma^1{}_{12}\Gamma^2{}_{11} - \Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^2{}_{12}\Gamma^2{}_{21} + \Gamma^3{}_{13,1} - \Gamma^1{}_{11}\Gamma^3{}_{13} - \Gamma^2{}_{11}\Gamma^3{}_{23} + \Gamma^3{}_{13}\Gamma^3{}_{31}$$

(3.74)

$$R_{22} = R^0_{220} + R^1_{221} + R^2_{222} + R^3_{223} \tag{3.75}$$

$$R_{22} = \Gamma^0{}_{20,2} + \Gamma^0{}_{20}\Gamma^0{}_{02} - \Gamma^1{}_{22}\Gamma^0{}_{10} + \Gamma^1{}_{21,2} - \Gamma^1{}_{22,1} + \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^1{}_{22}\Gamma^1{}_{11} - \Gamma^2{}_{12}\Gamma^1{}_{22}$$

$$+\Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^0_{22}\Gamma^2_{02} - \Gamma^0_{22}\Gamma^2_{02} + \Gamma^1_{22}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{22} + \Gamma^3_{22}\Gamma^2_{32} - \Gamma^3_{22}\Gamma^2_{32}$$

$$+\Gamma^3{}_{23,2} - \Gamma^1{}_{22}\Gamma^3{}_{13} + \Gamma^3{}_{23}\Gamma^3{}_{32} \tag{3.76}$$

$$R_{33} = R^0_{330} + R^1_{331} + R^2_{332} + R^3_{333} \tag{3.77}$$

$$R_{33} = -\Gamma^1{}_{00}\Gamma^3{}_{13} - \Gamma^2{}_{00}\Gamma^3{}_{23} - \Gamma^1{}_{33,1} - \Gamma^1{}_{33}\Gamma^1{}_{11} - \Gamma^2{}_{33}\Gamma^1{}_{21} + \Gamma^3{}_{31}\Gamma^1{}_{33} - \Gamma^2{}_{33,2} - \Gamma^1{}_{33}\Gamma^2{}_{12} + \Gamma^3{}_{32}\Gamma^2{}_{33}$$

$$+\Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^0_{33}\Gamma^3_{03} - \Gamma^0_{33}\Gamma^3_{03} + \Gamma^1_{33}\Gamma^3_{13} - \Gamma^1_{33}\Gamma^3_{13} + \Gamma^2_{33}\Gamma^3_{23} - \Gamma^2_{33}\Gamma^3_{23} + \Gamma^3_{33}\Gamma^3_{33} - \Gamma^3_{33}\Gamma^3_{33} \tag{3.78}$$

3.3 Construction of the Einstein's Field Equation Exterior to $\boldsymbol{f(t,r,\theta)}$ Field

The well-known Einstein's field equation exterior to spherical massive bodies is given by (Weinberg, 1972; Arfken, 1985;Bergmann, 1987; Howusu, 2009),

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \qquad (3.79)$$

where

$G_{\mu\nu}$ is the Einstein's tensors,

$R_{\mu\nu}$ is the Ricci tensors,

R is the Riemann scalar,

$g_{\mu\nu}$ is the covariant metric tensor

The Einstein's field equation exterior to spherical massive bodies whose tensor field varies with time, radial distance and polar angle which is the third objective of this research work was constructed using Riemann scalar and the covariant metric tensors for this field.

The Riemann scalar is given as (Weinberg, 1972; Arfken, 1985; Bergmann, 1987; Howusu, 2009)

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \qquad (3.80)$$

Using equations (3.12)-(3.15) and (3.72), (3.74), (3.76) and (3.78), the Riemann scalar for this field is given as:

$$R = g^{00}\{\Gamma^{0}_{00,0} - \Gamma^{0}_{00,0} + \Gamma^{0}_{00}\Gamma^{0}_{00} - \Gamma^{0}_{00}\Gamma^{0}_{00} + \Gamma^{1}_{00}\Gamma^{0}_{10}\Gamma^{1}_{01,0} - \Gamma^{1}_{00}\Gamma^{0}_{10}$$

$$-\Gamma^{2}_{00}\Gamma^{0}_{20} - \Gamma^{2}_{00}\Gamma^{0}_{20} + \Gamma^{3}_{00}\Gamma^{0}_{30} - \Gamma^{3}_{00}\Gamma^{0}_{30} - \Gamma^{1}_{00,1} + \Gamma^{0}_{01}\Gamma^{1}_{00} - \Gamma^{0}_{00}\Gamma^{1}_{01} + \Gamma^{1}_{01}\Gamma^{1}_{10}$$

$$-\Gamma^{1}_{00}\Gamma^{1}_{11} - \Gamma^{2}_{00}\Gamma^{1}_{21} - \Gamma^{2}_{00,2} + \Gamma^{0}_{02}\Gamma^{2}_{00} - \Gamma^{1}_{00}\Gamma^{2}_{12} - \Gamma^{1}_{00}\Gamma^{3}_{13} - \Gamma^{2}_{00}\Gamma^{3}_{23}\}$$

$$+g^{11}\{\Gamma^{0}_{10,1} - \Gamma^{0}_{11,0} + \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{0}_{11}\Gamma^{0}_{00} + \Gamma^{1}_{10}\Gamma^{0}_{11} - \Gamma^{1}_{11}\Gamma^{0}_{10}$$

$$-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^0_{11}\Gamma^1_{01}-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}$$

$$+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}\}$$

$$+g^{22}\{\Gamma^0{}_{20,2}+\Gamma^0{}_{20}\Gamma^0{}_{02}-\Gamma^1{}_{22}\Gamma^0{}_{10}+\Gamma^1{}_{21,2}-\Gamma^1{}_{22,1}+\Gamma^1{}_{21}\Gamma^1{}_{12}-\Gamma^1{}_{22}\Gamma^1{}_{11}-\Gamma^2{}_{12}\Gamma^1{}_{22}$$

$$+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}$$

$$+\Gamma^3{}_{23,2}-\Gamma^1{}_{22}\Gamma^3{}_{13}+\Gamma^3{}_{23}\Gamma^3{}_{32}\}+g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}-\Gamma^1{}_{33,1}-\Gamma^1{}_{33}\Gamma^1{}_{11}-\Gamma^2{}_{33}\Gamma^1{}_{21}$$

$$+\Gamma^3{}_{31}\Gamma^1{}_{33}-\Gamma^2{}_{33,2}-\Gamma^1{}_{33}\Gamma^2{}_{12}+\Gamma^3{}_{32}\Gamma^2{}_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}$$

$$+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\} \qquad (3.81)$$

The Einstein's exterior field equations for this field are given using the Ricci tensors, Riemann scalar and the covariant metric tensor as:

$$G_{00}=\Gamma^0_{00,0}-\Gamma^0_{00,0}+\Gamma^0_{00}\Gamma^0_{00}-\Gamma^0_{00}\Gamma^0_{00}+\Gamma^1_{00}\Gamma^0_{10}\Gamma^1_{01,0}-\Gamma^1_{00}\Gamma^0_{10}-\Gamma^2_{00}\Gamma^0_{20}$$

$$-\Gamma^2_{00}\Gamma^0_{20}+\Gamma^3_{00}\Gamma^0_{30}-\Gamma^3_{00}\Gamma^0_{30}-\Gamma^1{}_{00,1}+\Gamma^0{}_{01}\Gamma^1{}_{00}-\Gamma^0{}_{00}\Gamma^1{}_{01}+\Gamma^1{}_{01}\Gamma^1{}_{10}-\Gamma^1{}_{00}\Gamma^1{}_{11}$$

$$-\Gamma^2{}_{00}\Gamma^1{}_{21}-\Gamma^2{}_{00,2}+\Gamma^0{}_{02}\Gamma^2{}_{00}-\Gamma^1{}_{00}\Gamma^2{}_{12}-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}$$

$$-\frac{1}{2}[g^{00}\{\Gamma^0_{00,0}-\Gamma^0_{00,0}+\Gamma^0_{00}\Gamma^0_{00}-\Gamma^0_{00}\Gamma^0_{00}+\Gamma^1_{00}\Gamma^0_{10}\Gamma^1_{01,0}-\Gamma^1_{00}\Gamma^0_{10}-\Gamma^2_{00}\Gamma^0_{20}-\Gamma^2_{00}\Gamma^0_{20}$$

$$+\Gamma^3_{00}\Gamma^0_{30}-\Gamma^3_{00}\Gamma^0_{30}-\Gamma^1{}_{00,1}+\Gamma^0{}_{01}\Gamma^1{}_{00}-\Gamma^0{}_{00}\Gamma^1{}_{01}+\Gamma^1{}_{01}\Gamma^1{}_{10}-\Gamma^1{}_{00}\Gamma^1{}_{11}-\Gamma^2{}_{00}\Gamma^1{}_{21}$$

$$-\Gamma^2{}_{00,2}+\Gamma^0{}_{02}\Gamma^2{}_{00}-\Gamma^1{}_{00}\Gamma^2{}_{12}-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}\}+g^{11}\{\Gamma^0{}_{10,1}-\Gamma^0{}_{11,0}+\Gamma^0{}_{10}\Gamma^0{}_{01}$$

$$-\Gamma^0{}_{11}\Gamma^0{}_{00}+\Gamma^1{}_{10}\Gamma^0{}_{11}-\Gamma^1{}_{11}\Gamma^0{}_{10}-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^0_{11}\Gamma^1_{01}-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}$$

$$-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}$$

$$+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}\}+g^{22}\{\Gamma^0{}_{20,2}+\Gamma^0{}_{20}\Gamma^0{}_{02}-\Gamma^1{}_{22}\Gamma^0{}_{10}$$

$$+\Gamma^1{}_{21,2}-\Gamma^1{}_{22,1}+\Gamma^1{}_{21}\Gamma^1{}_{12}-\Gamma^1{}_{22}\Gamma^1{}_{11}-\Gamma^2{}_{12}\Gamma^1{}_{22}+\Gamma^2{}_{22,2}-\Gamma^2{}_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}$$

$$-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}+\Gamma^3{}_{23,2}-\Gamma^1{}_{22}\Gamma^3{}_{13}+\Gamma^3{}_{23}\Gamma^3{}_{32}\}+g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13}$$

$$-\Gamma^2{}_{00}\Gamma^3{}_{23}-\Gamma^1{}_{33,1}-\Gamma^1{}_{33}\Gamma^1{}_{11}-\Gamma^2{}_{33}\Gamma^1{}_{21}+\Gamma^3{}_{31}\Gamma^1{}_{33}-\Gamma^2{}_{33,2}-\Gamma^1{}_{33}\Gamma^2{}_{12}+\Gamma^3{}_{32}\Gamma^2{}_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}$$

$$+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\}]g_{00}=0 \qquad (3.82)$$

$$G_{11}=\Gamma^0{}_{10,1}-\Gamma^0{}_{11,0}+\Gamma^0{}_{10}\Gamma^0{}_{01}-\Gamma^0{}_{11}\Gamma^0{}_{00}+\Gamma^1{}_{10}\Gamma^0{}_{11}-\Gamma^1{}_{11}\Gamma^0{}_{10}-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1{}_{11,1}-\Gamma^1{}_{11,1}$$

$$+\Gamma^0_{11}\Gamma^1_{01}-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}$$

$$+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}-\frac{1}{2}[g^{00}\{\Gamma^0_{00,0}-\Gamma^0_{00,0}$$

$$+\Gamma^0_{00}\Gamma^0_{00}-\Gamma^0_{00}\Gamma^0_{00}+\Gamma^1_{00}\Gamma^0_{10}\Gamma^1{}_{01,0}-\Gamma^1_{00}\Gamma^0_{10}-\Gamma^2_{00}\Gamma^0_{20}-\Gamma^2_{00}\Gamma^0_{20}+\Gamma^3_{00}\Gamma^0_{30}-\Gamma^3_{00}\Gamma^0_{30}-\Gamma^1{}_{00,1}+\Gamma^0{}_{01}\Gamma^1{}_{00}$$

$$-\Gamma^0{}_{00}\Gamma^1{}_{01}+\Gamma^1{}_{01}\Gamma^1{}_{10}-\Gamma^1{}_{00}\Gamma^1{}_{11}-\Gamma^2{}_{00}\Gamma^1{}_{21}-\Gamma^2{}_{00,2}+\Gamma^0{}_{02}\Gamma^2{}_{00}-\Gamma^1{}_{00}\Gamma^2{}_{12}-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}\}$$

$$+g^{11}\{\Gamma^0{}_{10,1}-\Gamma^0{}_{11,0}+\Gamma^0{}_{10}\Gamma^0{}_{01}-\Gamma^0{}_{11}\Gamma^0{}_{00}+\Gamma^1{}_{10}\Gamma^0{}_{11}-\Gamma^1{}_{11}\Gamma^0{}_{10}-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1{}_{11,1}-\Gamma^1{}_{11,1}+\Gamma^0_{11}\Gamma^1_{01}$$

$$-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}$$

$$+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}\}+g^{22}\{\Gamma^0{}_{20,2}+\Gamma^0{}_{20}\Gamma^0{}_{02}-\Gamma^1{}_{22}\Gamma^0{}_{10}+\Gamma^1{}_{21,2}-\Gamma^1{}_{22,1}$$

$$+\Gamma^1{}_{21}\Gamma^1{}_{12}-\Gamma^1{}_{22}\Gamma^1{}_{11}-\Gamma^2{}_{12}\Gamma^1{}_{22}+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}$$

$$+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}+\Gamma^3{}_{23,2}-\Gamma^1{}_{22}\Gamma^3{}_{13}+\Gamma^3{}_{23}\Gamma^3{}_{32}\}+g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}-\Gamma^1{}_{33,1}-\Gamma^1{}_{33}\Gamma^1{}_{11}$$

$$-\Gamma^2{}_{33}\Gamma^1{}_{21}+\Gamma^3{}_{31}\Gamma^1{}_{33}-\Gamma^2{}_{33,2}-\Gamma^1{}_{33}\Gamma^2{}_{12}+\Gamma^3{}_{32}\Gamma^2{}_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}$$

$$+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\}]g_{11}=0 \qquad (3.83)$$

$$G_{22}=\Gamma^0_{20,2}+\Gamma^0_{20}\Gamma^0_{02}-\Gamma^1_{22}\Gamma^0_{10}+\Gamma^1_{21,2}-\Gamma^1_{22,1}+\Gamma^1_{21}\Gamma^1_{12}-\Gamma^1_{22}\Gamma^1_{11}-\Gamma^2_{12}\Gamma^1_{22}$$

$$+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}$$

$$+\Gamma^3_{23,2}-\Gamma^1_{22}\Gamma^3_{13}+\Gamma^3_{23}\Gamma^3_{32}-\frac{1}{2}[g^{00}\{\Gamma^0_{00,0}-\Gamma^0_{00,0}+\Gamma^0_{00}\Gamma^0_{00}-\Gamma^0_{00}\Gamma^0_{00}+\Gamma^1_{00}\Gamma^0_{10}\Gamma^1_{01,0}-\Gamma^1_{00}\Gamma^0_{10}$$

$$-\Gamma^2_{00}\Gamma^0_{20}-\Gamma^2_{00}\Gamma^0_{20}+\Gamma^3_{00}\Gamma^0_{30}-\Gamma^3_{00}\Gamma^0_{30}-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^0_{00}\Gamma^1_{01}+\Gamma^1_{01}\Gamma^1_{10}-\Gamma^1_{00}\Gamma^1_{11}-\Gamma^2_{00}\Gamma^1_{21}$$

$$-\Gamma^2_{00,2}+\Gamma^0_{02}\Gamma^2_{00}-\Gamma^1_{00}\Gamma^2_{12}-\Gamma^1_{00}\Gamma^3_{13}-\Gamma^2_{00}\Gamma^3_{23}\}+g^{11}\{\Gamma^0_{10,1}-\Gamma^0_{11,0}+\Gamma^0_{10}\Gamma^0_{01}-\Gamma^0_{11}\Gamma^0_{00}$$

$$+\Gamma^1_{10}\Gamma^0_{11}-\Gamma^1_{11}\Gamma^0_{10}-\Gamma^2_{11}\Gamma^0_{20}+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^0_{11}\Gamma^1_{01}-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}$$

$$+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2_{12,1}-\Gamma^2_{11,2}+\Gamma^1_{12}\Gamma^2_{11}-\Gamma^1_{11}\Gamma^2_{12}+\Gamma^2_{12}\Gamma^2_{21}+\Gamma^3_{13,1}-\Gamma^1_{11}\Gamma^3_{13}-\Gamma^2_{11}\Gamma^3_{23}+\Gamma^3_{13}\Gamma^3_{31}\}$$

$$+g^{22}\{\Gamma^0_{20,2}+\Gamma^0_{20}\Gamma^0_{02}-\Gamma^1_{22}\Gamma^0_{10}+\Gamma^1_{21,2}-\Gamma^1_{22,1}+\Gamma^1_{21}\Gamma^1_{12}-\Gamma^1_{22}\Gamma^1_{11}-\Gamma^2_{12}\Gamma^1_{22}+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}$$

$$-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}+\Gamma^3_{23,2}-\Gamma^1_{22}\Gamma^3_{13}+\Gamma^3_{23}\Gamma^3_{32}\}+g^{33}\{-\Gamma^1_{00}\Gamma^3_{13}$$

$$-\Gamma^2_{00}\Gamma^3_{23}-\Gamma^1_{33,1}-\Gamma^1_{33}\Gamma^1_{11}-\Gamma^2_{33}\Gamma^1_{21}+\Gamma^3_{31}\Gamma^1_{33}-\Gamma^2_{33,2}-\Gamma^1_{33}\Gamma^2_{12}+\Gamma^3_{32}\Gamma^2_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}$$

$$-\Gamma^0_{33}\Gamma^3_{03}+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\}]g_{22}=0 \qquad (3.84)$$

$$G_{33}=-\Gamma^1_{00}\Gamma^3_{13}-\Gamma^2_{00}\Gamma^3_{23}-\Gamma^1_{33,1}-\Gamma^1_{33}\Gamma^1_{11}-\Gamma^2_{33}\Gamma^1_{21}+\Gamma^3_{31}\Gamma^1_{33}-\Gamma^2_{33,2}-\Gamma^1_{33}\Gamma^2_{12}+\Gamma^3_{32}\Gamma^2_{33}$$

$$+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}-\frac{1}{2}[g^{00}\{\Gamma^0_{00,0}-\Gamma^0_{00,0}$$

$$+\Gamma^0_{00}\Gamma^0_{00}-\Gamma^0_{00}\Gamma^0_{00}+\Gamma^1_{00}\Gamma^0_{10}\Gamma^1{}_{01,0}-\Gamma^1_{00}\Gamma^0_{10}-\Gamma^2_{00}\Gamma^0_{20}-\Gamma^2_{00}\Gamma^0_{20}+\Gamma^3_{00}\Gamma^0_{30}-\Gamma^3_{00}\Gamma^0_{30}-\Gamma^1{}_{00,1}+\Gamma^0{}_{01}\Gamma^1{}_{00}$$

$$-\Gamma^0{}_{00}\Gamma^1{}_{01}+\Gamma^1{}_{01}\Gamma^1{}_{10}-\Gamma^1{}_{00}\Gamma^1{}_{11}-\Gamma^2{}_{00}\Gamma^1{}_{21}-\Gamma^2{}_{00,2}+\Gamma^0{}_{02}\Gamma^2{}_{00}-\Gamma^1{}_{00}\Gamma^2{}_{12}-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}\}$$

$$+g^{11}\{\Gamma^0{}_{10,1}-\Gamma^0{}_{11,0}+\Gamma^0{}_{10}\Gamma^0{}_{01}-\Gamma^0{}_{11}\Gamma^0{}_{00}+\Gamma^1{}_{10}\Gamma^0{}_{11}-\Gamma^1{}_{11}\Gamma^0{}_{10}-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^0_{11}\Gamma^1_{01}$$

$$-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}$$

$$+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}\}+g^{22}\{\Gamma^0{}_{20,2}+\Gamma^0{}_{20}\Gamma^0{}_{02}-\Gamma^1{}_{22}\Gamma^0{}_{10}+\Gamma^1{}_{21,2}-\Gamma^1{}_{22,1}$$

$$+\Gamma^1{}_{21}\Gamma^1{}_{12}-\Gamma^1{}_{22}\Gamma^1{}_{11}-\Gamma^2{}_{12}\Gamma^1{}_{22}+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}$$

$$+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}+\Gamma^3{}_{23,2}-\Gamma^1{}_{22}\Gamma^3{}_{13}+\Gamma^3{}_{23}\Gamma^3{}_{32}\}+g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}-\Gamma^1{}_{33,1}-\Gamma^1{}_{33}\Gamma^1{}_{11}-\Gamma^2{}_{33}\Gamma^1{}_{21}$$

$$+\Gamma^3{}_{31}\Gamma^1{}_{33}-\Gamma^2{}_{33,2}-\Gamma^1{}_{33}\Gamma^2{}_{12}+\Gamma^3{}_{32}\Gamma^2{}_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}$$

$$-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\}]g_{33}=0 \tag{3.85}$$

$$G_{\mu\nu}=0\text{ ,otherwise} \tag{3.86}$$

3.3.1 Solution of the Einstein's Field Equation

In order to solve the Einstein's field equations (3.82)-(3.85), we assumed a solution in the form of

$$f(t,r,\theta)=\sum_{n=0}^{\infty}R_n(r)\exp n\left(t-\frac{r\theta}{c}\right) \tag{3.87}$$

3.4 Motion of Particles and Photons in the Equatorial Plane of $f(t, r, \theta)$ Field

The motion of particles and photons in the equatorial plane was studied using the explicit result of (3.87) and Lagrangian which is given in the space time exterior to any astrophysical body (Peter, 2000; Chifu, 2010; Chifu *et al.*, 2011) as

$$L = \frac{1}{c}\left(-g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}\right)^{\frac{1}{2}} \tag{3.88}$$

Thus in this field, the Lagrangian, that is equation (3.50) becomes

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^{2} - g_{11}\left(\frac{dr}{d\tau}\right)^{2} - g_{22}\left(\frac{d\theta}{d\tau}\right)^{2} - g_{33}\left(\frac{d\phi}{d\tau}\right)^{2}\right)^{\frac{1}{2}} \tag{3.89}$$

CHAPTERFOUR

RESULT

4.1 Construction of the Riemann-Christofel Tensor for $f(t,r,\theta)$ Field

The Riemann-Christoffel tensors for this field are given explicitly by equation (4.1)-(4.16) after a series of calculations as shown by appendix A1-A31

$$R^{0}{}_{000} = 0 \tag{4.1}$$

$$R^{1}{}_{001} = \frac{4}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 - \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}$$

$$-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial^2 f(t,r,\theta)}{\partial r^2} + \frac{1}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 \tag{4.2}$$

$$R^{2}{}_{002} = -\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} + \frac{1}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$-\frac{1}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial f(t,r,\theta)}{\partial r} \tag{4.3}$$

$$R^{3}{}_{003} = -\frac{1}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial f(t,r,\theta)}{\partial r} - \frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta} \tag{4.4}$$

$$R^{0}{}_{110} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial r^2} - \frac{4}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2$$

$$+\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\frac{\partial^2 f(t,r,\theta)}{\partial t^2} - \frac{1}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 \tag{4.5}$$

$$R^{1}{}_{111} = 0 \tag{4.6}$$

$$R^2{}_{112} = \frac{3}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{1}{c^2 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}\right)$$

$$+\frac{1}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right) \qquad (4.7)$$

$$R^3{}_{113} = \frac{1}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right) - \frac{\cot\theta}{c^2 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta}$$

(4.8)

$$R^0{}_{220} = -\frac{1}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}\right)$$

$$+\frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} \qquad (4.9)$$

$$R^1{}_{221} = \frac{3}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}\right)$$

$$+\frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} \qquad (4.10)$$

$$R^2{}_{222} = 0 \qquad (4.11)$$

$$R^3{}_{223} = \frac{2f(t,r,\theta)}{c^2} \qquad (4.12)$$

$$R^0{}_{330} = \frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{\sin\theta\cos\theta}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta} \qquad (4.13)$$

$$R^1{}_{331} = \frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{\sin\theta\cos\theta}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta} \qquad (4.14)$$

$$R^{2}{}_{332} = \frac{2f(t,r,\theta)}{c^2} \tag{4.15}$$

$$R^{3}{}_{333} = 0 \tag{4.16}$$

4.2 Construction of the Ricci Tensors for $f(t, r, \theta)$ Field

After series of mathematical steps as shown in appendix B1-B12, the Ricci tensors are given explicitly by equation (4.17)-(4.10) as

$$R_{00} = \frac{4}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 - \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}$$

$$-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial^2 f(t,r,\theta)}{\partial r^2} + \frac{2}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$-\frac{2}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\left(\frac{\partial f(t,r,\theta)}{\partial r}\right) - \frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} - \frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta} \tag{4.17}$$

$$R_{11} = \frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial r^2} - \frac{4}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2$$

$$+\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\frac{\partial^2 f(t,r,\theta)}{\partial t^2} + \frac{2}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$-\frac{1}{c^2 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} + \frac{2}{c^2 r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)$$

$$-\frac{\cot\theta}{c^2 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta} \tag{4.18}$$

$$R_{22} = \frac{2}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2f(t,r,\theta)}{c^2} \quad (4.19)$$

$$R_{33} = \frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2\sin^2\theta f(t,r,\theta)}{c^2} \quad (4.10)$$

4.3 Construction of the Einstein's Field Equation Exterior to $f(t,r,\theta)$ Field

The Riemann's scalar R is given explicitly after a series mathematical steps as shown in appendix B13-B15 as

$$R = \frac{8}{c^4}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{2}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial t^2} - \frac{2}{c^2}\frac{\partial^2 f(t,r,\theta)}{\partial r^2}$$

$$-\frac{2}{c^4 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2 - \frac{4}{c^2 r^2} f(t,r,\theta) \quad (4.11)$$

After series of mathematical steps as shown in appendix C1-C12, the Einstein's field equations are given explicitly as

$$G_{00} = \frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r} + \frac{3}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$

$$-\frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta} + \frac{2f(t,r,\theta)}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right) = 0 \quad (4.12)$$

$$G_{11} = \frac{1}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} - \frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)$$

$$-\frac{2f(t,r,\theta)}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1} = 0 \tag{4.13}$$

$$G_{22} = -\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) = 0 \tag{4.14}$$

$$G_{33} = -\frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{4r^2\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{r^2\sin^2\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2\sin^2\theta}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$-\frac{\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 = 0 \tag{4.15}$$

4.3.1 Solution of the Einstein's Field Equations

The sixteen (16) Einstein's exterior geometrical field equations reduce to four equations given by (4.12)-(4.15). This is because of the symmetric nature of the field equations.

Equations (4.14) reduces in the order of c^0 to equation (4.16) as shown in appendix D.10

$$\nabla^2 f(t,r,\theta) + \frac{\partial^2 f(t,r,\theta)}{\partial t^2} = 0 \tag{4.16}$$

It may be noted that in the order of c^{-2} the wave equation (4.14) in the limit of weak fields reduces to

$$\nabla^2 f(t,r,\theta) - \frac{1}{r^2c^2}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2 - \frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{4}{c^2} f(t,r,\theta)\left[\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right] + \frac{\partial^2 f(t,r,\theta)}{\partial t^2} = 0 \tag{4.17}$$

As shown in appendix D.6 equation (4.17) can also be written as

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2} + \frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$

$$-\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 + \frac{\partial^2 f(t,r,\theta)}{\partial t^2} = 0 \tag{4.18}$$

Let us now seek a solution of equation (4.18) in the form

$$f(t,r,\theta) = \sum_{n=0}^{\infty} R_n(r)\exp n\left(t - \frac{r\theta}{c}\right) \tag{4.19}$$

where $R_n(r)$ functions of r only. By obtaining the first and second derivatives partially of equation (4.19) for $f(t,r,\theta)$; it can be shown trivially that the separate terms of the expanded equation can be shown in the equations below:

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2} = R_0^{11}(r) + \left(R_1^{11}(r) - \frac{2\theta}{c} R_1^1(r) - \frac{\theta^2}{c^2} R_1(r) \right) \exp\left(t - \frac{r\theta}{c} \right)$$

$$+ \left(R_2^{11}(r) - \frac{2.2\theta}{c} R_2^1(r) - \frac{2^2\theta^2}{c^2} R_2(r) \right) \exp 2\left(t - \frac{r\theta}{c} \right) + \left(R_3^{11}(r) - \frac{2.3\theta}{c} R_3^1(r) - \frac{3^2\theta^2}{c^2} R_3(r) \right) \tag{4.20}$$

$$\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r} = \frac{2}{r} R_0^1(r) + \frac{2}{r} R_1^1(r) \exp\left(t - \frac{r\theta}{c} \right) + \frac{2}{r} R_2^1(r) \exp 2\left(t - \frac{r\theta}{c} \right) + \frac{2}{r} R_3^1(r) \exp 3\left(t - \frac{r\theta}{c} \right)$$

$$- \frac{2\theta}{cr} R_1(r) \exp\left(t - \frac{r\theta}{c} \right) - \frac{2.2\theta}{cr} R_2(r) \exp 2\left(t - \frac{r\theta}{c} \right) - \frac{2.3\theta}{cr} R_3(r) \exp 3\left(t - \frac{r\theta}{c} \right) + \ldots \tag{4.21}$$

$$\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta} \right)^2 = \frac{1}{c^4} R_1^2(r) \exp 2\left(t - \frac{r\theta}{c} \right) + \frac{2^2}{c^4} R_2^2(r) \exp 4\left(t - \frac{r\theta}{c} \right) + \frac{3^2}{c^4} R_3^2(r) \exp 6\left(t - \frac{r\theta}{c} \right) + \ldots \tag{4.22}$$

$$\frac{4 f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2} \right) = \frac{4}{c^2} R_0(r)R_1(r) \exp\left(t - \frac{r\theta}{c} \right) + \frac{4.2^2}{c^2} R_0(r)R_2(r) \exp 2\left(t - \frac{r\theta}{c} \right)$$

$$+ \frac{4.3^2}{c^2} R_0(r)R_3(r) \exp 3\left(t - \frac{r\theta}{c} \right) + \frac{4}{c^2} R_1^2(r) \exp 2\left(t - \frac{r\theta}{c} \right) + \frac{20}{c^2} R_1(r)R_2(r) \exp 3\left(t - \frac{r\theta}{c} \right)$$

$$+ \frac{40}{c^2} R_1(r)R_3(r) \exp 4\left(t - \frac{r\theta}{c} \right) + \frac{4.2^2}{c^2} R_2^2(r) \exp 4\left(t - \frac{r\theta}{c} \right) + \frac{52}{c^2} R_2(r)R_3(r) \exp 6\left(t - \frac{r\theta}{c} \right) + \ldots \tag{4.23}$$

$$\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 = \frac{4}{c^2}R_1^{\,2}(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_2^{\,2}(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{4.3^2}{c^2}R_3^{\,2}(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\dots$$

(4.24)

$$\frac{\partial^2 f(t,r,\theta)}{\partial t^2} = R_1(r)\exp\left(t-\frac{r\theta}{c}\right)+2^2R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)+3^2R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\dots$$

(4.25)

Comparing coefficients of $\exp(0)$

$$R_0^{\,11}(r)+\frac{2}{r}R_0^{\,1}(r)=0 \tag{4.26}$$

Hence, we can choose the most convenient astrophysical solution for (4.26) as shown by appendix D.17-D.22, as

$$R_0(r)\approx -\frac{k}{r} \tag{4.27}$$

where $k = GM_0$; deducing from Schwarzschild's metric and Newton's dynamical theory of gravitation, G is the universal gravitational constant and M_0 is the total mass of the spherical body.

Equating coefficients of $\exp\left(t-\frac{r\theta}{c}\right)$ yields

$$R_1^{\,11}(r)+2\left[\frac{1}{r}-\frac{\theta}{c}\right]R_1^{\,1}(r)-\frac{\theta}{c}\left[\theta+\frac{2}{r}+\frac{4}{c}R_0(r)\right]R_1(r)=0 \tag{4.28}$$

This is the exact differential equation for R_1 and it determines R_1 in terms of R_0 . Thus, the solution assumes an exact wave equation, which in the order of c^0 reduces to

$$f(t,r,\theta) \approx -\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right) \tag{4.29}$$

4.4 Motion of Particles in the Equatorial Plane of $f(t,r,\theta)$ Field

The Lagrangian in the space time exterior to any astrophysical body is defined (Chifu, 2010; Chifu, 2011) as

$$L = \frac{1}{c}\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)^{\frac{1}{2}} \tag{4.30}$$

Thus in this field, the Lagrangian, becomes

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right)^{\frac{1}{2}} \tag{4.31}$$

Considering orbit in the equatorial plane of a homogeneous spherical mass, then,

$$\theta \equiv \frac{\pi}{2}$$

the Lagrangian equation reduces to (4.32) by substituting equation (3.7) and (3.8) into (4.31):

$$L = \frac{1}{c}\left(-\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{r}^2\right)^{\frac{1}{2}} \tag{4.32}$$

It is an established fact that $L=\in$, with $\in=1$ for time like orbits and $\in=0$, for null orbits (Chifu, 2012). Setting $L=\in$, in equation (4.32) and squaring both sides yields;

$$c^2 \in^2 = -\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{r}^2 \tag{4.33}$$

Orbital shape (which is a function of azimuthal angle) is paramount in most applications of general relativity. Therefore it is very important to transform equation (4.33) in terms of the azimuthal angle ϕ. After a series of transformation and simplification as shown in appendix E.6-E.9, equation (4.33) becomes

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left(1+\frac{2f(t,r,\theta)}{c^2}\right)-\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{t}^2 - c^2 \in^2 = 0 \tag{4.34}$$

Substituting equation (4.29) into equation (4.34), and for time like orbits, $\in=1$, equation (4.34) reduced to (3.35) as shown in appendix E.9-E.11;

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2 - \left(1-\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-2}\dot{t}^2 - \left(1-\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-1}c^2 = 0 \tag{4.35}$$

Since light travels on null geodesics, we have $\in=0$ and equation (4.34) reduces to (4.36) as shown in appendix E.12-E.16

$$\dot{t}^2\left(1+\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right) = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left(1-\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right) \tag{4.36}$$

Simplifying equation (4.36) gives (4.37) as shown in appendix E.12-E.21,

$$\dot{t}^2 = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{4k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)+\frac{4k}{c^4 r^2}\exp 2\left(t-\frac{r\theta}{c}\right)\right] \tag{4.37}$$

In the order of c^0 : equation (4.37) reduces to

$$\dot{t}^{2} = \frac{1}{\left(1+u^{2}\right)^{2}}\left(\frac{du}{d\phi}\right)^{2} \tag{4.38}$$

which on further simplification reduces to (4.39) as shown in appendix E.22-E.23,

$$\dot{t} = \frac{1}{\left(1+u^{2}\right)}\left(\frac{du}{d\phi}\right) \tag{4.39}$$

In order of c^{-2} : equation (4.37) reduces to equation (4.40) as shown in appendix E.24-E.27

$$\dot{t} = \frac{1}{1+u^{2}}\frac{du}{d\phi}\left[1-\frac{2k}{c^{2}r}\left(\exp t-\frac{r\theta}{c}\right)\right] \tag{4.40}$$

This is the photon equation of motion in this region

4.5 DISCUSSION

In all the sixteen (16) components of the Ricci tensors calculated from the mathematically most simply and astrophysically most satisfactory gravitational metric tensors (Howusu, 2007; Howusu, 2010; Lumbi *et al.*, 2014) equations (3.7)-(3.10), only R_{00} in the order of c^0 reduces to the well-known Laplacian operator $R_{00}\left\{g_{\mu\nu}\right\} = -\frac{1}{c^2}\left\{\nabla^2 f\right\}$ where ∇^2 is the well known Laplacian operator (Howusu, 2009; Howusu, 2010) as shown in appendix B.1-B5. In Minkwoski coordinate and in the limit of linear term, the R_{00} reduces to the well-known D'Alembertian operator as shown in appendix B.1-B.6, given by $R_{00}\left\{g_{\mu\nu}\right\} = -\frac{1}{c^2}\left\{\Box^2 f\right\}$ where $\Box^2$ is the well-known D'Alembertian operator (Howusu, 2009; Howusu, 2010). In general and in Minkowski coordinates, the well-known D'Alembertian operator acting on f contained hitherto unknown perfectly consistent additional terms of all order of c^{-2}. It is most interesting and instructive to note that, out of the 16 nonzero components of the Ricci tensors it is only R_{00} that has all the properties mentioned above. Hence, through the R_{00} Ricci tensor component and our mathematically most simply and astrophysically most satisfactory gravitational metric tensor, (Howusu, 2010) we may have discovered one natural, theoretically consistent and physically applicable and interesting information contained in the gravitational metric tensor an extension of the well-known D'Alembertian operator from pure empty space-time to space-time with gravitational fields in Minkowski coordinate.

Interestingly, we discover that the solution we obtained, that is equation (4.29) has a particular link to the pure Newtonian gravitational scalar potential for the gravitational field and hence put Einstein's geometrical gravitational field on the same level with the Newtonian dynamical theory of gravitation (Chifu, 2009; Chifu *et al.,* 2009; Chifu, 2012).

Equation (4.16) admits a wave solution with a phase velocity V (Howusu, 2007; Chifu, 2009) given by

$$V = i \qquad (4.18)$$

where $i = \sqrt{-1}$. Hence, such a wave does not occur in physical or astrophysical reality, only in imagination! And it has a speed of only one meter per second.

Thus equation (4.18) is the wave equation of a wave having an imagary speed ic propagating in vacuum.

Equation (4.35) is the planetary equation of motion for particle in the region of a rotating homogenous spherical mass. When solved, it will reveal the perihelion precision of planetary orbits within this gravitational field.

Equation (4.39) is the time equation of motion for photons (Chifu, *et al.*, 2009) in the equatorial plane of a rotating homogenous spherical mass

In the order of c^{-2} equation (4.40) is the time equation of motion for photons in the equatorial plane of a rotating homogenous spherical mass. This equation contains some additional terms not found in Schwarzschild's equation, which indicate theoretically the effects of gravity, radial distance and polar angle to the time equation of motion of photons in the equatorial plane of a rotating homogenous spherical mass.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

Recall that in chapter one we defined gravity and stated the two paramount theories of gravity. Statement of the problem, aim and objectives were also stated.

Chapter two was a review of works related to gravity as well as Einstein's theory of gravitation and solution to Einstein's field equation.

Chapter three described the methods used in constructing the Riemann-Christoffel tensors, the Ricci tensor, the Einstein's exterior field equations as well as the study of motion of test particles and photons within the equatorial plane of the field, which are the set objectives of this research work.

Chapter four gave the results of the Riemann-Christoffel tensors, the Ricci tensor, the Einstein's exterior field equations as well as the equation of motion for test particles and photons within the equatorial plane of the field.

5.2 Conclusion

Out of the sixteen (16) components of the Ricci tensor, it is only the R_{00} that can reduced to the well-known laplacian operator in the limit of c^0 and also reduces to the well known D'Alembertian operator in the limit of the linear term and in Minkowski coordinates as shown in equation (). This is in perfect agreement with what Howusu proposed in two of his books titled "THE METRIC TENSORS FOR GRAVITATIONAL FIELDS AND THE MATHEMATICAL PRINCIPLES OF

RIEMANIAN THEORETICAL PHYSICS" and "EXACT ANALYTICAL SOLUTIONS OF EINSTEIN'S GEOMETRICAL FIELD EQUATIONS" (Howusu, 2009; Howusu, 2010)

We also introduced the dependence of geometrical gravitational field on one and only one dependant function f, comparable to one and only one gravitational scalar potential in Newtonian dynamical theory of gravitation (Chifu, 2009; Chifu, *et al*., 2009).

Thus, we have obtained an analytical solution of Einstein's geometrical field equation in a gravitational field whose tensor field varies with time, radial distance and polar angle only. Our metric tensor, which is the basic parameter in this field, is defined completely.

5.3 Recommendations

This research work can be use in the study rotating astrophysical bodies within a spherical geometry whose tensor field varies with time, radial distance and polar angle. Example of such bodies is rotating stars such as Neutron star, Wolf-Rayet e.t.c.

The D'Alembertian operator found in this work can be use to generalized Maxwell's electric and magnetic wave field equations in gravitational field.

The door is open for the study of other astrophysical phenomenon within this field such as gravitational red-shift by the sun, time dilation, length contraction, just to mention a few.

5.4 Limitations

This work is only limited to astrophysical bodies within a spherical geometry whose tensor varies with time, radial distance and polar angle only. It can be extended to include bodies whose tensor field varies with time, radial distance, polar angle and azimuthal angle.

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APPENDICES

Appendix A

The Riemann-Christoffel tensors are given as

$$R^{0}_{000} = 0 \tag{A.1}$$

$$R^{1}_{001} = \frac{\partial}{\partial t}\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right] - \frac{\partial}{\partial r}\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$+\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$-\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right]\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right]+\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}\right]\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$-\left[\frac{1}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}\right]\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta}\right] \tag{A.2}$$

$$R^{1}_{001} = \frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}$$

$$-\frac{2}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial r}\right]^2 - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2} + \frac{1}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial r}\right]^2$$

$$+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 + \frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$+\frac{1}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial r}\right]^2+\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2 \tag{A.3}$$

$$R^1_{001}=\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}$$

$$-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}+\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2 \tag{A.4}$$

$$R^2_{002}=-\frac{\partial}{\partial\theta}\left[\frac{1}{c^2r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]+\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]\left[\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)\right]$$

$$-\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}\right]\frac{1}{r} \tag{A.5}$$

$$R^2_{002}=-\frac{1}{c^2r^2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}+\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{1}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r} \tag{A.6}$$

$$R^3_{003}=-\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}\right]\left[\frac{1}{r}\right]-\left(\frac{1}{c^2r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}\right)\cot\theta \tag{A.7}$$

$$R^3_{003}=-\frac{1}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}-\frac{\cot\theta}{c^2r^2}\frac{\partial f(t,r,\theta)}{\partial\theta} \tag{A.8}$$

$$R^0_{110}=\frac{\partial}{\partial r}\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]-\frac{\partial}{\partial t}\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\frac{\partial f(t,r,\theta)}{\partial t}\right]$$

$$+\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]^2-\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\frac{\partial f(t,r,\theta)}{\partial t}\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right]$$
$$+\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\right]\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$-\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$-\left[\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta}\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial \theta}\right] \qquad \text{(A.9)}$$

$$R^0_{110}=-\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)^2+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$-\frac{6}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}$$

$$+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)^2+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2$$

$$+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)^2$$

$$-\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 \qquad \text{(A.10)}$$

$$R^0_{110}=-\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 \qquad \text{(A.11)}$$

$$R^1_{111}=0 \qquad \text{(A.12)}$$

$$R^2_{112}=\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{\partial}{\partial \theta}\left[\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta}\right]$$

$$+\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]\left[\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]$$

$$-\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2 \quad \text{(A.13)}$$

$$R^2_{112}=\frac{4}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}$$

$$-\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{1}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r} \quad \text{(A.14)}$$

$$R^2_{112}=\frac{3}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{1}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2} \quad \text{(A.15)}$$

$$R^3_{113}=\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]\left(\frac{1}{r}\right)-\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)\cot\theta+\left(\frac{1}{r}\right)^2$$
(A.15)

$$R^3_{113}=\frac{1}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{\cot\theta}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right) \quad \text{(A.16)}$$

$$R^0_{220}=\frac{\partial}{\partial\theta}\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]+\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]^2$$

$$-\left[-r\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right] \quad \text{(A.17)}$$

$$R^0_{220} = -\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2} + \frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} \tag{A.18}$$

$$R^1_{221} = \frac{\partial}{\partial\theta}\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right] - \frac{\partial}{\partial r}\left[-r\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right] + \left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]^2 \tag{A.19}$$

$$R^1_{221} = \frac{3}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2 - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2} + \frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} \tag{A.20}$$

$$R^2_{222} = 0 \tag{A.21}$$

$$R^3_{223} = \frac{\partial}{\partial\theta}(\cot\theta) - \left[-r\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]\left(\frac{1}{r}\right) + (\cot\theta)^2 \tag{A.22}$$

$$R^3_{223} = -\cos ec^2\theta + \left(1+\frac{2f(t,r,\theta)}{c^2}\right) + \cot^2\theta \tag{A.23}$$

$$R^3_{223} = \frac{2f(t,r,\theta)}{c^2} \tag{A.24}$$

$$R^0_{330} = -\left[-r\sin^2\theta\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$

$$-(-\sin\theta\cos\theta)\left[\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right] \tag{A.25}$$

$$R^0_{330} = \frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{\sin\theta\cos\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta} \tag{A.26}$$

$$R^{1}_{331} = -\frac{\partial}{\partial r}\left[-r\sin^2\theta\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right] - \left[-r\sin^2\theta\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\right]$$
$$-\left(-\sin\theta\cos\theta\right)\left[-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}\right] + \left(\frac{1}{r}\right)\left[-r\sin^2\theta\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]$$
(A.27)

$$R^{1}_{331} = \frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{\sin\theta\cos\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta} \quad \text{(A.28)}$$

$$R^{2}_{332} = -\frac{\partial}{\partial\theta}\left(-\sin\theta\cos\theta\right) - \left[-r\sin^2\theta\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\right]\left(\frac{1}{r}\right) + \cot\theta\left(-\sin\theta\cos\theta\right)$$

(A.29)

$$R^{2}_{332} = \frac{2\sin^2\theta f(t,r,\theta)}{c^2} \quad \text{(A.30)}$$

$$R^{3}_{333} = 0 \quad \text{(A.31)}$$

Appendix B

The Ricci Tensors and The Riemann scalar

The Ricci tensors are given as

$$R_{00} = \frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}$$

$$-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2} + \frac{1}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2$$

$$-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} + \frac{1}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{1}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}$$
$$-\frac{1}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r} - \frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta} \qquad \text{(B.1)}$$

$$R_{00} = \frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}$$

$$-\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2$$

$$-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2} - \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2} - \frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta} \qquad \text{(B.2)}$$

Equation (B.2) can also be written as

$$R_{00} = \frac{1}{c^2}\left[\frac{4}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 - \left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right.$$

$$-\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial^2 f(t,r,\theta)}{\partial r^2} + \frac{2}{c^2 r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$-\frac{2}{r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}\right] \qquad \text{(B.3)}$$

$$R_{00}=\frac{1}{c^2}\left[\left\{1+\frac{2f(t,r,\theta)}{c^2}\right\}\frac{\partial^2 f(t,r,\theta)}{\partial r^2}+\frac{2}{r}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\left(\frac{\partial f(t,r,\theta)}{\partial r}\right)\right.$$

$$+\frac{1}{r^2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}-\frac{2}{c^2r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$

$$\left.-\frac{4}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right] \qquad \text{(B.4)}$$

In the limit of c^0, equation (B.4) reduces to the well-known Euclidean laplacian operator given as

$$R_{00}=\frac{1}{c^2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(t,r,\theta)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] \qquad \text{(B.5)}$$

Equation (B.4) also reduces to the well-known D'Alembertian operator in the other of linear terms given as

$$R_{00}=\frac{1}{c^2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(t,r,\theta)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial t}\left(\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial}{\partial t}\right)\right]$$

(B.6)

$$R_{11} = -\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{1}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$+\frac{3}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{1}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}\frac{1}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right) \tag{B.7}$$

$$R_{11} = -\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) + \frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$+\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$

$$-\frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right) \tag{B.8}$$

$$R_{22} = -\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$

$$+\frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{3}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}$$

$$+\frac{r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2} \qquad \text{(B.9)}$$

$$R_{22}=\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2} \qquad \text{(B.10)}$$

$$R_{33}=\frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{\sin\theta\cos\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}+\frac{r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{\sin\theta\cos\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}+\frac{2\sin^2\theta f(t,r,\theta)}{c^2} \qquad \text{(B.11)}$$

$$R_{33}=\frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2\sin^2\theta f(t,r,\theta)}{c^2} \qquad \text{(B.12)}$$

The Riemann scalar is given as

$$R=\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\left[\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right.$$

$$-\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial\theta}\right]^2-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}$$

$$\left.-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}-\frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}\right]$$

$$-\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\left[-\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)\right.$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$\left.-\frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}-\frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)\right]$$

$$-\frac{1}{r^2}\left[\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2}\right]$$

$$-\frac{1}{r^2\sin^2\theta}\left[\frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2\sin^2\theta f(t,r,\theta)}{c^2}\right] \tag{B.13}$$

$$R=\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{1}{c^2}\frac{\partial^2 f(t,r,\theta)}{\partial r^2}-\frac{2}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$+\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial\theta}\right]^2-\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}$$

$$-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}-\frac{\cot\theta}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}$$

$$+\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2-\frac{1}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{2}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}$$
$$+\frac{\cot\theta}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{2}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}$$
$$-\frac{2f(t,r,\theta)}{c^2r^2}-\frac{2}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2f(t,r,\theta)}{c^2r^2} \tag{B.14}$$

$$R=\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{8}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2r^2} \tag{B.15}$$

Appendix C

Einstein's Field Equations

The Einstein's field equations for this field are given as

$$G_{00}=\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}-\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}$$
$$+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}-\frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}$$
$$-\frac{1}{2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right.$$
$$\left.-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2-\frac{4f(t,r,\theta)}{c^2 r^2}\right]=0 \quad \text{(C.1)}$$

$$G_{00}=\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial^2 f(t,r,\theta)}{\partial r^2}-\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}$$
$$+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{\partial f(t,r,\theta)}{\partial \theta}\right]^2-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial^2 f(t,r,\theta)}{\partial t^2}$$
$$-\frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}-\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{4}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}$$
$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2+\frac{2f(t,r,\theta)}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)=0 \qquad \text{(C.2)}$$

$$G_{00}=\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\frac{\partial f(t,r,\theta)}{\partial r}+\frac{3}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2-\frac{1}{c^2 r^2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$
$$-\frac{\cot\theta}{c^2 r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}+\frac{2f(t,r,\theta)}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)=0 \qquad \text{(C.3)}$$

$$G_{11} = -\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$+\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$

$$-\frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)+\frac{1}{2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2\right.$$
$$-\frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$\left.-\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2-\frac{4f(t,r,\theta)}{c^2 r^2}\right]=0 \qquad \text{(C.4)}$$

$$G_{11} = -\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 + \frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$+\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)+\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$

$$+\frac{2}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial \theta^2}$$

$$-\frac{\cot\theta}{c^2 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)+\frac{4}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{4}{c^2 r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$

$$-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$

$$-\frac{2f(t,r,\theta)}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}=0 \qquad \text{(C.5)}$$

$$G_{11}=\frac{1}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{2}{c^2r}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\frac{\partial f(t,r,\theta)}{\partial r}$$

$$-\frac{1}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\frac{\partial^2 f(t,r,\theta)}{\partial\theta^2}-\frac{\cot\theta}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)$$

$$-\frac{2f(t,r,\theta)}{c^2r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}=0 \qquad \text{(C.6)}$$

$$G_{22}=\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2}$$

$$+\frac{1}{2}r^2\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{8}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right.$$
$$\left.-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2r^2}\right]=0 \qquad \text{(C.7)}$$

$$G_{22}=\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2}$$

$$+\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{2f(t,r,\theta)}{c^2}=0 \qquad \text{(C.8)}$$

$$G_{22} = -\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) = 0 \qquad \text{(C.9)}$$

$$G_{33} = \frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2\sin^2\theta f(t,r,\theta)}{c^2} + \frac{1}{2}r^2\sin^2\theta\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2\right.$$

$$-\frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$\left.-\frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{4f(t,r,\theta)}{c^2 r^2}\right] = 0 \qquad \text{(C.10)}$$

$$G_{33} = \frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2\sin^2\theta f(t,r,\theta)}{c^2} + \frac{4r^2\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{4r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{r^2\sin^2\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2\sin^2\theta}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$-\frac{\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{2\sin^2\theta f(t,r,\theta)}{c^2} = 0 \qquad \text{(C.11)}$$

$$G_{33} = -\frac{2r\sin^2\theta}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{4r^2\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$

$$-\frac{r^2\sin^2\theta}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2\sin^2\theta}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)$$

$$-\frac{\sin^2\theta}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 = 0 \qquad \text{(C.12)}$$

Appendix D

Solution of Einstein's Field Equation

Solution of equation (C.9)

Equation (C.9) can be rewritten as

$$-\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$
$$-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)=0 \qquad \text{(D.1)}$$

Multiplying equation (D.1) with a negative sign gives

$$\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$

$$-\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)=0 \qquad \text{(D.2)}$$

$$\left[r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+2r\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$

$$\left.-\frac{4r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+r^2\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0 \qquad \text{(D.3)}$$

$$\left[r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+2r\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2}\left(1-\frac{4f(t,r,\theta)}{c^2}+\ldots\right)\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$

$$\left.-\frac{4r^2}{c^2}\left(1-\frac{6f(t,r,\theta)}{c^2}+\ldots\right)\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+r^2\left(1-\frac{4f(t,r,\theta)}{c^2}+\ldots\right)\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0$$

(D.4)

$$\left[r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+2r\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$

$$\left.-\frac{4r^2}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24r^2 f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{4r^2 f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0$$

(D.5)

$$\left[\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$
$$\left.-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0$$

(D.6)

But

$$\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\right\}f(t,r,\theta)=\nabla f(t,r,\theta) \qquad \text{(D.7)}$$

Equation (D.7) is the well-known Euclidean Laplacian operator

$$\nabla f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$

$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0 \quad \text{(D.8)}$$

$$\nabla f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0$$

(D.9)

In the limit of c^0 equation (D.9) reduced to

$$\nabla f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)=0$$

(D.10)

$$f(t,r,\theta)=\sum_{n=0}^{\infty}R_n(r)\exp n\left(t-\frac{r\theta}{c}\right)$$

(D.11)

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2}=R_0^{11}(r)+\left(R_1^{11}(r)-\frac{2\theta}{c}R_1^1(r)-\frac{\theta^2}{c^2}R_1(r)\right)\exp\left(t-\frac{r\theta}{c}\right)$$

$$+\left(R_2^{11}(r)-\frac{2.2\theta}{c}R_2^1(r)-\frac{2^2\theta^2}{c^2}R_2(r)\right)\exp 2\left(t-\frac{r\theta}{c}\right)+\left(R_3^{11}(r)-\frac{2.3\theta}{c}R_3^1(r)-\frac{3^2\theta^2}{c^2}R_3(r)\right)$$

(D.12)

$$\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}=\frac{2}{r}R_0^1(r)+\frac{2}{r}R_1^1(r)\exp\left(t-\frac{r\theta}{c}\right)+\frac{2}{r}R_2^1(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{2}{r}R_3^1(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$

$$-\frac{2\theta}{cr}R_1(r)\exp\left(t-\frac{r\theta}{c}\right)-\frac{2.2\theta}{cr}R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)-\frac{2.3\theta}{cr}R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\ldots$$

(D.13)

$$\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2 = \frac{1}{c^4}R_1^{\,2}(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{2^2}{c^4}R_2^{\,2}(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{3^2}{c^4}R_3^{\,2}(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots$$

(D.14)

$$\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) = \frac{4}{c^2}R_0(r)R_1(r)\exp\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_0(r)R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)$$

$$+\frac{4.3^2}{c^2}R_0(r)R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\frac{4}{c^2}R_1^{\,2}(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{20}{c^2}R_1(r)R_2(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$

$$+\frac{40}{c^2}R_1(r)R_3(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_2^{\,2}(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{52}{c^2}R_2(r)R_3(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots$$

(D.15)

$$\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 = \frac{4}{c^2}R_1^{\,2}(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_2^{\,2}(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{4.3^2}{c^2}R_3^{\,2}(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots$$

(D.16)

$$\frac{\partial^2 f(t,r,\theta)}{\partial t^2} = R_1(r)\exp\left(t-\frac{r\theta}{c}\right)+2^2R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)+3^2R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\ldots$$

(D.17)

$$R_0^{\,11}(r)+\frac{2}{r}R_0^{\,1}(r)=0$$

(D.18)

The auxiliary equation of equation (D.17) is given as

$$m_0{}^2(r) + \frac{2}{r}m_0(r) = 0 \qquad \text{(D.19)}$$

Where $m_0(r) = R_0(r)$

Equation (D.19) is a quadratic equation where $a = 1, b = \frac{2}{r}, c = 0$. Solving (D.19) quadratically, we have

$$m_0(r) = \frac{-\frac{2}{r} \pm \sqrt{\left(-\frac{2}{r}\right)^2}}{2} \qquad \text{(D.20)}$$

$$m_0(r) = -\frac{4}{2r}$$

$$m_0(r) = -\frac{2}{r}$$

$$R_0(r) = m_0(r) = -\frac{2}{r} \qquad \text{(D.21)}$$

Comparing our solution with Newton's gravitational scalar potential

$$-\frac{k}{r} = -\frac{2}{r}$$

$$k = 2 \qquad \text{(D.22)}$$

Hence,

$$R_0(r) = -\frac{k}{r} \qquad \text{(D.23)}$$

Substituting (D.22) into (D.10) we have,

$$f(t,r,\theta) \approx -\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right) \tag{D.24}$$

Appendix E

Motion of Particles and Photons in Equatorial Plane

The Lagrangian is given as,

$$L = \frac{1}{c}\left(-g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}\right)^{\frac{1}{2}} \tag{E.1}$$

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right)^{\frac{1}{2}} \tag{E.2}$$

For

$$\theta \equiv \frac{\pi}{2} \tag{E.3}$$

Equation (E.2) reduces to

$$L = \frac{1}{c}\left(-\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{r}^2\right)^{\frac{1}{2}} \tag{E.4}$$

It is an established fact that $L = \in$, with $\in = 1$ for time like orbits and $\in = 0$, for null orbits. Setting $L = \in$, in equation (4.118) and squaring both sides yields;

$$c^2 \in^2 = -\left(1+\frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{r}^2 \tag{E.5}$$

Using the following transformation, with $r = r(\phi)$ and $u(\phi) = \frac{1}{r(\phi)}$ then,

$$\dot{r} = \dot{\phi}\frac{dr}{d\phi} \text{ or } \dot{r} = \frac{l}{1+r^2}\frac{dr}{d\phi} \tag{E.6}$$

But

$$\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} \quad \text{or} \quad \frac{dr}{d\phi} = -u^2\frac{du}{d\phi} \tag{E.7}$$

And thus,

$$\dot{r} = \frac{l}{1+r^2}\frac{dr}{d\phi} \tag{E.8}$$

Substituting (E.6)-(E.8) into (E.5), the Lagrangian becomes

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left(1+\frac{2f(t,r,\theta)}{c^2}\right)-\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{t}^2 - c^2 \in^2 = 0 \tag{E.9}$$

Substituting (D.23) into (E.9), and for $\in = 1$, we have

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right]-\left[1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1}\dot{t}^2 - c^2 = 0 \tag{E.10}$$

Multiplying (E.10) by $\left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-1}$ we have

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2 - \left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-2}\dot{t}^2 - \left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-1}c^2 = 0 \tag{E.11}$$

For $\in = 0$ which correspond to the equation of motion of light on null geodesic and substituting (D.23) into (E.9) we have,

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1+\frac{2}{c^2}\left(-\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right)\right]-\left[1+\frac{2}{c^2}\left(-\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right)\right]^{-1}\dot{t}^2=0$$

(E.12)

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]-\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1}\dot{t}^2=0 \qquad \text{(E.13)}$$

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]-\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)+\ldots\right]\dot{t}^2=0 \qquad \text{(E.14)}$$

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]-\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]\dot{t}^2=0 \qquad \text{(E.15)}$$

$$\dot{t}^2\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right] \qquad \text{(E.16)}$$

Multiplying (E.16) by $\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1}$ we have,

$$\dot{t}^2=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1} \qquad \text{(E.17)}$$

$$\dot{t}^2=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)+\ldots\right] \qquad \text{(E.18)}$$

$$\dot{t}^2=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right] \qquad \text{(E.19)}$$

$$\dot{t}^2 = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)+\frac{4}{c^4}\frac{k^2}{r^2}\exp 2\left(t-\frac{r\theta}{c}\right)\right]$$

(E.20)

$$\dot{t}^2 = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{4}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)+\frac{4}{c^4}\frac{k^2}{r^2}\exp\left(t-\frac{r\theta}{c}\right)\right] \quad \text{(E.21)}$$

In order of c^0, (E.21) reduced to

$$\dot{t}^2 = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2$$

(E.22)

Taking the square root of (E.22), we have

$$\dot{t} = \frac{1}{\left(1+u^2\right)}\left(\frac{du}{d\phi}\right) \quad \text{(E.23)}$$

In order of c^{-2}: (E.21) reduces to

$$\dot{t}^2 = \frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{4}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right] \quad \text{(E.24)}$$

Taking the square root of both sides of equation (E.24) gives

$$\dot{t} = \frac{1}{\left(1+u^2\right)}\left(\frac{du}{d\phi}\right)\left[1-\frac{4}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{\frac{1}{2}} \quad \text{(E.25)}$$

Applying binomial expansion to equation (E.25) gives

$$\dot{t} = \frac{1}{\left(1+u^2\right)}\left(\frac{du}{d\phi}\right)\left[1-\frac{4}{2c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)+...\right] \tag{E.26}$$

On further simplification, (E.26) reduces to

$$\dot{t} = \frac{1}{1+u^2}\frac{du}{d\phi}\left[1-\frac{2k}{c^2 r}\left(\exp t-\frac{r\theta}{c}\right)\right] \tag{E.27}$$