CONSTRUCTION AND ANALYSIS OF QUANTUM STOCHASTIC DYNAMICS USING NONCOMMUTATIVE $L_P -$ SPACES

BY

YUSUF IBRAHIM

DEPARTMENT OF MATHEMATICS,

AHMADU BELLO UNIVERSITY,

ZARIA, NIGERIA

AUGUST, 2011

CONSTRUCTION AND ANALYSIS OF QUANTUM STOCHASTIC DYNAMICS USING NONCOMMUTATIVE $L_P -$ SPACES

BY

YUSUF IBRAHIM B.Sc

(BUK 1994), M.Sc (BUK 1999)

Ph.D/Scien/36887/02-03

A DISSERTATION SUBMITTED TO THE

POSTGRADUATE SCHOOL,

AHMADU BELLO UNIVERSITY,

ZARIA, NIGERIA

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS

DEPARTMENT OF MATHEMATICS,

AHMADU BELLO UNIVERSITY,

ZARIA, NIGERIA

AUGUST, 2011

DECLARATION

I declare that the work in the dissertation entitled 'Construction and Analysis of Quantum Stochastic Dynamics using Noncommutative $L_P -$ spaces' has been performed by me in the Department of Mathematics under the supervision of Dr. A. A. Tijjani.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any university to the best of my knowledge.

<u>Yusuf Ibrahim</u>	____________	**<u>August, 2011</u>**
Name of student	**Signature**	**Date**

CERTIFICATION

This dissertation entitled 'CONSTRUCTION AND ANALYSIS OF QUANTUM STOCHASTIC DYNAMICS USING NONCOMMUTATIVE $L_P -$ SPACES' by Yusuf Ibrahim meets the regulations governing the award of the degree of Doctor of Philosophy of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

______________________________ Date ______________
Dr. A. A. Tijjani
Chairman, Supervisory Committee

______________________________ Date ______________
Prof. D. Singh
Member, Supervisory Committee

______________________________ Date ______________
Dr. B. Sani
Member, Supervisory Committee

______________________________ Date ______________
Prof. G.O.S. Ekhaguere
External examiner

______________________________ Date ______________
Dr. A. A. Tijjani
Head of Department

______________________________ Date ______________
Prof. A. Joshua
Dean, Postgraduate School

ACKNOWLEDGMENTS

First and foremost, I give thanks to Allah and his prophet, peace and blessings of Allah be upon him.

I am grateful to my supervisor Dr. A. A. Tijjani for his advice and constructive criticisms during the course of the work. Despite his overwhelming responsibilities and work load both as the Head of Department and the Deputy Dean of the faculty, he was able to assists me in getting the work done. This work will not have seen the light of the day, if not for the help and assistance of all the members of staff of the department of Mathematics Ahmadu Bello University Zaria. I thank them for their encouragement and most at times their scholarly advice. In particular I wish to thank, Dr. A. M. Ibrahim the postgraduate coordinator of the department, Dr. B. Sani and Dr. A. O. Ajibade. It is also my pleasure, in particular to thank Professor D. Singh for his kind words. The postgraduate students of the Department of Mathematics, Ahmadu Bello University Zaria, I am grateful for the support and help they rendered to me. Among them, I wish to, most especially, thank Mrs. I. Onyezulli, Mr. S. Isa and Mr. M. Madugu. I thank them all.

I am grateful to the following mathematicians, Professors, B. Zegarlinski, A. Isar and O. Bratteli for sending me materials that helped me a lot, and being a constant source of inspiration for me.

I am highly indebted to my friends in the Department of Mathematics, Nigerian Defence Academy, they are my pillars of support during my ups and downs. I will like to express my gratitude to the following individuals, Professor S. C. Osuala, Associate Professor K. B. Yuguda, Dr. B. Ali, and Dr. E. Andi.

I take this opportunity to depart from the norm followed in every acknowledgment. I thank Professor G.O.S Ekhaguere who was my external examiner, not only for that reason alone, but because he taught me in seven hours what I could not learn in seven years. I am humbled, thank you sir.

My family has been very supportive to me, I wish to use this opportunity to thank my wife for understanding and caring during the course of this work.

ABSTRACT

A noncommutative algebra involving operators of the form $\rho_{\Lambda}^{\alpha(t)} \cdot \rho_{\Lambda}^{\alpha(t)}$ is defined. Using the noncommutative L_p −spaces technique, we give a constructive approach to quantum Markov evolution of infinite system, based on the notion of the thermodynamic limit. The infinite Markov time evolution is constructed as the thermodynamic limit of the corresponding finite volume (dynamics) evolution. The extended Markov time evolution is then studied, with a view of addressing questions of exponential stability and ergodicity . In quantum spin system the existence of non local physical correlation at a phase transition is a manifestation of the entanglement among the constituents parts. We studied asymptotic entanglement within the frame work of open quantum systems for two independent quantum harmonic oscillators interacting with an environment. The question of separability is addressed using the Peres-Simon equation.

Table of Contents

CHAPTER 1

GENERAL INTRODUCTION

In this chapter we state the research problem as well as the justification and objectives for the investigation. We also give some definitions and results needed in this work. First of all, we begin with a general introduction.

In the last two decades, more and more interest arose about the problems of dissipation in quantum mechanics. The quantum description of dissipation is important in physics. For example, dissipative processes play a basic role in the theory of laser and that of atomic nucleus. The irreversible dissipative behaviour of the vast majority of physical phenomenon come into a contradiction with reversible nature of our basic models. The very restrictive principles of conservative and isolated systems are unable to deal with this type of situations. The fundamental quantum dynamical laws are of the reversible type. The dynamics of a closed system is governed by the Hamiltonian, a self adjoint operator that represents its total energy and is a constant of motion. The paradox of irreversibility arises; the reversibility of microscopic dynamics contrasting with the irreversibility of the macroscopic behaviour we are trying to deduce from it. One way to solve this paradox of irreversibility is to use models to which the Hamiltonian dynamics and Liouville theorem do not apply but the irreversible behaviour is clearly present even in the microscopic dynamical description. The reason for replacing Hamiltonian dynamics and Liouville's theorem is that no system is truly isolated being subject to uncontrollable random influences from outside. For this reason these models are called quantum open system (Isar, etal.1994). The aim of quantum open system theory is to study the interaction of simple quantum system interacting with very large ones. In general the properties that one is seeking are to exhibit the dissipation of the small system in favour of the large one, to identify when this interaction gives rise to a return to equilibrium or a thermalization of the small system (Attal and Joyce, 2006). There are two ways of studying these systems. The first approach is the Hamiltonian approach. Here the complete quantum system formed by the small system and the reservoir is studied through a Hamiltonian describing the free evolution. The associated unitary group gives rise to a group of *-endomorphism of a von Neumann algebra of observable together with a state for the system constitute a quantum dynamical system. The aim is to then to study the ergodic properties of that quantum dynamical system. The second approach is the Markovian approach. In this approach one gives up the idea of modelizing the reservoir and concentrates on the effective dynamics of the small system. This evolution is supposed to be described by a semigroup of completely positive maps. These semigroups admit a generator which is of the Lindblad form (Attal and Joyce, 2006).

Quantum spin system introduced in 1961 in the discussion of magnetic properties of crystalline substance could also be studied within the frame work of open quantum systems. Basically a quantum spin system consists of a set of particles confined to a lattice and interacting at distance. There are two physical interpretations of these models, either as a lattice gas or as a spin system. In the spin system it is assume that every lattice site is permanently occupied by a particle but the particle have various internal degrees of freedom e.g. the particles could have an intrinsic spin with several possible orientation. The interaction between the particles then follows from the coupling of the internal degrees of freedom and this yield an evolution in which the spin orientation are constantly changing. (Bratteli and Robinson, 1979).

1.1 Statement of the research problem

In noncommutative analysis a major problem is the construction of a dissipative quantum dynamical semigroup. The description of infinite quantum spin systems is far less advanced than in the commutative case and there is no satisfactory description of quantum stochastic dynamics especially for spin systems at high temperature or for one dimensional lattice with finite interaction at arbitrary finite temperature (Majewski and Zegarlinski ,1996).

The purpose of statistical mechanics is the description of the mechanics of large or even infinite systems. We recall that infinite systems in statistical mechanics arose as a result of the thermodynamic limit (the general name to the limit $\Lambda \to \infty$, where Λ is a finite subset of the d -dimensional lattice $\mathbb{Z}^d, d \geq 1$). The aim of this is to give an unambiguous meaning of such concept as temperature, pressure, and phase transition (Ruelle, 1969).

An attempt to use the theory of noncommutative L_p spaces for the construction and analysis of quantum stochastic dynamics for spin systems was initiated by Majewski and Zegarlinski (1996). A constructive approach for the construction of quantum Markov evolution of infinite system based on the notion of the thermodynamic limit, is addressed in this work. By a constructive approach, we mean one in which existence of the evolution of the extended system is not postulated, as in pure semigroup approach, but constructed on the basis of the local character of the evolution in the bounded regions Λ_n i.e the evolution is constructed as the thermodynamic limit of the corresponding finite volume dynamics with an appropriate control of the convergence. In order words we adopt the view that the dynamics of extended quantum system has to be derived from the limit of the time evolution P_t^{X,Λ_n} as $\Lambda_n \to \infty$. If it exists in the appropriate topology and posses some necessary properties, then the time evolution of the system can be defined as

$$P_t^X = \lim_{\Lambda_n} P_t^{X,\Lambda_n} \quad .$$

No work has been done in a constructive approach to quantum Markov evolution of infinite systems, before the work of Majewski and Zegarlinski (1996), except for a few models. In this work we consider a noncommutative algebra, i.e a von Neumann algebra $\mathcal{M}_0$ with elements of the form $\rho^{\alpha(t)}.\rho^{\alpha(t)}$. This will made clear in chapter three. The finite time evolution satisfies the following equation,

$$\frac{d}{dt} P_t^{X,\Lambda} = \mathcal{L}_{X,\Lambda} P_t^{X,\Lambda} \; ; \quad P_0^{X,\Lambda} = id \tag{1.1.1}$$

where $P_t^{X,\Lambda}$ is the finite volume stochastic dynamics and $\mathcal{L}_{X,\Lambda}$ the generator, with $X \subset \Lambda$.

Here $\mathbb{Z}^d$ is the d-dimensional lattice, $\mathcal{F}$ is the collection of all the finite subsets of $\mathbb{Z}^d$, $\Lambda \in \mathcal{F}$. The extended time evolution P_t^X is then studied with a view of addressing questions of exponential stability and ergodicity of the extended time evolution P_t^X.

In quantum spin system, it was realized that the existence of non local physical correlation at a phase transition is a manifestation of the entanglement among the constituents parts (Its etal., 2008).

We studied entanglement within the frame work of open quantum systems, the question of separability of two Harmonic oscillators interacting with an environment is addressed using the Peres-Simon equation.

1.2 Justification

In classical mechanics, a commutative dynamical system is a triple (X, T_t, μ), where X a measurable space, is the phase space of the system. T_t is the time evolution expressed as a one parameter family of transformation on the phase space X, and μ an invariant measure for T_t. Having a commutative dynamical system, it is natural to ask fundamental questions on ergodicity, return to equilibrium and a proper description of the classical Markov semigroup. However the measurable structure of the phase space X is too weak for a study of such questions. There is need for an additional structure. This is the point where we introduce the L_p −spaces. We associate with the triple (X, T_t, μ) the $L_p(X, \mu)$ spaces and study the time evolution as a family of transformation on $L_p(X, \mu)$. The L_p −spaces plays an essential role in the construction and analysis of classical Markov evolution.

In order to generalized the classical L_p −spaces technique, to the quantum setting we need a noncommutative L_p −spaces, this is realized by a von Neumann algebra. The triple (X, T_t, μ) is then replaced with a quantum counterpart of a dynamical system, namely, the triple $(\mathcal{M}_0, P_t^{X,\Lambda}, \varphi)$, where $\mathcal{M}_0$ is a von Neumann algebra, $P_t^{X,\Lambda}$ is the finite volume stochastic dynamics, and φ is a faithful normal state. This constitutes a basis for a description of infinite quantum system. The general properties of L_p −spaces can then be applied. The advantage lies in the fact that functional analysis technique could be employed to get a proper description of the infinite volume quantum dynamics for spin systems, as well as a study to the questions of ergodicity and stability of the infinite quantum system. An example of a physical application, is the Heisenberg model of a ferromagnetic material.

In quantum information science, entanglement is indispensable and play an important role in quantum computation and other related fields (Nielsen and Chuang, 2000). Despite the potential application of quantum entangled states, the theory of quantum entanglement itself is far from being complete.

1.3 Objective

Non-commutative L_p −spaces over a von Neumann algebra with respect to a faithful normal semi-finite trace was constructed by Segal (1953).Since then, various types of L_p spaces have been constructed,(Trunov,1979), (Zolotarev,1982), (Haagerup,1979),

,(Terp,1981),(Yeadon,1975), (Kosaki,1984). We study the Trunov L_p-spaces involving closed operators of the form $\rho_n^{\alpha(t)}.\ \rho_n^{\alpha(t)}$.

Conditional expectations, which are projections of norm one, have been studied by Umegaki (1954,1956) and Tomiyama (1957,1958). Takesaki (1972), established the necessary and sufficient conditions for the existence of such conditional expectations. We use the generalized conditional expectation formulated in Majewski and Zegarlinski (1996) to define a pre-markov generator which is symmetric, bounded and $*-$invariant. This makes sense for spin systems which interact over a finite range.

In Majewski and Zegarlinski (1996), the construction of a dynamics on the inductive limit C^*algebra was presented. In this work, we study dynamics on an inductive limit von Neumann algebra $\mathcal{M}_0$, and formulate a strong ergodicity condition for the dynamics of spin system on a lattice. We derive a coordinate form of the Lindblad-type generator and the Simon-Peres type equation in terms of the variance and covariance. This is then applied in addressing the question of separability.

The outline of the work is as follows. In chapter one we give an introduction containing a statement of the problem. Chapter two is a review of the works of some authors. In chapter three, we give a definition of a noncommutative algebra, and study stochastic dynamics for spin system on a lattice, in chapter four. Within the Lindblad theory of open quantum systems, we study entanglement and derived the equations of motions in terms of the variance and covariance of the coordinates q_x, q_y and momenta p_x, p_y operators, of two harmonic oscillators interacting with an environment in chapter five. While chapter six gives the summary and conclusion.

1.4 Mathematical Tools

Here we give some definitions and theorems.

Definition 1.4.1

Let $\mathfrak{U}$ be a vector space over $\mathbb{C}$ .The space $\mathfrak{U}$ is called an algebra if it is equipped with a multiplication which associates the product AB to each pair $A, B \in \mathfrak{U}$ such that

1 $A(BC) = (AB)C, \quad A, B, C \in \mathfrak{U}$

2 $A(B + C) = AB + AC, \quad A, B, C \in \mathfrak{U}$

3 $\alpha\beta(AB) = (\alpha A)(\beta B), \quad A, B \in \mathfrak{U}, \quad \alpha, \beta \in \mathbb{C}$ (1.4.1)

Definition 1.4.2

The algebra $\mathfrak{U}$ is a normed algebra if to each element $A \in \mathfrak{U}$ there is associated a real number $\|A\|$, the normed of A, satisfying

(i) $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.

(ii) $\|\alpha A\| = |\alpha|\|A\|, \quad \alpha \in \mathbb{C}$

(iii) $\|A + B\| \leq \|A\| + \|B\|, \quad A, B \in \mathfrak{U}$

(iv) $\|AB\| \leq \|A\|\|B\|, \quad A, B \in \mathfrak{U}$ (1.4.2)

If the algebra $\mathfrak{U}$ is complete with respect to the norm, that is, if $\mathfrak{U}$ is also a Banach space, then it is called a Banach algebra.

Definition 1.4.3

A mapping A→ A^* of $\mathfrak{U}$ into itself is called an involution or adjoint operation of the algebra $\mathfrak{U}$ if it has the following properties

(i) $(A^*)^* = A, \quad A \in \mathfrak{U}$

(ii) $(AB)^* = B^*A^*, \quad A, B \in \mathfrak{U}$

(iii) $(A + B)^* = A^* + B^*, \quad A, B \in \mathfrak{U}$

(iv) $(\beta A)^* = \bar{\beta}A^*, \quad A, B \in \mathfrak{U}, \quad \beta \in \mathbb{C}$

An algebra with an involution * is called a *- algebra.

A Banach algebra $\mathfrak{U}$ with an involution * is called a Banach * -algebra.

A Banach * -algebra $\mathfrak{U}$ is called a C* algebra if it satisfies $\|x^*x\| = \|x\|^2$, for $x \in \mathfrak{U}$.

Definition 1.4.4 (Sunders, 1987)

Let $\mathfrak{H}$ be a complex Hilbert Space, $\mathcal{B}(\mathfrak{H})$the algebra of all bounded linear operators on $\mathfrak{H}$. A von Neumann algebra is a *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathfrak{H})$ which is self-adjoint, contain the identity operator 1 and is closed in the weak operator topology. Let $\mathcal{M}_+$ denote the positive elements of $\mathcal{M}$, i.e $\mathcal{M}_+ = \{x \in \mathcal{M}: x \geq 0\}$.

Example;

$\mathcal{B}(\mathfrak{H})$ is an example of a von Neumann algebra.

Definition 1.4.5

A linear functional φ on $\mathcal{M}$ is said to be positive if $\varphi(x^*x) \geq 0$ for each $x \in \mathcal{M}$.

Definition 1.4.6 Topologies on $\mathcal{M}$ (Sunders, 1987)

The strong operator topology is the locally convex topology induced by the family of semi norms $\{p_\xi\}$ defined on $\mathcal{M}$ by $p_\xi(x) = \|x\xi\|$, with $x \in \mathcal{M}, \xi \in \mathfrak{H}$.

The $\sigma -$ strong operator topology is the locally convex topology induced by the family of semi norms $\{p_{\xi_n}\}$ defined on $\mathcal{M}$ by $p_{\xi_n}(x) = (\sum\|x\xi_n\|^2)^{1/2}$, with $x \in \mathcal{M}$, $\{\xi_n\} \subset \mathfrak{H}$ such that $\sum\|\xi_n\|^2 < \infty$.

The $\sigma -$weak operator topology is the locally convex topology induced by the family of seminorms $\{p_{\xi_{n,}\eta_n}\}$ defined on $\mathcal{M}$ by $p_{\xi_{n,}\eta_n}(x) = \sum|\langle x\xi_n, \eta_n\rangle|$, with $x \in \mathcal{M}$, $\{\xi_n\}, \{\eta_n\} \subset \mathfrak{H}$, such that $\sum\|\xi_n\|^2 < \infty$, $\sum\|\eta_n\|^2 < \infty$.

The weak operator topology is induced by the family of semi norms $\{p_{\xi,\eta}\}$ defined on $\mathcal{M}$ by $p_{\xi,\eta}(x) = \sum|\langle x\xi, \eta\rangle|$, with $x \in \mathcal{M}, \xi, \eta \in \mathfrak{H}$.

Definition 1.4.7

The predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is the space of all σ- weakly continuous linear functionals on $\mathcal{M}$. We denote the positive part of $\mathcal{M}_*$ by $\mathcal{M}_{*,+}$.

Definition 1.4.8 (Commutant)

Let $\mathcal{M}$ be a subset of $\mathcal{B}(\mathfrak{H})$.We put $\mathcal{M}' = \{x \in \mathcal{B}(\mathfrak{H}); xy = yx \; for \; all \; y \in \mathcal{M}\}$. The space $\mathcal{M}'$ is called commutant the of $\mathcal{M}$ and we denote by $\mathcal{M}'' = (\mathcal{M}')'$ the bicommutant of $\mathcal{M}$.

Proposition 1.4.1 (Bratteli and Robinson, 1979)

For every subset $\mathcal{M}$ of $\mathcal{B}(\mathfrak{H})$ we have

(i) $\mathcal{M}'$ is weakly closed

(ii) $\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(5)} = \cdots$

and $\mathcal{M} \subset \mathcal{M}'' = \mathcal{M}^{(4)} = \cdots$

Proposition 1.4.2 (Bratteli and Robinson, 1979)

Let $\mathcal{M}$ be a self-adjoint subset of $\mathcal{B}(\mathfrak{H})$. Let $\mathfrak{H}_1$ be a closed subspace of the Hilbert space $\mathfrak{H}$ and P be the orthogonal projection onto $\mathfrak{H}_1$. Then $\mathfrak{H}_1$ is invariant under $\mathcal{M}$(in the sense $x\mathfrak{H}_1 \subset \mathfrak{H}_1$ for all $x \in \mathcal{M}$) if and only if $P \in \mathcal{M}'$.

Theorem 1.4.1 (von Neumann density theorem)

Let $\mathcal{M}$ be a $*-$subalgebra of $\mathcal{B}(\mathfrak{H})$ which contains the identity I. Then $\mathcal{M}$ is weakly (strongly) dense in $\mathcal{M}''$.

Theorem 1.4.2 (Bicommutant theorem)

Let $\mathcal{M}$ be a $* -$ sub algebra of $\mathcal{B}(\mathfrak{H})$ which contains the identity I. The following conditions on $\mathcal{M}$ are equivalent.

(i) $\mathcal{M}$ is weakly (strongly) closed.

(ii) $\mathcal{M} = \mathcal{M}''$.

Definition 1.4.9

The center of a von Neumann algebra is the abelian von Neumann subalgebra $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$,

Definition 1.4.10

Given an element $x \in \mathcal{M}$, the smallest projection $P \in \mathcal{M}\ with\ Px = x$ is called the left support of x and is denoted by $s_l(x)$. Similarly, the right support is the smallest projection $Q \in \mathcal{M}\ with\ xQ = x$ and is denoted by $s_r(x)$.

Definition 1.4.11

Let $\mathcal{Z}$ be the center of a von Neumann algebra $\mathcal{M}$.Then $\mathcal{Z}$ is weakly closed. Let P be a projection in $\mathcal{M}$.There exists a least central projection in $\mathcal{M}$ majorizing P.This central projection is called the central support of P, and is denoted by $c(P)$.

Definition 1.4.12

Two projections $P \, and \, Q$ in a von Neumann algebra $\mathcal{M}$ are said to be equivalent if there exist a partially isometric operator $u \in \mathcal{M}$ whose initial domain is the range of P and whose terminal domain is the range of Q.

Definition 1.4.13

A projection P in a von Neumann algebra $\mathcal{M}$ is said to be finite if P$\sim Q \precsim P$ implies $P = Q, \; otherwise$ it is said to be infinite. A projection P is said to be purely infinite if there is no nonzero finite projection $Q \precsim P \in \mathcal{M}$. If zP is infinite for every central projection $z \in \mathcal{M}$ with $zP \neq 0$, then P is called properly infinite.

Definition 1.4.14

A von Neumann algebra $\mathcal{M}$ is said to be finite if the identity projection is finite , infinite if the identity projection is infinite, properly infinite if the identity projection is properly infinite.

Definition 1.4.15

A non-zero projection P in a von Neumann algebras $\mathcal{M}$ is said to abelian if $P\mathcal{M}P$ is commutative.

Proposition 1.4.3 (Sakai,1971)

An abelian projection P is finite.

Definition 1.4.16

(i) A von Neumann algebra $\mathcal{M}$ is said to be of type I if every nonzero central projection in $\mathcal{M}$ majorizes a non zero abelian projection in $\mathcal{M}$.

(ii) If there is no nonzero finite projection in$\mathcal{M}$, that is, if $\mathcal{M}$ is purely infinite then it is said to be of type III .

(iii) If $\mathcal{M}$ has no nonzero abelian projection and if every non zero central projection in $\mathcal{M}$ majorizes a non zero finite projection of $\mathcal{M}$ then it is said to be of type II.

(iv) If $\mathcal{M}$ is finite and of type II , then it is said to be of type II_1.

(v) If $\mathcal{M}$ is of type II and has no nonzero central finite projection, then $\mathcal{M}$ is to be of type II_∞ .

Definition 1.4.17

A von Neumann algebra $\mathcal{M}$ is said to be countably decomposable if every family of mutually orthogonal non-zero projections in $\mathcal{M}$ is at most countable. A projection P in $\mathcal{M}$ is said to be countably decomposable if $P\mathcal{M}P$ is countably decomposable.

Proposition 1.4.4 (Bratteli and Robinson, 1979).

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ .Then the following four conditions are equivalent.

(i) $\mathcal{M}$ is $\sigma - finite.$

(ii) There exists a countable subset of $\mathfrak{H}$ which is separating for $\mathcal{M}.$

(iii) There exists a faithful normal state on $\mathcal{M}.$

(iv) $\mathcal{M}$ is isomorphic with a von Neumann algebra $\pi(\mathcal{M})$ which admits a separating and cyclic vector.

Definition 1.4.18

$\mathcal{M}$ is a quasi-local algebra if there is a net $\{\mathcal{M}_{\Lambda}\}_{\Lambda\in\mathbb{Z}^d}$ of von Neumann algebras such that,

(i) If $\Lambda_1 \geq \Lambda_2$ then $\mathcal{M}_{\Lambda_1} \supseteq \mathcal{M}_{\Lambda_2}$

(ii) $\mathcal{M} = \overline{\cup \mathcal{M}_{\Lambda}}$ where the bar is the uniform closure

(iii) The algebras $\mathcal{M}_{\Lambda}$ have a common identity I.

Definition 1.4.19

A linear map φ on $\mathcal{M}_+$ defined by $\varphi: \mathcal{M}_+ \to [0,\infty]$ Satisfying

(i) $\varphi(x+y) = \varphi(x) + \varphi(y)$ for $x, y \in \mathcal{M}_+$

(ii) $\varphi(\lambda x) = \lambda\varphi(x)$ for $x \in \mathcal{M}_+, \lambda \geq 0$

is called a weight.

(iii) If $\varphi(x) = 0 \Rightarrow x = 0$.Then the weight φ is said to be faithful

(iv) If $\varphi(x) = sup_i \varphi(x_i)$ whenever x is the supremum of a monotone increasing net $\{x_i\}$ in $\mathcal{M}_+$ then the weight φ is said to be normal.

(v) If $\varphi(x^*x) = \varphi(x\,x^*)$ then the weight φ is called a trace

(vi) If $\forall x \in \mathcal{M}_+, \exists\, x_i \in \mathcal{M}$ with $x_i \uparrow x$ $\sigma-$ strongly and $\varphi(x_i) < \infty$. Then the weight φ is semi-finite.

Definition 1.4.20

A state φ on $\mathcal{M}$ is a weight such that $\|\varphi\| = 1$.

A weight φ is said to be finite if $\varphi(x) < \infty$ for all $x \in \mathcal{M}_+$.

Definition 1.4.21 Fundamental to the study of a weight φ is an analysis of certain subspaces of $\mathcal{M}$ defined as follows,

$P_\varphi = \{x \in \mathcal{M}_+ : \varphi(x) < +\infty\}$

$N_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}$

$m_\varphi = N_\varphi^* N_\varphi = \left\{\sum_{i=1}^{n} x_i^* y_i :\ x_i, y_i \in N_{\varphi,}\ \ n = 1,2 \dots\right\}$

P_φ is a hereditary positive cone , that is , $x, y \in P_\varphi$ and $\lambda \in [0, \infty)$ implies that

$\lambda x + y \in P_\varphi$ and $x \in P_\varphi$, $z \in \mathcal{M}_+$with $z \leq x, \Rightarrow z \in P_\varphi$

N_φ is a left ideal in $\mathcal{M}$

m_φ is a self adjoint sub algebra of $\mathcal{M}$

$P_\varphi = m_\varphi^+ = m_\varphi \cap \mathcal{M}_+$, and every element of m_φ is a linear combination of four elements of P_φ.

Theorem 1.4.3 (Haagerup, 1975)

The following theorem characterizes normal weights.

For a weight φ on $\mathcal{M}$, the following conditions are equivalent:

(i) φ is normal

(ii) There exist a monotone increasing net $\{\omega_i : i \in I\} \subseteq \mathcal{M}_{*,+}$ such that

$\omega_i(x) \uparrow \varphi(x)$ for all $x \in \mathcal{M}_+$.

(iii) There exist a family$\{\psi_i : i \in I\} \subseteq \mathcal{M}_{*+}$ such that $\varphi(x) = \sum_{i\in I} \psi_i(x)$ for all $x \in \mathcal{M}_+$.

(iv) $\varphi(x)$ is σ-weakly lower semi continuous, that is if $x_i \to x$, $\sigma -$ weakly,

$x_i, x \in \mathcal{M}_+$ then $\varphi(x) \leq \liminf \varphi(x_i)$.

Definition 1.4.22 (Trunov,1982)

A weight φ is called **locally finite** if $\forall x \in \mathcal{M}_+, \varphi(x) = \infty, \exists\, y \in \mathcal{M}_+$ with $y \leq x$ such that $0 < \varphi(y) < \infty$.

Definition 1.4.23 (Trunov,1982)

A weight φ is called **regular**, if $\forall \omega \in \mathcal{M}_{*+}, \omega \neq 0, \exists\, \omega' \in \mathcal{M}_{*+}, \omega' \neq 0$,

with $\omega' \leq \omega$ such that $\omega' \leq \varphi$.

Definition 1.4. 24

A representation of a von Neumann algebra $\mathcal{M}$ is a pair $(\mathfrak{H}, \pi)$ where $\mathfrak{H}$ is a complex Hilbert space and π is a $* -$ homomorphism of $\mathcal{M}$ into $\mathcal{B}(\mathfrak{H})$.The representation $(\mathfrak{H}, \pi)$ is said to be faithful if and only if π is a $* -$isomorphism between $\mathcal{M}$ and $\pi(\mathcal{M})$.

Theorem1.4.4 (Sunders, 1987)

Let $\varphi \in \mathcal{M}_{*+}$ be a faithful, normal state on $\mathcal{M}$.Then there exist a triple $(\mathfrak{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ where,

(i) π_φ is a $* -$ algebra homomorphism of $\mathcal{M}$ into $\mathcal{B}(\mathfrak{H})$

(ii) $\Omega_\varphi \in \mathfrak{H}_\varphi$ and $\mathfrak{H}_\varphi = \overline{\pi_\varphi(\mathcal{M})\Omega_\varphi}$

(iii) $\langle \pi_\varphi(x)\Omega_\varphi, \Omega_\varphi \rangle = \varphi(x)$, for all $x \in \mathcal{M}$

The image $\pi_\varphi(\mathcal{M})$ is a von Neumann algebra of operators on $\mathfrak{H}_\varphi$, π_φ is an isometric (norm-preserving) $\sigma -$ weakly continuous homomorphism of $\mathcal{M}$ onto $\pi_\varphi(\mathcal{M})$.

Theorem 1.4.5 (Sunders, 1987)

Let $\varphi \in \mathcal{M}_{*+}$ be a faithful, normal, semifinite weight on $\mathcal{M}$. Let $P_\varphi, N_\varphi, and\ m_\varphi$ be the associated subspaces of $\mathcal{M}$, as in definition 1.4.21.Then there exist a triple $(\mathfrak{H}_\varphi, \pi_\varphi, \eta_\varphi)$ where

(i) $\mathfrak{H}_\varphi$ is a Hilbert space

(ii) π_φ is a *- algebra homomorphism of $\mathcal{M}$ into $\mathcal{B}(\mathfrak{H})$

(iii) $\eta_\varphi : N_\varphi \to \mathfrak{H}_\varphi$ is a linear map such that

$\langle \eta_\varphi(x), \eta_\varphi(y) \rangle = \varphi(y^* x)$, such that $\{\pi_\varphi(z)\eta_\varphi(x): x \in N_\varphi,\ z \in \mathcal{M}\}$ is dense in $\mathfrak{H}_\varphi$. For all $x, y \in N_\varphi$ and $z \in \mathcal{M}$. Such a representation is unique up to unitary isomorphism.

Definition 1.4.25

A set $S \subset \mathfrak{H}$ is said to be

(i) cyclic for $\mathcal{M}$ if $[\mathcal{M}S] = \mathfrak{H}$

(ii) separating for $\mathcal{M}$ if for $x \in \mathcal{M}, x = 0$ if and only if $xS = \{0\}$

where $[\mathcal{M}S]$ denotes the closed linear span of $\mathcal{M}S$.

Proposition 1.4.5 (Sunders,1987)

Let S_0 and F_0 be the conjugate-linear operators, with domains $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$, respectively,with Ω cyclic for $\mathcal{M}$ defined by $S_0(x\Omega) = x^*\Omega, F_0(x\Omega) = x^*\Omega$. Then S_0 and F_0 are densely defined closable operators; their closures, denoted by S and F, respectively, satisfy $S = \overline{S_0} = F_0^*$ and $F = \overline{F_0} = S_0^*$

Definition 1.4.26

Let $\Delta = FS = S^*S$. Then Δ is invertible,with inverse $\Delta^{-1} = SF = SS^*$. S and Δ have dense range and thus $\Delta^{1/2}$ have dense range, we have the following $S = J\ (S^*S)^{1/2}$, where J is a self-adjoint partial anti -isometry operator on $\mathfrak{H}$.

(i) $S = J\Delta^{1/2}$, polar decomposition of S.

(ii) $F = S^* = \Delta^{1/2}J$.

(iii) Let x be in the domain of S. Then $I = J^2$ and $\Delta^{-1} = J\Delta J$.

Δ is called the modular operator and J the modular conjugation.

Theorem 1.4.6 Tomita –Takesaki

(i) $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad \forall\ t \in \mathbb{R}$

(ii) $J\mathcal{M}J = \mathcal{M}'$

Put $\sigma_t^{\varphi}(x) = \left(\Delta_{\varphi}^{it}\, x\, \Delta_{\varphi}^{-it}\right)$ $x \in \mathcal{M},\ \forall\ \ t \in \mathbb{R}$

This defines a one parameter group of automorphisms of $\mathcal{M}$.

Definition 1.4.27 (Sunders)

Let $\mathfrak{U}$ be an involutive algebra over the complex number field $\mathbb{C}$ with the involution

$\xi \in \mathfrak{U} \longrightarrow \xi^{\#} \in \mathfrak{U}$. $\mathfrak{U}$ is called a generalized Hilbert algebra if $\mathfrak{U}$ admits an inner product $\langle \xi, \eta \rangle$ satisfying following conditions:

(i) $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^{\#}\zeta \rangle$, $\xi, \eta, \zeta \in \mathfrak{U}$

(ii) For each $\xi \in \mathfrak{U}$,the map $\eta \in \mathfrak{U} \longrightarrow \xi\eta \in \mathfrak{U}$ is continuous.

(iii) The subalgebra $\mathfrak{U}^{2} = \{\sum_{i=1}^{n} \xi_i \eta_i : \xi_i, \eta_i \in \mathfrak{U}, n = 1,2, \dots\}$ of $\mathfrak{U}$, is spanned by the elements $\xi\eta$ with $\xi, \eta \in \mathfrak{U}$, is dense in $\mathfrak{U}$.

(iv) The involution $\xi \longrightarrow \xi^{\#} \in \mathfrak{U}$ is preclosed.

Suppose $\mathfrak{U}$ is a generalized Hilbert algebra. Let $\mathfrak{H}$ be the Hilbert space obtained by completion of $\mathfrak{U}$. To each $\xi \in \mathfrak{U}$, there corresponds a unique bounded operator $\pi(\xi)$ on $\mathfrak{H}$ defined by $\pi(\xi)\eta = \xi\eta$, $\eta \in \mathfrak{U}$. $\pi(\mathfrak{U})$, the set of bounded operators satisfying condition (i)-(iii) is a non-degenerate self-adjoint algebra of operators on $\mathfrak{H}$. $\pi(\mathfrak{U})''$ is the von Neumann algebra generated by $\pi(\mathfrak{U})$. $\pi(\mathfrak{U})''$ is called a left von Neumann algebra denoted by $\mathfrak{L}(\mathfrak{U})$.

Since the involution $\xi \in \mathfrak{U} \longrightarrow \xi^{\#} \in \mathfrak{U}$ with domain $\mathfrak{D}^{\#} = \{\xi \in \mathfrak{U} \colon \xi^{\#} \in \mathfrak{U}\}$ is preclosed and therefore has a closure whose adjoint involution is given by $\eta \in \mathfrak{U} \rightarrow \eta^{b} \in \mathfrak{U}$ with domain $\mathfrak{D}^{b} = \{\eta \in \mathfrak{U} \colon \eta^{b} \in \mathfrak{U}\}$, we have the following definition.

Definition 1.4.28

A vector $\eta \in \mathfrak{D}^{b}$ is said to be π'-bounded if $\xi \in \mathfrak{U} \longrightarrow \pi(\xi)\eta$ is bounded. If this is the case, then the map $\xi \in \mathfrak{U} \longrightarrow \pi(\xi)\eta \in \mathfrak{H}$ extends to a bounded operator $\pi'(\eta)$.The set of π'- bounded elements is denoted by $\mathfrak{U}^{b}$ and given by

$\mathfrak{U}^{b} = \{\eta \in \mathfrak{D}^{b}; \exists c > 0 \ni \|\pi(\xi)\eta\| \le c\|\xi\| \ \ \forall\, \xi \in \mathfrak{U}\}$

and also the set of π – bounded elements is denoted by $\mathfrak{U}^{\#}$ and given by

$\mathfrak{U}^{\#} = \{\xi \in \mathfrak{D}^{\#}; \exists c > 0 \ni \|\pi'(\eta)\xi\| \le c\|\eta\| \ \ \forall\, \eta \in \mathfrak{U}^{b}\}$

Definition 1.4.29

The generalized Hilbert algebra $\mathfrak{U}$ is said to be achieved if $\mathfrak{U} = \mathfrak{U}^{\#}$.

Definition 1.4.30 Kubo-Martins-Schwinger boundary conditions

A σ_t- invariant positive linear functional φ of $\mathcal{M}$ is said to satisfy the Kubo-Martins-Schwinger boundary condition for $(\sigma_t)_{t\in\mathbb{R}}$ if for any pair $x, y \in \mathcal{M}$,there exist a bounded function $F_{xy}: \{\lambda \in \mathbb{C}: 0 \leq im\,\lambda < 1\} \longrightarrow \mathbb{C}$ continuous on and analytic in the strip $0 \leq im\,\lambda < 1$ with boundary values

$$F_{xy}(t) = \varphi(\sigma_t(x)y)\ and\ F_{xy}(t+i) = \varphi(y\sigma_t(x))$$

Lemma 1.4.1 (Sunders,1987)

$\varphi \in \mathcal{M}_{*+}$ satisfies the KMS conditions with respect to the σ -weakly continuous one parameter group $\left(\sigma_t^{\varphi}\right)_{t\in\mathbb{R}}$ if φ is invariant with respect to σ_t, that is $\varphi \circ \sigma_t^{\varphi} = \varphi$.

Definition 1.4.31

Let φ be a linear map from a von Neumann algebra $\mathcal{M}$ to a von Neumann algebra $\mathcal{N}$ and consider the algebra of $n\times n$ matrices with entries from $\mathcal{M}$ and $\mathcal{N}$ denoted by$M_n(\mathcal{M})$ and $M_n(\mathcal{N})$respectively. The linear map φ is called completely positive, if

$\varphi_n \otimes I: M_n(\mathcal{M}) \to M_n(\mathcal{N})$, defined by $\varphi_n\left(x \otimes E_{ij}\right) = \varphi(x) \otimes E_{ij}$, is positive for all n. Where E_{ij}, i, j =1,2,...n are matrix units spanning $M_n(\mathbb{C})$.

Theorem 1.4.7 (Takesaki, 1979)

(a) Let $\mathcal{N}$ be a $C*-$ algebra and $\mathfrak{H}$ a Hilbert space. If $\{\pi, \mathfrak{R}\}$ is a representation of $\mathcal{N}$,

V is a bounded linear operator of $\mathfrak{H}$ into $\mathfrak{R}$, then the map $T: x \in \mathcal{N} \to V^*\pi(x)V \in \mathcal{B}(\mathfrak{H})$ is completely positive .

(b) If T is a completely positive map of a $C*-$ algebra A into another $C*-$ algebra B, then we have $T(a)^*T(a) \leq \|T\|T(a^*a)$, $a \in A$.

Definition 1.4.32

(i) A closed, densely defined linear operator on $\mathfrak{H}$ is said to be affiliated to $\mathcal{M}$ if $ux \subseteq xu$ for every unitary $u \in \mathcal{M}^{'}$. If x is affiliated to $\mathcal{M}$, we write $x\, \eta\mathcal{M}$. If x is bounded then we say that x is in $\mathcal{M}$

(ii) A linear set $\mathfrak{D}$ in $\mathfrak{H}$ is said to be associated with $\mathcal{M}$, if $u(\mathfrak{D}) \subseteq \mathfrak{D}$ for every unitary $u \in \mathcal{M}^{'}$. We write $\mathfrak{D}\eta\mathcal{M}$.

(iii) Let $\mathfrak{D}$ be a linear subset of $\mathfrak{H}$.Then $\mathfrak{D}$ is said to be strongly dense in $\mathfrak{H}$ with respect to $\mathcal{M}$ if

(a) $\mathfrak{D}\eta\mathcal{M}$.

(b) There is a sequence $\{\mathfrak{D}_n\}$ of subspaces of $\mathfrak{H}$, with $\mathfrak{D}_n\eta\mathcal{M}$ such that $\mathfrak{D}_n \subset \mathfrak{D}$ and,

(c) the projection operator of $\mathfrak{H}$ onto the orthogonal complement $\mathfrak{D}_n^{\perp}$ of $\mathfrak{D}_n$ is a finite projection in $\mathcal{M}$ and $\mathfrak{D}_n^{\perp} \downarrow 0$. We say that $\{\mathfrak{D}_n\}$ defines $\{\mathfrak{D}\}$.

(iv) A closed, densely defined linear operator x on $\mathfrak{H}$ is said to be measurable with respect to $\mathcal{M}$ provided that

(a) $x\, \eta\mathcal{M}$ and

(b) x has a strongly dense domain

Definition 1.4.33

(a) Let $\mathcal{M}_{proj}$ be the set of projections in $\mathcal{M}$, a measure τ is a nonnegative mapping $\tau: \mathcal{M}_{proj} \to \mathbb{R}_+$ such that,

(i) $\tau(0) = 0$

(ii) $\tau(\sum P_n) = \sum \tau(P_n)$, for any countable set $\{P_n\}$ of mutually orthogonal projections in $\mathcal{M}_{proj}$.

(b) An integral φ on $(\mathfrak{H}, \mathcal{M})$ is a faithful, nonnegative linear functional $\varphi: \mathcal{M} \to \mathbb{C}$, such that the restriction of φ to $\mathcal{M}_{proj}$ is a measure.

(c) A sequence $\{x_n\}$ of measurable operators(see definition, 1.4.32) is said to converge in measure to a measurable operator x if given $\delta > 0$, there is a sequence $\{P_n\}$ of projections in $\mathcal{M}_{proj}$ such that $\|(x_n - x) \times P_n\| < \delta$ and $\varphi(P_n^{\perp}) \to 0$, where $\|\cdot\|$ is the operator norm on $\mathcal{M}$.

(d) Suppose h is a densely defined closed operator associated with $\mathcal{M}$, h is said to be locally measurable (with respect to $\mathcal{M}$) if the following equivalent conditions hold:

(i) There exists a sequence (e_n) of projections in the center of $\mathcal{M}$ such that $e_n \nearrow I$ and all operators he_n are measurable in the sense of Segal.

(ii) For each $x \in \mathcal{M}$, the operator xh is closed.

A gauge space Γ is a system $(\mathfrak{H}, \mathcal{M}, \tau)$ composed of a complex Hilbert space $\mathfrak{H}$, a von Neumann algebra $\mathcal{M}$ and a normal trace τ.The normal trace τ associated with Γ is called a gauge, and the gauge space Γ is finite if $\tau(I) < \infty$, where I denotes the identity operator.

Definition 1.4.34

A projection P will be said to be associated with Γ if it is $\mathcal{M}$, and is said to be metrically finite or τ −finite if $\tau(P) < \infty$.

Definition 1.4.35

A gauge space is called regular when the only projection of gauge zero is the zero projection.

A projection on which a gauge vanishes is called a null projection.

Definition 1.4.36

Let N be the maximal central null projection then $(I - N)\mathfrak{H}$ is called the carrier of τ.

Definition 1.4.37

Let $(\mathfrak{H}, \mathcal{M}, \tau)$ be a gauge space. Then a sequence $\{T_n\}$ of measurable operators is said to converge nearly everywhere (n.e) to a measurable operator T if given $\varepsilon > 0$, there is a sequence $\{P_n\}$ of projections in $\mathcal{M}$ such that $P_n \uparrow I$ as $n \uparrow \infty$, $\|(T_n - T) \times P_n\| < \varepsilon$ and $P_n^{\perp}$ is algebraically finite.

Definition 1.4.38

A sequence $\{T_n\}$ of measurable operators converges metrically nearly everywhere (m.n.e) to a measurable operator T with respect to a gauge space $(\mathfrak{H}, \mathcal{M}, \tau)$, if it converges nearly everywhere (n.e)to T on the carrier of τ.

Definition 1.4.39

The rank of an operator T with respect to the gauge space is defined as the gauge of the closure of the range of T. T is said to be elementary, if it is everywhere defined and its rank is finite, or nearly everywhere zero.

Definition 1.4.40

A measurable operator T on the gauge space is called integrable if it is the limit metrically nearly everywhere (m.n.e) of a sequence $\{T_n\}$ of elementary operators that is Cauchy in the set of all integrable operators $L_1(\mathfrak{H}, \mathcal{M}, \tau)$. The integral or trace of T, denoted by $\tau(T)$ is defined as $\lim_n \tau(T_n)$.

Definition 1.4.41

A strongly continuous one parameter semigroup $(P_t)_{t \geq 0}$ on a Hilbert space $\mathfrak{H}$ is a family P_t of linear maps satisfying,

$P_0 = 1$

$P_s \, P_t = P_{s+t}$

$\lim_{t \to 0} P_t u = u,$ where $u \in \mathfrak{H}$

Definition 1.4.42

A semigroup $(P_t)_{t \geq 0}$ on a von Neumann algebra $\mathcal{M}$ is said to be

(i) a contraction semigroup if $\|P_t x\| \leq \|x\| \;\; \forall \, x \in \mathcal{M}, t \geq 0$.

(ii) uniformly continuous if $\lim_{t\to 0}\|P_t - I\| = 0.$

(iii) strongly continuous or C_0 −semigroup if $\lim_{t\to 0}\|P_t x - x\| = 0$,

$\forall\ x \in \mathcal{M}.$

Lemma 1.4.2 (Davies,1976)

Every weakly continuous one parameter contraction semigroup $(P_t)_{t\geq 0}$ on a Hilbert space $\mathfrak{H}$ is also strongly continuous.

Definition 1.4.43

A linear operator G on $\mathcal{M}$ is said to be the generator of a C_0 −semigroup $(P_t)_{t\geq 0}$ if:

$G(f) = \lim_{t\to 0}\frac{1}{t}(P_t f - f)$, exists for each f in

$\mathfrak{D}(G) = \left\{f \in \mathcal{M}: \lim_{t\downarrow 0}\frac{1}{t}(P_t f - f)\ exists\ in\ \mathcal{M}\right\}$

Lemma 1.4.3 (Ahmed,1991)

If $G \in \mathcal{B}(\mathfrak{H})$, then $P_t = e^{tG},\ \ t \geq 0$, is a uniformly continuous semigroup of operators on $\mathfrak{H}$ and its infinitesimal generator is G.

Theorem 1.4.8(Ahmed,1991)

Let X be a Banach space and $(P_t)_{t\geq 0}$ a C_o- semigroup on X, then there exists constants $M \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|P_t\| \leq Me^{\alpha t},\ \ \forall t > 0.$

Theorem 1.4.9 (Ahmed, 1991)

If $(P_t)_{t\geq 0}$ is a C_o- semigroup on a Banach space X, then for each $x \in X,\ \ t \to P_t x$ is a continuous X-valued function on $[0, \infty)$.

Theorem 1.4.10 (Ahmed, 1991)

Let X be a Banach space and $(P_t)_{t\geq 0}$ a C_o- semigroup on X with G as its infinitesimal generator. Then

i. For $x \in X,\ t \in [0,\infty),\ \lim_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} P_\theta\,(x)d\theta = P_t x.$

ii. For $x \in X,\ t > 0,\qquad \int_0^t P_\theta(x)\,d\theta\ \in D(G).$

iii. For $x \in \mathfrak{D}(G), P_t x \in \mathfrak{D}(G)$ and $\frac{d}{dt}(P_t x) = GP_t x = P_t\,Gx.$

iv. For $x \in \mathfrak{D}(G),\quad t \geq s \geq 0\,, P_t x - P_s x = \int_s^t GP_t x\,d\tau = \int_s^t P_t\,Gx\ d\tau.$

v. The domain of G is dense in X that is $\overline{\mathfrak{D}(A)} = X.$

vi. G is a closed operator or equivalently the

$Graph\,(G) = \{(x,y) \in X \times X : y = Gx\}$ is a closed subset of $X \times X$.

Definition 1.4.44

Let A be a linear, not necessarily bounded, operator on $\mathfrak{H}$. The set $\rho(A)$ defined by $\{\lambda \in \mathbb{C}: (\lambda I - A)^{-1} \in \mathcal{B}(\mathfrak{H})\}$ is called the resolvent set of the operator A. If $\rho(A)$ is non-empty then for $\lambda \in \rho(A)$, $R(\lambda, A) \equiv (\lambda I - A)^{-1}$is called the resolvent of the operator A corresponding to λ.

Lemma 1.4.4 (Ahmed, 1991)

Let G be the infinitesimal generator of a C_o- semigroup $(P_t)_{t\geq 0}$ of contractions on a Banach space $\mathfrak{H}$ then,

(a) For every $\lambda > 0$ the operator $R_\lambda x = \int_0^\infty e^{-\lambda t} P_t\,(x)dt$ is defined on all of $\mathfrak{H}$ and

$R_\lambda x \in D(G)$, for every $x \in \mathfrak{H}$.

(b) $R(\lambda, G) \equiv (\lambda I - G)^{-1} = R_\lambda$ for $\lambda > 0$, $\rho(G) \supset (0,\infty)$ and $\|R(\lambda, G)\| < \frac{1}{\lambda}$, for

$\lambda > 0.$

Theorem Hille-Yosida 1.4.11 (Ahmed, 1991)

Let $\mathfrak{H}$ be a Hilbert space and G be a linear, not necessarily bounded, operator on $\mathfrak{H}$ with domain $D(G)$ and range $R(G)$ in $\mathfrak{H}$. Then G is the infinitesimal generator of a C_o-semigroup of contractions $(P_t)_{t\geq 0}$ on $\mathfrak{H}$ if and only if,

(i) G is closed, $D(G))$ is dense in $\mathfrak{H}$.

(ii) $\rho(G) \supset (0, \infty)$ and $\|R(\lambda, G)\| < \frac{1}{\lambda}$, for $\lambda > 0$.

Definition 1.4.45

A quantum sub-Markov semigroup, or quantum dynamical semigroup (q.d.s) on a von Neumann algebra $\mathcal{M}$, is a one parameter family $(P_t)_{t\geq 0}$ of linear maps of $\mathcal{M}$ into itself satisfying.

(a) $P_t(x) = x$ for all $x \in \mathcal{M}$.

(b) Each $P_t(.)$ is completely positive.

(c) $P_t(P_s) = P_{t+s}$ for all t, s$\geq$ 0.

(d) $P_t(\mathrm{I}) \leq \mathrm{I}\ for\ all\ t \geq 0$.

(e) For each $x \in \mathcal{M},\ the\ map\ t \to P_t(x)$ is σ-weakly continuous on $\mathcal{M}$

(f) P_t is a normal operator on $\mathcal{M}$ for all $t \geq 0$, i.e. for every increasing net $(a_\alpha)_\alpha$ in $\mathcal{M}$ with l.u.$b\ a_\alpha = a \in \mathcal{M}$, we have l.u.$b\ P_t(a_\alpha) = P_t(a)$.

Definition 1.4.46

(i) A quantum dynamical semigroup is called a quantum Markov semigroup if $P_t(1) = 1\ \ for\ all\ t \geq 0$.

(ii) A von Neumann subalgebra $p\mathcal{M}p$ reduces a quantum Markov semigroup P_t if and only if $P_t(p) = p$ for all positive t, where p is a projection.

(iii) A dynamical semigroup is said to be irreducible if it is not reduced by any proper a von Neumann subalgebra.

CHAPTER 2

LITERATURE REVIEW

2.0 Introduction

In this chapter we discussed the development of noncommutative L_p-spaces, conditional expectations, quantum dynamical semigroups and quantum entanglement. Segal developed a noncommutative theory of integration in which the measure is required to be unitarily invariant and hence central. This is an extension of classical L_p-spaces. We present other types of such extensions in this chapter. A noncommutative extension of the theory of conditional expectations by Umegaki (1954), together with the notions of the generalized conditional expectations developed by Accardi and Cecchini (1982) is also discussed. Conditional expectation is necessary for a formal study of quantum stochastic dynamics for spin system on a lattice. We give a brief account of quantum stochastic dynamics and quantum entanglement, together with a few related results respectively.

2.1 Noncommutative L_p-spaces

In 1953, Irving Segal initiated and developed the theory of noncommutative L_p- spaces for a semifinite von Neumann algebra $\mathcal{M}$ having a faithful normal semifinite trace τ. He defined the noncommutative $L_p -$ spaces as follows;

Definition 2.1

For a pair $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a semifinite von Neumann algebra together with a faithful normal semifinite trace τ defined on $\mathcal{M}$, let $L_1(\tau)$ be the space of integrable operators, this is a Banach space of closed, densely defined (in general unbounded) linear operators on $\mathfrak{H}$ affiliated with $\mathcal{M}$, the norm $\|x\|_1 = \underset{\|s\|\leq 1}{lub}\, \tau|sx|$, $s \in \mathcal{M}$, is called the L_1-norm. The collection of all square integrable operators $L_2(\tau)$ is the Banach space defined by the set

$$L_2(\tau) = \{x \in \mathcal{M} : \|x\|_2 < \infty\}, \quad \text{with the } L_2\text{-norm} \quad \|x\|_2 = \tau(|x|^2)^{\frac{1}{2}}\,. \;.$$

For 1$\leq p < \infty$, we have the Banach space of pth- integrable operators defined by the set

$$L_p(\tau) = \{x \in \mathcal{M} : |x|^p \in L_1(\tau)\},$$

with the $L_p -$ norm $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$, for $p = \infty$, we have $\|x\|_\infty = \|x\|$, and $L_{\infty,}(\tau)$ is identified with $\mathcal{M}$.

Much later, after a decade and half, Nelson (1974) realized a simplified approach to the construction of the Segal spaces, this construction is based on Stinespring's notion of convergence in measure of measurable operators. Yeadon (1974), in his paper on noncommutative $L_p -$spaces, defined the spaces concretely as spaces of unbounded operators. With the celebrated theory of Tomita –Takesaki, Haagerup (1979), developed L_p -spaces on the cross product von Neumann algebra $\mathcal{M}$. We have the following:

Definition 2.2

Let $R(\mathcal{M}, \sigma_t^\varphi)$ denote the cross product of $\mathcal{M}$ with the modular automorphism group of φ, that is the von Neumann algebra acting in $L_2(\mathbb{R}; \mathfrak{H}) \backsimeq L_2(\mathbb{R}) \otimes \mathfrak{H}$ and generated by the operators $\pi(x)$, $x \in \mathcal{M}$ and $\lambda(s), s \in \mathbb{R}$, where

$$(\pi(x)\xi)(t) = \sigma_{-t}^\varphi(x)\xi(t)$$

$$(\lambda(s)\xi) = \xi(t - s)\;, \quad \text{for } \xi \in L_2(\mathbb{R}; \mathfrak{H}).$$

Denote by $(\theta_s)_{s\in\mathbb{R}}$ the dual action of $\mathbb{R}$ on $R\big(\mathcal{M}, \sigma_t^\varphi\big)$ defined by

$$\theta_s\big(\pi(x)\big) = \pi(x)$$

$$\theta_s\big(\lambda(s)\big) = e^{ist}\lambda(s).$$

There is a canonical normal faithful semifinite trace τ on $R(\mathcal{M}, \sigma_t^{\varphi})$ satisfying $\tau \circ \theta_s = e^{-s}\tau$. If ϕ is a normal weight on $\mathcal{M}$, we denote by $\tilde{\phi}$ the corresponding dual weight on $R(\mathcal{M}, \sigma_t^{\varphi})$. Let h_ϕ be the generalized positive operator satisfying $\tilde{\phi} = \tau(h_\phi \cdot)$. Then $\lambda(t) = h_\varphi^{it}$, for $t \in \mathbb{R}$.

Put $tr(h_\phi) = \phi(I)$.

Denote by $\widehat{\mathcal{M}}$ the algebra of τ- measurable operators associated with $R(\mathcal{M}, \sigma_t^{\varphi})$ equipped with the measure topology. An affiliated operator h is called a τ- measurable operator if its domain is τ- dense, i.e $\forall \delta > 0\ \exists P \in \widehat{\mathcal{M}}_{proj} : P\mathfrak{H} \subseteq D(h) and\ \ \tau(I - P) \leq \delta$, where $\widehat{\mathcal{M}}_{proj}$ is the lattice of projections in $\widehat{\mathcal{M}}$.

The L_p spaces of Haagerup are defined by $L_p(\mathcal{M}) = \{x \in \widehat{\mathcal{M}} : \forall s \in \mathbb{R},\ \theta_s(x) = e^{-s/p}x\}$, $p \in [1, \infty]$, with norm $\|x\|_p = tr(|x|^p)^{1/p}$ and $x \in L_p(\mathcal{M})$. Then $L_1^+(\mathcal{M}) = \{h_\phi : \phi \in \mathcal{M}_*^+\}$, and $L_\infty(\mathcal{M}) = \pi(\mathcal{M}) \simeq \mathcal{M}$, with norm $\|x\|_\infty$.

The Haagerup L_p -space has application in the general description of dynamics of infinite quantum systems.

Trunov (1979), studied the Segal space and extended it to the L_p −spaces of bilinear forms using the representation of a faithful normal state φ, that is, $\varphi(x) = \tau(x.h_\varphi) = \tau(h_\varphi x)$, $x \in \mathcal{M}$, where $h_\varphi \geq 0$ is a uniquely determined nonsingular self-adjoint operator in $L_1(\tau)$. For $x \in \mathcal{M}$ and $1 \leq p < \infty$,we have $h_\varphi^{\frac{1}{2p}}.x.h_\varphi^{\frac{1}{2p}} \in L_p(\tau)$, where $L_p(\tau)$ is the Segal $L_p -$ space. We have the following;

Definition 2.3

For each $1 \leq p < \infty$ let the function,

$$x \to \|x\|_{L_p(\varphi)} = \tau\left(\left|h_\varphi^{\frac{1}{2p}}.x.h_\varphi^{\frac{1}{2p}}\right|^p\right)^{1/p}$$

be defined on $\mathcal{M}$, and write $\|x\|_\infty = \|x\|$, $x \in \mathcal{M}$.

Remark 2.1

This mapping $x \to \|x\|_p$ does not depend on the choice of the faithful normal semi-finite trace τ and is a norm on $\mathcal{M}$.

Trunov then applies the Tomita-Takesaki theory of modular Hilbert algebras to define his functional $\gamma(x) \in \mathcal{M}_*$ that is, the canonical embedding $\gamma: \mathcal{M} \to \mathcal{M}_*$ determined by the state φ, for each $x \in \mathcal{M}$, the functional $\gamma(x) \in \mathcal{M}_*$ is defined by the equation

$\gamma(x)y = \langle J\pi(y)^* J\hat{x}, \hat{1}\ \rangle,\ \ y \in \mathcal{M}$. The mapping γ is a positive linear bijection of $\mathcal{M}$ onto a dense subset of the space $\mathcal{M}_*$, for any $x, y \in \mathcal{M}$, where $\pi: \mathcal{M} \to \mathcal{B}(\mathfrak{H})$ is defined by $\pi(x)\hat{y} = \widehat{xy}$, $x, y \in \mathcal{M}$, is a faithful normal $*$ −representation of $\mathcal{M}$ induced by φ and J is an antilinear isometry on $\mathfrak{H}$.

Remark 2.2

From the above definition we see that for $x, y \in \mathcal{M}$

$$\gamma(x)y = \tau\left(h_\varphi^{\frac{1}{2}}.x.h_\varphi^{\frac{1}{2}}\ y\right),$$

and $\|x\|_{L_1(\varphi)} = \|\gamma(x)\| = \tau\left|h_\varphi^{\frac{1}{2}}.x.h_\varphi^{\frac{1}{2}}\right|$, with $L_1(\varphi) = \{x \in \mathcal{M}: \|x\|_1 < \infty\}$.

The inner product of the $L_2(\varphi)$ space is given by $\langle x, y\rangle = \tau\left(\left(h_\varphi^{\frac{1}{4}}.y.h_\varphi^{\frac{1}{4}}\right)^* \left(h_\varphi^{\frac{1}{4}}.x.h_\varphi^{\frac{1}{4}}\right)\right)$

and the norm is given by

$$\|x\|_2 = \tau\left(\left(h_\varphi^{\frac{1}{4}}.x.h_\varphi^{\frac{1}{4}}\right)^* \left(h_\varphi^{\frac{1}{4}}.x.h_\varphi^{\frac{1}{4}}\right)\right)^{\frac{1}{2}}, \quad L_2(\varphi) = \{x \in \mathcal{M}: \|x\|_2 < \infty\}$$

Corollary 2.1

If $1 \leq p < q \leq \infty, then\ L_q(\varphi) \subset L_p(\varphi) and\ \|a\|_p \leq \|a\|_q.$

$a \in L_q(\varphi)$.

Kosaki (1981) gave a construction using an injective map of $\mathcal{M}$ the von Neumann algebra into its predual $\mathcal{M}_*$, that is, $x \to x.\varphi$ and then applied the theory of complex interpolation spaces. In the same paper Kosaki showed that the spaces he constructed are isomorphic to the Haagerup spaces. Terp (1981) studied the interpolation spaces between the von Neumann algebra $\mathcal{M}$ and its predual $\mathcal{M}_*$. Tikhonov (1982), constructed L_P spaces with respect to a weight on a von Neumann algebra. Zolotarev (1982) studied the L_p spaces of Trunov and used the idea of Kosaki to give a construction of L_p-spaces with respect to a state over a semifinite von Neumann algebra.

2.2 Conditional Expectations

The notion of conditional expectation was first extended to the noncommutative case by Umegaki (1954). In his extensive work, he showed the existence of a conditional expectation when the von Neumann algebra has a faithful normal tracial state τ .He stated the properties as follows.

Theorem 2.1 (Umegaki, 1954)

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}_1$ a von Neumann subalgebra of $\mathcal{M}$.

The mapping $x \in \mathcal{M} \rightarrow E(x) \in \mathcal{M}_1$, have the following properties:

(i) $E(\alpha x + \beta y) = \alpha \mathrm{E}(x) + \beta \mathrm{E}(y), \quad \alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{M}$

(ii) $x \geq 0 \Rightarrow \mathrm{E}(x) \geq 0, \quad x \in \mathcal{M}$

(iii)$x \geq 0$ and $\mathrm{E}(x) = 0$ $implies$ $x = 0, \quad x \in \mathcal{M}$

(iv)$\|E(x)\| \leq \|x\|, \quad x \in \mathcal{M}$

(vii) $\mathrm{E}(x)^* \mathrm{E}(x) \leq \mathrm{E}(x^* x), \quad x \in \mathcal{M}$

(viii)$\mathrm{E}(\mathrm{E}(x)y) = \mathrm{E}\big(x\mathrm{E}(y)\big) = \mathrm{E}(x)\mathrm{E}(y)$, with $x, y \in \mathcal{M}$

(ix) If x_i weakly strongly to x,($x_i \nearrow x$) implies that $\mathrm{E}(x_i)$ converges weakly to $\mathrm{E}(x)$ that is $\mathrm{E}(x_i) \nearrow \mathrm{E}(x)$. $E(x)$ is strongly and weakly continuous on the unit sphere of $\mathcal{M}$.

(x) $\mathrm{E}(xy) = \mathrm{E}(yx)$ for $x \in L_1(\mathcal{M}) \quad y \in \mathcal{M}_1' \cap \mathcal{M}$

(xi) $\tau\big(xE(\mathrm{y})\big) = \tau(\mathrm{E}(x)\mathrm{y})$, τ is a faithful normal tracial state.

On the other hand, Tomiyama(1957) showed that each projection of norm one of a C^* algebra onto its C^* subalgebra has most of the properties of a conditional expectation stated as follows:

Theorem 2.2 (Tomiyama,1957)

Let $\mathcal{A}$ be a C* algebra with a unit and $\mathcal{B}$ a C* subalgebra of $\mathcal{A}$.If P is a projection of norm one from $\mathcal{M}$ onto $\mathcal{N}$, then

(i) $x \geq 0 \Rightarrow P(x) \geq 0, \quad x \in \mathcal{A}$

(ii) $P(\,axb\,) = aP(x)b, \quad x \in \mathcal{A} \quad a, b \in \mathcal{B}$

(iii) $P(x^*)P(x) \leq P(x^* x), \quad x \in \mathcal{A}$

(iv) $P(x^*) = P(x)^*$, $x \in \mathcal{A}$

The theory of conditional expectations developed by Umegaki depends on the existence of tracial states and hence is not applicable to von Neumann algebras of type III. Takesaki (1971) gave the necessary and sufficient conditions for the existence of conditional expectations that are characterized as projections of norm one. His version of proof uses the modular algebra developed by himself and Tomita in 1970. His fundamental result is,

Theorem 2.3 (Takesaki, 1971)

Let $\mathcal{M}$ be a von Neumann algebra, φ a faithful normal semifinite weight on $\mathcal{M}_+$, and $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ on which $\varphi_{/\mathcal{N}} = \dot{\varphi}$ is a semifinite weight. Then the following two statements are equivalent:

(i) $\mathcal{N}$ is invariant under the modular automorphism group σ_t associated with φ .

(ii) There exists a $\sigma - weakly$ continuous faithful projection E of norm one from $\mathcal{M}$ onto $\mathcal{N}$ such that $\dot{\varphi}(x) = \dot{\varphi} \circ E(x), \quad \forall\, x \in m_\varphi$.

where m_φ is a self adjoint sub algebra of $\mathcal{M}$.

The projection E of norm one of $\mathcal{M}$ onto $\mathcal{N}$ is called the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ. The conditional expectation E of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ is also determined by $\varphi(x^* yz) = \varphi\ (x^* E(y) z\,), \quad x, z \in m_\varphi \cap \mathcal{N}, \quad y \in \mathcal{M}$.

with the following properties;

i. $E(x^* x) \geq 0$ $x \in \mathcal{M}$

ii. $E(\, axb\,) = aE(x)b$ $a, b \in \mathcal{N} and\ x \in \mathcal{M}$

iii. $E(x^*)E(x) \leq E(x^* x)$ $x \in \mathcal{M}$.

He gave the form of the conditional expectation as follows:

$$E(x) = \pi_{\mathcal{N}}^{-1}(P \pi_{\mathcal{M}}(x)\, P), \qquad x \in \mathcal{M},$$

where π_M (resp., π_N) is the isomorphism of $\mathcal{M}$(resp., $\mathcal{N}$) onto the left von Neumann algebra $\mathfrak{L}(\mathfrak{U})$ of $\mathfrak{U}$ (resp., the left von Neumann algebra $\mathfrak{L}(\mathfrak{B})$ of $\mathfrak{B}$).

The result of Takesaki was independently proved by Golodez (1972) .

It was observed that for questions in the theory of quantum stochastic process, the characterization of conditional expectation as projections was inadequate, because for a general von Neumann algebra, a normal faithful norm one projection with those properties described by Umegaki rarely exists. Accardi and Cecchini (1982) constructed a conditional expectation that always exist. This conditional expectation known as the generalized conditional expectation is no longer a projection. The form is given by,

$$E(x) = U^* x U, \qquad\qquad x \in \mathcal{M}$$

where U is a partial isometry from the Hilbert space $\mathfrak{H}$ of $\mathcal{M}$ into the Hilbert space $\mathcal{K}$ of $\mathcal{N}$.

Theorem 2.4 (Accardi and Cecchini, 1982)

Let $\mathcal{M}$, $\mathcal{N}$ be von Neumann algebras with $\mathcal{N} \subseteq \mathcal{M}$ and let φ be a faithful normal semifinite weight on $\mathcal{M}_+$ whose restriction φ_0 on $\mathcal{N}_+$ is semifinite. Let $\mathfrak{U} \supseteq \mathfrak{B}$ be the achieved generalized Hilbert algebras with completion $\mathfrak{H}$, $\mathcal{K}$, associated to $\mathcal{M}, \varphi$ and $\mathcal{N}, \varphi_0$ respectively: Let $J_{\mathfrak{U}}$, $J_{\mathfrak{B}}$ be the corresponding conjugation operators called the Tomita involutions and P the orthogonal projection from $\mathfrak{H}$ onto $\mathcal{K}$. Then the map

$$a \in \mathfrak{U} \longrightarrow \pi_0^{-1}\left(J_{\mathfrak{B}}\ P\, J_{\mathcal{U}}\ \pi(a) J_{\mathcal{U}}\ J_{\mathfrak{B}} P\right) \in \mathfrak{B}$$

is well defined and extends to a faithful normal completely positive identity preserving map satisfying $\varphi(m) = \varphi_0\big(E(m)\big)$; $\varphi(m) < \infty$; $\qquad m \in \mathcal{M}_+$

2.3 Quantum Dynamical Semigroups

Lindblad (1976) gave the first complete characterization of the infinitesimal generators of quantum dynamical semigroups. He assumed the following axioms.

Let $\mathcal{M}$ be a von Neumann algebra, a quantum dynamical semigroup $(P_t)_{t\geq 0}$ is a one parameter family of maps of $\mathcal{M}$ into itself satisfying:

(i) P_t is positive

(ii) $P_t(1) = 1$

(iii) $P_s.P_t = P_{s+t}$

(iv) $lim_{t\downarrow 0}\|P_t - I\| = 0$

(v)P_t=$e^{t\mathcal{L}}$

(vi) $lim_{t\downarrow 0}\|\mathcal{L} - t^{-1}(P_t - I)\| = 0$

(vii) $\mathcal{L}$ is ultraweakly continuous.

The form of the infinitesimal generator of quantum dynamical semigroup is given by

$$\mathcal{L}(x) = \psi(x) - \frac{1}{2}\{\psi(I), x\} + i[H, x],$$

where ψ is a completely positive map.

We collect some results on quantum dynamical semigroups.

Definition 2.4 (Lindblad,1976)

A bounded map $\mathcal{L}: \mathcal{M} \rightarrow \mathcal{M}$ satisfying the following properties;

i. $\mathcal{L}(1) = 0$,

ii. $\mathcal{L}(x^*) = \mathcal{L}(x)^* \quad \forall \quad x \in \mathcal{M}$ and

iii. $\mathcal{L}_n(x^*y) - \mathcal{L}_n(x^*)y - x^*\mathcal{L}_n(y) \geq 0, \qquad \forall \ x \in M_n(\mathcal{M})$

is said to be completely dissipative, where $\mathcal{L}_n = \mathcal{L} \otimes I_n$ and I_n the $n \times n$ identity matrix.

Theorem 2.5 (Lindblad, 1976)

If $\mathcal{L}$ a bounded map on $\mathcal{M}$ into $\mathcal{M}$ is a generator of the semigroup $P_t = e^{t\mathcal{L}}$, then the semigroup P_t is a completely positive map on $\mathcal{M}$. And $P_t(1) = 1$ if and only if $\mathcal{L}$ is completely dissipative.

Theorem 2.6 (Lindblad, 1976)

$\mathcal{L}$ is completely dissipative if and only if it is of the form

$\mathcal{L}(x) = \sum_j \left(V_j^* x V_j - \frac{1}{2}\{V_j^* V_j, x\}\right) + i[H, x]$, where V_j, $\sum V_j^* V_j \in \mathcal{B}(\mathfrak{H})$, where H is a self-adjoint operator in $\mathcal{B}(\mathfrak{H})$ and V_j is a bounded linear operator on $\mathfrak{H}$

Theorem 2.7 (Evans, 1976)

Let $\mathcal{A}$ be a C* algebra and $e^{t\mathcal{L}}$ a strongly continuous one parameter semigroup of positive maps on $\mathcal{A}$ such that

(i) D($\mathcal{L}$) is a subalgebra of $\mathcal{A}$

(ii) $\mathcal{L}(x^* x) - x^* \mathcal{L}(x) - \mathcal{L}(x)^* x \geq 0$, $\forall\ x \in D(\mathcal{L})$

then $e^{t\mathcal{L}}(x^* x) \geq e^{t\mathcal{L}}(x^*) e^{t\mathcal{L}}(x)$, $\forall\, x \in \mathcal{A},\ t > 0$.

Evans (1976) studied the irreducible ergodic properties of dynamical semigroups with Lindblad-type generators with particular reference to locally completely positive maps, that is those maps φ satisfying the Kadison-Schwarz inequality, defined formally as follows

$$\varphi(x^* x) \geq \varphi(x)^*\, \varphi(x),\ \ \forall\, x \in \mathcal{A}.$$

In that paper, Evans, showed that a dynamical semigroup of locally completely positive maps on a von Neumann algebra is irreducible if and only if the largest von Neumann algebra in the fixed point set is trivial.

Theorem 2.8 (Evans,1976)

Let $(P_t)_{t \geq 0}$ be a dynamical semigroup of locally completely positive maps on a von Neumann algebra $\mathcal{M}$. Then

(i) The set $\{x \in \mathcal{M} : P_t(x^* x) = x^* x,\ P_t(x) = x\}$ is a weakly closed subalgebra of $\mathcal{M}$.

(ii) The von Neumann subalgebra $p\mathcal{M}p$ reduces P_t if and only if

$P_t(yp) = P_t(y)p$ for all positive $t, and\ y \in \mathcal{M}$, where p is a projection in $\mathcal{M}$.

(iii) P_t is irreducible if and only if the set of fixed points

$\mathcal{M}(T) = \{x \in \mathcal{M}:\ P_t(x) = x \quad \forall t \geq 0\}$ consist of scalars.

Frigerio (1977) derived the equivalence between irreducibility and the uniqueness of the equilibrium state. He gave a sufficient condition for approach to equilibrium. Frigerio investigation is restricted to the class of dynamical semigroups possessing a faithful normal stationary state.

Theorem 2.9 (Frigerio, 1978)

Let $(P_t)_{t\geq 0}\ be$ a dynamical semigroup of a von Neumann algebra $\mathcal{M}$ with a faithful normal stationary state ψ, then there exists a unique P_t –invariant normal conditional expectation $E: \mathcal{M} \to \mathcal{M}(T)$ defined by $E(x) = w^* - \lim_{\lambda \to 0} \lambda \int_0^\infty dte^{-\lambda t} P_t(x)$, $x \in \mathcal{M},$

where $\mathcal{M}(T)$ is the fixed point set of P_t which is also a von Neumann subalgebra of $\mathcal{M}$.

Theorem 2.10 (Frigerio,1978)

A state $\varphi \in \mathcal{M}_*$ is P_t -invariant and majorized by a scalar multiple of ψ if and only if it is of the form $\varphi(x) = \psi(y)^{-1}(Jy\Omega, x\Omega)$, for some positive $y \in \mathcal{M}(T)$, J being an antiunitary involution on $\mathfrak{H}$,such that $J\Omega = \Omega$, $J\mathcal{M}J = \mathcal{M}'$.

Christensen (1978) showed that any generator of a norm continuous semigroup of completely positive normal maps on a von Neumann $\mathcal{M}$ can be decomposed in the following theorem:

Theorem 2.11 (Christensen, 1978)

Let $(P_t)_{t\geq 0}$ be a uniformly continuous semigroup of completely normal maps on a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathfrak{H}$. Then there exist an $x \in \mathcal{B}(\mathfrak{H})$ and a completely positive normal map $\psi: \mathcal{M} \longrightarrow \mathcal{B}(\mathfrak{H})$ such that the generator $\mathcal{L}$ of P_t has the form $\mathcal{L}(m) = \psi(m) + x^* m + mx, \quad m \in \mathcal{M}.$

Theorem 2.12 (Majewski and Zegarlinski ,1996)

Suppose $\left\| \partial_X \gamma_{X+j} \right\| < \alpha e^{-Md(k,j)}$, with some positive constants M, α and a metric d, then the infinite volume stochastic dynamics $P_t^X = e^{t\mathcal{L}^X}$ is well defined. Moreover the semigroup P_t^X is strongly ergodic in the sense that there is a unique P_t^X- invariant state ψ for which we have $\| P_t^X f - \psi f \| \leq 2e^{-mt} \sum_{j \in \mathbb{Z}^d} \left\| \partial_j f \right\|$

with some $m > 0$ and $\partial_j f \equiv f - Tr_j f$.

Another approach of constructing stochastic dynamics using Dirichlet forms was considered in (Cipriani, et al., 2000).

2.4 Quantum Entanglement

Quantum entanglement is a possible property of a quantum mechanical state of a system of two or more objects in which the quantum states of the constituting objects are linked together so that one object can no longer be adequately described without full mention of its counterpart even though the individual objects are spatially separated.

Definition 2.5

Suppose there are two noninteracting systems A and B, with states φ_A, φ_B and Hilbert spaces $\mathfrak{H}_A$,$\mathfrak{H}_B$ respectively. Let $\mathfrak{H}$ be the Hilbert space of the composite system given by $\mathfrak{H} = \mathfrak{H}_A \otimes \mathfrak{H}_B$. Then any state of the composite system that cannot be cast into the form $\varphi = \sum_i P_i (\varphi_A \otimes \varphi_B)$ will be called entangled, where P_i is a projection.

In 1964 John Bell showed that quantum entangled systems are systems correlated in a way that classical systems cannot. The fundamental question in quantum entanglement theory is, which states are entangled and which are not. This question is not trivial. The simplest is the case of pure bipartite states. Peres (1996) stated the condition for separable states for continuous variables of two harmonic oscillators. The condition for separable states was discovered independently by Simon (Horodecki, etal., 2007).

However our understanding of mixed state entanglement is much less complete and most proven results are restricted to situations where the constituents parts are quantum two level systems. Studies on various aspects of entanglement have been carried out by several authors in (Li ,etal 2008), (Paratharasthy,2004),(Peres,1996) to mention a few.

CHAPTER 3

NONCOMMUTATIVE L_P –SPACES AND

FINITE VOLUME QUANTUM STOCHASTIC DYNAMICS

3.0 Introduction

In this chapter a noncommutative L_p-spaces over a von Neumann algebra involving operators of the form $\rho_{\mathrm{n}}^{\alpha(t)}.\rho_{\mathrm{n}}^{\alpha(t)}$ is defined. We give a formulation on how such operators are realized. To get a nontrivial analogue of classical stochastic dynamics, we consider a quantum generalization of conditional expectation as was done in Majewski and Zegarlinski (1996). We define and state the properties of the generalized conditional expectation $E_{X,\Lambda}$. The generator $\mathcal{L}_{X,\Lambda}$ of the finite volume quantum stochastic dynamics on a finite set $X \subseteq \Lambda$, is defined with respect to the map $E_{X,\Lambda}$ and is of the form

$$\mathcal{L}_{\Lambda,X}(.) = E_{X,\Lambda}(.) - \tfrac{1}{2}\{E_{X,\Lambda}(I),.\}.$$

To have a dynamics that describes irreversible processes like dissipation, we proceed as follows, let $X + \boldsymbol{j}$ be a translate of the set X by a vector $\boldsymbol{j} \in \mathbb{Z}^d$, the generator for a finite volume quantum stochastic dynamics for spin systems is defined as a self adjoint operator $\mathcal{L}^{X,\Lambda} = \sum_{\boldsymbol{j}\in\Lambda} \mathcal{L}_{\Lambda,X+\boldsymbol{j}}$, such that the infinite sum converges for $X \subseteq \Lambda$. Then the corresponding finite volume stochastic dynamics for spins systems is defined as

$$P_t^{X,\Lambda} = e^{t\mathcal{L}^{X,\Lambda}}.$$

3.1 Quasilocal von Neumann Algebra and Non Commutative L_P –Spaces

In this section the operators of the form $\rho_{\mathrm{n}}^{\alpha(t)}.\rho_{\mathrm{n}}^{\alpha(t)}$ are defined, they constitute the elements of a von Neumann algebra $\mathcal{M}_0$. This will be made clear in the following:

Let $\mathcal{M}$ be a von Neumann algebra and ρ a closed positive self adjoint operator affiliated to $\mathcal{M}$. Let e be a projection on the Hilbert space $\mathfrak{H}$ such that $e\rho \subset \rho e$ and ρe is a positive bounded everywhere –defined operator on $\mathfrak{H}$, we say that e is a bounding projection for ρ. Now let (e_n) be an increasing sequence of projections each of which is bounding for ρ and $\bigvee_{n=1}^{\infty} e_n = I$, we say that (e_n) is a bounding sequence for ρ and ρe_n is a positive bounded everywhere –defined operator on $\mathfrak{H}$. (Kadison and Ringrose, 1983). Let ρe_n considered as a bounded operator be denoted by ρ_n and its' spectrum by $sp(\rho_n)$. Let $C\big(sp(\rho_n)\big)$ be the space of all continuous real valued functions on $sp(\rho_n)$ and let $\mathcal{B}(\mathfrak{H})_+$ be the set of positive operators in $\mathcal{B}(\mathfrak{H})$. Using functional calculus ,we introduce the operator ρ_n^a in $\mathcal{B}(\mathfrak{H})_+$ for $a \in \mathbb{R}_+$ as follows; Let $f_a \in C\big(sp(\rho_n)\big)$ be defined by $f_a(s) = s^a$, $s \in sp(\rho_n)$. With ρ_n^a define as $f_a(\rho_n)$ for real values of a.

Thus $$f_a(\rho_n) = \rho_n^{a} \ \in \mathcal{B}(\mathfrak{H})_+ .$$

Replace $a \in \mathbb{R}_+$ with a positive real-valued function $\alpha(t)$ on the closed interval [0,1] such that $0 \le \alpha(t) \le \frac{1}{2}$, thus we have,

$$f_{\alpha(t)}(\rho_n) = \rho_n^{\alpha(t)} \in \mathcal{B}(\mathfrak{H})_+.$$

Remark: Note in particular, $\alpha(t)$ on [0,1] can be defined as $\alpha(t) = \frac{1-t}{2}$, for $t \in [0,1]$.

For a self-adjoint operator $x \in \mathcal{M}$, let $\left\{x.\rho_n^{\alpha(t)} : x \in \mathcal{M}, \rho_n^{\alpha(t)} \in \mathcal{B}(\mathfrak{H})_+\right\}$ be the set of strong product of bounded operators on $\mathfrak{H}$, this is a *-algebra when endowed with the operations of sum, product and involution defined as follows,

$$x.\rho_n^{\alpha(t)} + y.\rho_n^{\alpha(t)} = (x+y).\rho_n^{\alpha(t)}, \qquad x+y \in \mathcal{M}$$

$$\left(x.\rho_n^{\alpha(t)}\right)\left(y.\rho_n^{\alpha(t)}\right) = (x.y).\rho_n^{\alpha(t)}, \qquad x.y \in \mathcal{M}$$

$$\left(x.\rho_n^{\alpha(t)}\right)^* = x^*.\rho_n^{\alpha(t)}, \qquad x^* \in \mathcal{M}$$

for $x, y \in \mathcal{M}$ and $\rho_n^{\alpha(t)} \in \mathcal{B}(\mathfrak{H})_+$, then the set $\left\{x.\rho_n^{\alpha(t)} : x \in \mathcal{M}\right\}$ is clearly a *- subalgebra of $\mathcal{B}(\mathfrak{H})$. Denoting the strong product $x.\rho_n^{\alpha(t)}$ by $\tilde{x}$ and the set by $\widetilde{\mathcal{M}}$, we have

$$\widetilde{\mathcal{M}} = \left\{\tilde{x} : \ x.\rho_n^{\alpha(t)} = \tilde{x}, \ \ x \in \mathcal{M}, \rho_n^{\alpha(t)} \in \mathcal{B}(\mathfrak{H})_+\right\},$$

$\tilde{x}$ is a strong product of bounded operators on $\mathfrak{H}$ and is of the form $\rho_n^{\alpha(t)}.x.\rho_n^{\alpha(t)}$.

This is given by the following:

$$\rho_n^{\alpha(t)}x.\rho_n^{\alpha(t)} = \left(1.\rho_n^{\alpha(t)}\right)\left(x.\rho_n^{\alpha(t)}\right) = (1x).\rho_n^{\alpha(t)} = x.\rho_n^{\alpha(t)} = \tilde{x} \in \widetilde{\mathcal{M}}$$

This expression of $\tilde{x}$ implied that $\tilde{x}$ “commute” with $\rho_n^{\alpha(t)}$, this given by the following,

$$\rho_n^{\alpha(t)}.\tilde{x} = \rho_n^{\alpha(t)}.\left(x.\rho_n^{\alpha(t)}\right) = \left(I.\rho_n^{\alpha(t)}\right).\left(x.\rho_n^{\alpha(t)}\right)$$

$$= (I.x).\rho_n^{\alpha(t)}$$

$$= (x.I).\rho_n^{\alpha(t)} = \left(x.\rho_n^{\alpha(t)}\right)\left(I.\rho_n^{\alpha(t)}\right) = \tilde{x}.\rho_n^{\alpha(t)} \ .$$

The product $\rho_n^{\alpha(t)}x.\rho_n^{\alpha(t)}$ is also self-adjoint, that is

$$\tilde{x}^* = \left(\rho_n^{\alpha(t)}x.\rho_n^{\alpha(t)}\right)^* = \left(x.\rho_n^{\alpha(t)}\right)^*\rho_n^{\alpha(t)} = \rho_n^{\alpha(t)}x^*.\rho_n^{\alpha(t)} = x^*.\rho_n^{\alpha(t)} = x.\rho_n^{\alpha(t)} = \tilde{x}$$

since x and $\rho_n^{\alpha(t)}$are assumed to be self-adjoint.

The set $\widetilde{\mathcal{M}}$ is strongly closed, to see this, consider the net $\{\tilde{x}_i\} \subset \widetilde{\mathcal{M}}$, if the net $\{x_i\} \subset \mathcal{M}$ converges strongly to x, then the net $\tilde{x}_i = x_i\, \rho_n^{\alpha(t)}$ converges strongly to $\tilde{x} = x\rho_n^{\alpha(t)}$.

Thus for $\xi \in \mathfrak{H}$, we have

$$\lim_i \|(\tilde{x} - \tilde{x}_i)\xi\| = \lim_i \left\|(x - x_i)\rho_n^{\alpha(t)}\xi\right\| \le \lim_i \|(x - x_i)\xi\| \left\|\rho_n^{\alpha(t)}\xi\right\| \to 0 ,$$

hence $\widetilde{\mathcal{M}}$ is strongly closed and hence weakly closed since any strongly convergent sequence $\{x_i\}$ in $\mathcal{M}$, is also weakly convergent. Thus $\widetilde{\mathcal{M}}$ is a unital weakly closed *-subalgebra of $\mathcal{B}(\mathfrak{H})$ hence a von Neumann algebra.

Now let $\mathbb{Z}^d, d \ge 1$ be the d -dimensional lattice, whose sites are occupied by spin- $\frac{1}{2}$ particles. One associates with each point $\boldsymbol{j} = (j_1, j_2, \dots . j_d) \in \mathbb{Z}^d$ a Hilbert space $\mathfrak{H}_{\{\boldsymbol{j}\}}$ and with each finite subset $\Lambda \subset \mathbb{Z}^d$ the tensor product space $\mathfrak{H}_\Lambda = \bigotimes_{\boldsymbol{j}\in\Lambda}\mathfrak{H}_{\{\boldsymbol{j}\}}$. The self-adjoint operators at site $\boldsymbol{j} = (j_1, j_2, \dots . j_d) \in \mathbb{Z}^d$ are elements of the point algebra $\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$. The von Neumann algebra $\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$ is isomorphic to a 2×2 matrix algebra $\mathcal{M}_2(\mathbb{C})$. The algebra of self-adjoint operators localized to a finite region $\Lambda \subset \mathbb{Z}^d$, defined by $\mathcal{M}_\Lambda = \bigotimes_{\boldsymbol{j}\in\Lambda}\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$, is then the full matrix algebra $\mathcal{M}_{2^{|\Lambda|}}(\mathbb{C})$. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{Z}^d$ ordered by inclusion, and let $\Lambda_1, \Lambda_2 \in \mathcal{F}$ be two disjoint finite regions, that is $\Lambda_1 \cap \Lambda_2 = \emptyset$. Then $\mathfrak{H}_{\Lambda_1 \cup \Lambda_2} = \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2}$, and we write $\mathcal{M}_{\Lambda_1 \cup \Lambda_2} = \mathcal{M}_{\Lambda_1} \otimes \mathcal{M}_{\Lambda_2}$ for the matrix algebra.

$\mathcal{M}_{\Lambda_1}$ is isomorphic to the matrix subalgebra $\mathcal{M}_{\Lambda_1} \otimes I_{\Lambda_2}$ of $\mathcal{M}_{\Lambda_1 \cup \Lambda_2}$,where I_{Λ_2} denotes the identity on $\mathfrak{H}_{\Lambda_2}$. Identifying $\mathcal{M}_{\Lambda_1}$ and $\mathcal{M}_{\Lambda_1} \otimes I_{\Lambda_2}$ one concludes that the algebra $(\mathcal{M}_\Lambda)_{\Lambda\in\mathcal{F}}$ form an increasing family of matrix algebras, whose union $\bigcup_{\Lambda\in\mathcal{F}} \mathcal{M}_\Lambda$ is a normed *-algebra, which is incomplete because $\mathbb{Z}^d$ is infinite. The norm closure

$\overline{\bigcup_{\Lambda\in\mathcal{F}} \mathcal{M}_\Lambda}^{\|\ \|} = \mathcal{M}_0$ is a quasilocal von Neumann algebra (Bratteli and Robinson, 1979).

We have the following definitions for the lattice.

Definition: 3.1.1 (interaction)

An interaction Φ is defined as a function from the finite subsets $\mathrm{X} \subset \mathbb{Z}^d$ into the self adjoint elements of $\mathcal{M}_0$ such that $\Phi(\mathrm{X}) \in \widetilde{\mathcal{M}}_{\mathrm{X}}$. Each $\Phi(\mathrm{X})$ represents the energy of interaction of the set of all particles in the finite subset X. In a spin system the particles are considered to be fixed at the lattice sites and hence the total energy of interaction in a subset Λ consists of the interaction energy of all the subsystems. This total energy is defined to be the Hamiltonian $H_\Phi(\Lambda)$ associated with Λ.

Explicitly $H_\Phi(\Lambda) = \sum_{\mathrm{X} \subseteq \Lambda} \Phi(\mathrm{X})$, where $H_\Phi(\Lambda)$ is a self-adjoint element of $\mathcal{M}_\Lambda$.

Definition: 3.1.2(metric)

The metric on the lattice $\mathbb{Z}^d$ is defined by $d(\boldsymbol{l}, \boldsymbol{k}) = \max_{q=1,2,\ldots d} \left| l_q - k_q \right|$,

for the vectors $\boldsymbol{l} = (l_1, l_2 \ldots l_d)$, $\boldsymbol{k} = (k_1, k_2, \ldots . k_d) \in \mathbb{Z}^d$, and for the coordinates we have $d(l_1, k_1) = |l_1 - k_1|$ .This induces the lexicographic order on $\mathbb{Z}^d$,for any integers $m \geq n \Rightarrow d(l_m, k_1) \geq d(l_n, k_1)$.

Definition: 3.1.3

If $\mathbb{Z}^d$ is equipped with a metric $d(. \ , \ .)$ then, we say $\mathbb{Z}^d$ is homogeneous, if the metric has the following two properties;

(i) $d(\boldsymbol{j}, \boldsymbol{k}) \geq 1$ for all $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$.

(ii) For each $r \geq 1$ there is at most a finite number N_r of points k with $d(\boldsymbol{j}, \boldsymbol{k}) \leq r$ uniformly for $\boldsymbol{j} \in \mathbb{Z}^d$.

Definition: 3.1.4

An interaction Φ is defined to have a finite range if there exists a metric $d_\Phi \geq 1$ such that $\Phi(\mathrm{X}) = 0$ whenever $D(X) = \sup_{\boldsymbol{j},\boldsymbol{k} \in X} d(\boldsymbol{j}, \boldsymbol{k}) > d_\Phi$, where $D(X)$ is the diameter of the finite set X. The minimum possible value of d_Φ is called the range of Φ and there is no mutual interaction between particles whose separation is greater than this range.

Definition:3.1.5 (van Hove limit) (Robinson And Ruelle,1967)

Let $\Lambda \subset \mathbb{Z}^d$ be a bounded open set, i.e, a finite set. If $\boldsymbol{j} = (j_1, j_2, \dots . j_d) \in \mathbb{Z}^d$ with

$j_i > 0,$ we define $\Lambda(\boldsymbol{j})$ as the parallelepiped with edges of length $j_i - 1$:

$$\Lambda(\boldsymbol{j}) = \{\boldsymbol{k} \in \mathbb{Z}^d : 0 \leq k_i \leq j_i, for\ \ i = 1, \dots . d\}.$$

The translates $\Lambda_{\boldsymbol{n}} = \Lambda(\boldsymbol{j}) + \boldsymbol{nj}$ of $\Lambda(\boldsymbol{j})$ by vectors $\boldsymbol{nj} = (n_1 j_1, n_2 j_2, \dots \dots n_d j_d)$, with $\boldsymbol{n} \in \mathbb{Z}^d$ form a partition $\mathcal{P}_{\boldsymbol{j}}$ of $\mathbb{Z}^d$. We say that the sets tend to infinity in the sense of van Hove and we write $\Lambda(\boldsymbol{j}) \to \infty$ if for every partition $\mathcal{P}_{\boldsymbol{j}}$

$$\lim_{\Lambda(\boldsymbol{j}) \to \infty} \frac{n_\Lambda^+(\boldsymbol{j})}{n_\Lambda^-(\boldsymbol{j})} = 1$$

where $n_\Lambda^+(\boldsymbol{j})$ is the number of sets of the partition $\mathcal{P}_{\boldsymbol{j}}$ which have non-empty intersection with $\Lambda(\boldsymbol{j})$ and $n_\Lambda^-(\boldsymbol{j})$ is the number of sets of this partition which are contained in $\Lambda(\boldsymbol{j})$.

Definition 3.1.6

For any given finite set $X \subseteq \Lambda \in \mathcal{F}$, $\mathcal{M}_0 = \widetilde{\mathcal{M}}_X \otimes \widetilde{\mathcal{M}}_{X^c}$. A normalized partial trace on $\mathcal{M}_0$ is a completely positive map $Tr_X : \mathcal{M}_0 \to \widetilde{\mathcal{M}}_{X^c}$ satisfying the following conditions;

(i) $Tr_X(\tilde{x}^* \tilde{x}) \geq 0$

(ii) $Tr_X(1) = 1$

(iii) $Tr_X(Tr_X\ \tilde{x}) = Tr_X(\tilde{x})$

(iv) $Tr_X(\tilde{x}\tilde{y}) = Tr_X(\tilde{y}\tilde{x}),\ \ \tilde{x}, \tilde{y} \in \mathcal{M}_0.$

(v) $Tr_X\left(\tilde{g}\tilde{x}\tilde{f}\right) = \left(\tilde{g} Tr_X(\tilde{x})\tilde{f}\right),\ \tilde{x} \in \mathcal{M}_0$ and $,\tilde{g}, \tilde{f} \in \mathcal{M}_{X^c}$

We define the partial trace Tr_j at the point $j \in \mathbb{Z}^{\mathrm{d}}$ on the von Neumann algebra $\mathcal{M}_\Lambda$. Since the point algebra $\widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}}$ generates $\mathcal{M}_\Lambda$, we have $\mathcal{M}_\Lambda = \widetilde{\mathcal{M}}_{\{\boldsymbol{j}\}} \otimes \widetilde{\mathcal{M}}_{\{\mathbf{j}\}^{\mathrm{c}}}$.

Definition 3.1.7

A normalized partial trace $Tr_{\boldsymbol{j}}$ on $\mathcal{M}_\Lambda$ is a completely positive map $Tr_{\boldsymbol{j}} : \mathcal{M}_\Lambda \to \widetilde{\mathcal{M}}_{\{\mathbf{j}\}^{\mathrm{c}}}$ satisfying the following conditions for $\tilde{x} \in \mathcal{M}_\Lambda$,

(i) $Tr_{\boldsymbol{j}}(\tilde{x}^* \tilde{x}) \geq 0$

(ii) $Tr_j(1) = 1$

(iii) $Tr_j\big(Tr_j\,\tilde{x}\big) = Tr_j(\tilde{x})$

(iv) $Tr_j(\tilde{x}\tilde{y}) = Tr_j(\tilde{y}\tilde{x}), \quad \tilde{x}, \tilde{y} \in \mathcal{M}_\Lambda$

(v) $Tr_j\big(\tilde{g}.\tilde{x}.\tilde{f}\big) = \big(\tilde{g}.Tr_j(\tilde{x}).\tilde{f}\big), \quad \tilde{x} \in \mathcal{M}_\Lambda$ and $\tilde{g}, \tilde{f} \in \widetilde{\mathcal{M}}_{\{j\}^c}$.

Remark 3.1

We recall that a positivity and unit preserving map for which (v) holds is called a conditional expectation and is a projection from condition (iii). Let $Tr \equiv lim_{\mathcal{F}_o} Tr_\Lambda$ be the normalised trace on $\mathcal{M}$. Then we have $Tr(Tr_X(\tilde{f}^*)\tilde{g}) = Tr(\tilde{f}^* Tr_X(\tilde{g}))$. The $lim_{\mathcal{F}_o}$ is defined in the sense of van Hove convergence of Λ, i.e the convention that $|\Lambda| \to \infty$, indicates Λ eventually contains all finite subsets of $\mathbb{Z}^d$.

We defined L_p -spaces over the algebra $\mathcal{M}_0$ based on the L_p -spaces of Trunov (1978). Let τ be a faithful normal semifinite trace on $\mathcal{M}_0$. The set of positive nonsingular self-adjoint operators with a finite trace is given by $\{\tilde{h} \in \mathcal{M}_0 : \tau|\tilde{h}| < \infty\}$ with norm $\|\tilde{h}\|_1 = \tau(|\tilde{h}|)$, we denoted this set by $L_1(\mathcal{M}_0)$. Now we have from Segal (1953), the representation on $\mathcal{M}_0$ defined by

$$\varphi(\tilde{x}) = \tau(\tilde{x}.\tilde{h}) \ , \qquad \tilde{x} \in \mathcal{M}_0\ , \ \tilde{h} \in L_1(\mathcal{M}_0)$$

where φ is a faithful normal state on $\mathcal{M}_0$.

Thus, $\varphi(\tilde{x}) = \tau\big(\tilde{x}.\tilde{h}\big) = \tau\big(\tilde{h}.\tilde{x}\big) = \tau\left(\tilde{h}^{\frac{1}{2}}.\tilde{x}.\tilde{h}^{\frac{1}{2}}\right).$

This representation enables one to define for each $1 \le p < \infty$ a norm $\|.\|_p$ on $\mathcal{M}_0$.

For $\tilde{h} \in L_1(\mathcal{M}_0)$, we have the norm

$$\|\tilde{x}\|_p = \left(\tau\left|\tilde{h}^{\frac{1}{2p}}.\tilde{x}.\tilde{h}^{\frac{1}{2p}}\right|^p\right)^{\frac{1}{p}} = \left(\tau\left|\tilde{h}^{\frac{1}{2p}}.\left(\rho_n^{\alpha(t)} x.\rho_n^{\alpha(t)}\right).\tilde{h}^{\frac{1}{2p}}\right|^p\right)^{\frac{1}{p}},$$

the set $L_p(\mathcal{M}_0) = \{\tilde{x} \in \mathcal{M}_0 : \|\tilde{x}\|_p < \infty\}$ is a Banach space of pth-power integrable operators in $\mathcal{M}_0$. We set $L_\infty(\mathcal{M}_0) = \mathcal{M}_0$ and the predual $\mathcal{M}_* = L_1(\mathcal{M}_0)$

Remark 3.2

We have the basic properties for L_p spaces, for $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

i. $L_\infty(\mathcal{M}_0) \subset L_q(\mathcal{M}_0) \subset L_p(\mathcal{M}_0) \subset L_1(\mathcal{M}_0)$ and $\|\tilde{x}\|_p \leq \|\tilde{x}\|_q$ for

$\tilde{x} \in L_q(\mathcal{M}_0)$, and $q > p > 1$.

ii. The state $\varphi(\tilde{x}) = \tau(\tilde{h}\tilde{x})$ defines the following scalar product

$$\langle \tilde{x}, \tilde{y} \rangle = \varphi(\tilde{y}^* \tilde{x}) = \tau\left(\tilde{h}^{\frac{1}{2}} \tilde{y}^* \, \tilde{h}^{\frac{1}{2}} \, \tilde{x}\right) = \tau\left(\left(\tilde{h}^{\frac{1}{4}} \, \tilde{y}^* \, \tilde{h}^{\frac{1}{4}}\right)\left(\tilde{h}^{\frac{1}{4}} \, \tilde{x} \, \tilde{h}^{\frac{1}{4}}\right)\right).$$

where the Hilbert space $\mathfrak{H}$ is the completion of $\mathcal{M}_0$ with respect to this scalar product $\langle . , . \rangle$.

3.2 Finite Volume Quantum Stochastic Dynamics For Spins System

To get a nontrivial analogue of classical stochastic dynamics for spin systems on lattice, we need to consider a quantum generalization of the conditional expectation as was done in Majewski and Zegarlinski (1996). We begin with a definition of the generalized conditional expectation $E_{X,\Lambda}$ on the operators of the form $\tilde{x} = \rho_n^{\alpha(t)} x \, \rho_n^{\alpha(t)} \in \mathcal{M}_0$ for a finite set $X \subseteq \Lambda$. Let $E_{X,\Lambda} : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ be a map defined by,

$$E_{X,\Lambda}(\tilde{x}) = Tr_X\left(\gamma_{X,\Lambda}^* \;\tilde{x}\; \gamma_{X,\Lambda}\right), \qquad \gamma_{X,\Lambda} \in \mathcal{M}_0 \qquad (3.2.1)$$

where, $\gamma_{X,\Lambda} = \tilde{h}^{\frac{1}{2}}\left(Tr_X \tilde{h}\right)^{-\frac{1}{2}}$, $\gamma_{X,\Lambda}^* = \left(Tr_X \tilde{h}\right)^{-\frac{1}{2}} \tilde{h}^{\frac{1}{2}}$

We have the properties of $E_{X,\Lambda}$ in the following proposition.

Proposition 3.2.1

$E_{X,\Lambda}$ is a completely positive, unit preserving and *- invariant map on $\mathcal{M}_0$. If $E_{X,\Lambda}$ is of norm one then $E_{X,\Lambda}$ satisfies the Kadison-Schwarz inequality and is bounded. The extended map $E_{X,\Lambda}$ onto $L_2(\mathcal{M}_0)$, is symmetric with respect to the scalar product on $L_2(\mathcal{M}_0)$.

These properties are formally outline as follows:

(i) $E_{X,\Lambda}(\tilde{x}) \geq 0, \qquad \tilde{x} \in \mathcal{M}_0$

(ii) $E_{X,\Lambda}(1) = 1$

(iii) $\left(E_{X,\Lambda}(\tilde{x})\right)^* = E_{X,\Lambda}(\tilde{x})^*$,

(iv) The map $E_{X,\Lambda}$ satisfies the Kadison-Schwarz inequality,

$$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) \leq E_{X,\Lambda}(\tilde{x}^* \tilde{x})$$

(v) The map $E_{X,\Lambda}$ is bounded with respect to the norm $\|.\|$ on $\mathcal{M}_0$,

$$\left\|E_{X,\Lambda}(\tilde{x})\right\| \leq \|\tilde{x}\| , \qquad \tilde{x} \in \mathcal{M}_0 .$$

(vi) $E_{X,\Lambda}$ is symmetric with respect to the scalar product defined on $\mathfrak{H}$,

$$\langle E_{X,\Lambda}(\tilde{x}), \tilde{y} \rangle = \langle \tilde{x} \, , \, E_{X,\Lambda}(\tilde{y}) \rangle .$$

Proof:

(i) Let $\tilde{x} = \tilde{x}^{\frac{1}{2}} . \tilde{x}^{\frac{1}{2}}$, $\tilde{x} \geq 0$ then

$$E_{X,\Lambda}(\tilde{x}) = Tr_X\left(\gamma_{X,\Lambda}^* \, \tilde{x}^{\frac{1}{2}} \; \tilde{x}^{\frac{1}{2}} \; \gamma_{X,\Lambda}\right)$$

$$= Tr_X\left(\left(\tilde{x}^{\frac{1}{2}}\gamma_{X,\Lambda}\right)^* \left(\tilde{x}^{\frac{1}{2}}\gamma_{X,\Lambda}\right)\right)$$

$$= Tr_X\left(\left(\tilde{x}^{\frac{1}{2}}\gamma_{X,\Lambda}\right)^* \left(\tilde{x}^{\frac{1}{2}}\gamma_{X,\Lambda}\right)\right) \geq 0$$

since Tr_X is a completely positive map.

(ii) From $E_{X,\Lambda}(\tilde{x}) = Tr_X\left(\gamma_{X,\Lambda}^* \, \tilde{x} \, \gamma_{X,\Lambda}\right)$

we have, $E_{X,\Lambda}(1) = Tr_X\left(\gamma_{X,\Lambda}^* \; \gamma_{X,\Lambda}\right) = Tr_X\left(\left(Tr_X\tilde{h}\right)^{-\frac{1}{2}} \tilde{h}^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}\left(Tr_X\tilde{h}\right)^{-\frac{1}{2}}\right)$

$$= Tr_X\left(\left(Tr_X\tilde{h}\right)^{-\frac{1}{2}} \tilde{h}\left(Tr_X\tilde{h}\right)^{-\frac{1}{2}}\right)$$

$$= \left(Tr_X\tilde{h}\right)^{-\frac{1}{2}} Tr_X(\tilde{h})\left(Tr_X\tilde{h}\right)^{-\frac{1}{2}} = 1$$

(iii) $$\left(E_{X,\Lambda}(\tilde{x})\right)^* = Tr_X\left(\gamma_{X,\Lambda}^* \, \tilde{x} \, \gamma_{X,\Lambda}\right)^*$$

$$= Tr_X\left(\left(\tilde{x} \, \gamma_{X,\Lambda}\right)^* \gamma_{X,\Lambda}^{**}\right)$$

$$= Tr_X\left(\gamma_{X,\Lambda}^* \tilde{x}^* \, \gamma_{X,\Lambda}^{**}\right)$$

$$= Tr_X\left(\gamma_{X,\Lambda}^* \tilde{x}^* \; \gamma_{X,\Lambda}\right) = E_{X,\Lambda}(\tilde{x}^*)$$

(iv) From theorem 1.4.7 in chapter one, $E_{X,\Lambda}$ has the form $E_{X,\Lambda}(\tilde{x}) = V^*\pi(\tilde{x})\,V$, since $E_{X,\Lambda}$ is completely positive, hence we have,

$$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) = V^*\pi(\tilde{x})^* V V^*\pi(\tilde{x}) V$$

$$\leq \|V\|^2 V^* \pi(\tilde{x}^* \tilde{x})\ V$$

$$\leq \|V\|^2 E_{X,\Lambda}(\tilde{x}^* \tilde{x}) ,$$

from corollary 3.6 Takesaki (1979) we have that $\|V\|^2 \leq \|E_{X,\Lambda}\|$

thus, $$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) \leq \|E_{X,\Lambda}\|\ E_{X,\Lambda}(\tilde{x}^* \tilde{x})$$

$$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) \quad \leq E_{X,\Lambda}(\tilde{x}^*\ \tilde{x}\ ).$$

(v) Note by complete positivity of the map $E_{X,\Lambda}$ we have,

$$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) \leq \|E_{X,\Lambda}\|\ E_{X,\Lambda}\ (\tilde{x}^* \tilde{x})$$

$$\leq \|E_{X,\Lambda}\|\ Tr_X\left(\gamma_{X,\Lambda}^*\ \ \tilde{x}^*\ \tilde{x}\ \ \gamma_{X,\Lambda}\right)$$

$$\leq \|E_{X,\Lambda}\|\ \|\ \tilde{x}\|^2\ Tr_X\left(\gamma_{X,\Lambda}^*\ \gamma_{X,\Lambda}\right)$$

since $Tr_X\left(\gamma_{X,\Lambda}^*\ \gamma_{X,\Lambda}\right) = E_{X,\Lambda}(1) = 1,$

$$E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x})\ \leq \|E_{X,\Lambda}\|\ \|\ \tilde{x}\|^2$$

$$\left\|E_{X,\Lambda}(\tilde{x})\right\|^2 \leq\ \|E_{X,\Lambda}\|\|\tilde{x}\|^2$$

$$\left\|E_{X,\Lambda}(\tilde{x})\right\|^2 \leq\ \|\tilde{x}\|^2$$

$$\left\|E_{X,\Lambda}(\tilde{x})\right\| \leq\ \|\tilde{x}\|$$

(vi) $$\langle E_{X,\Lambda}(\tilde{x}), \tilde{y}\rangle = \langle Tr_X\left(\gamma_{X,\Lambda}^*\ \tilde{x}\ \gamma_{X,\Lambda}\right), \tilde{y}\rangle\ = \langle \gamma_{X,\Lambda}^*\ \ \tilde{x}\ \ \gamma_{X,\Lambda},\ Tr_X\ \tilde{y}\rangle$$

$$= \langle\ \ \tilde{x}\ \ \gamma_{X,\Lambda},\ \ \gamma_{X,\Lambda} Tr_X\ \tilde{y}\rangle$$

$$= \langle\ \gamma_{X,\Lambda}^* Tr_X\ \tilde{x}\ \ ,\ \ \gamma_{X,\Lambda}^*\ \tilde{y}\rangle$$

$$= \langle\ \gamma_{X,\Lambda}\ \tilde{y}^*\ \gamma_{X,\Lambda}^* Tr_X\ \tilde{x}\ , 1\ \ \rangle$$

$$= \langle\ \ \ Tr_X\ \tilde{x}\ ,\ \gamma_{X,\Lambda}^*\ \tilde{y}\ \gamma_{X,\Lambda}\ \rangle$$

$$= \langle \tilde{x} \, , \, Tr_X\big(\gamma_{X,\Lambda}^{*} \, \tilde{y} \, \gamma_{X,\Lambda}\big) \, \rangle$$

$$= \langle \tilde{x} \; , \, E_{X,\Lambda}(\tilde{y}) \, \rangle$$

3.2.1 The Lindblad-type Generator

To have a dynamics that describes irreversible processes like dissipation, we will need an operator for a finite set $X \subseteq \Lambda$ to be the map $\mathcal{L}_{X,\Lambda}: \mathcal{M}_0 \longrightarrow \mathcal{M}_0$, defined by

$$\mathcal{L}_{X,\Lambda}\,(\tilde{x}) = E_{X,\Lambda}(\tilde{x}) - \tfrac{1}{2}\{E_{X,\Lambda}(1), \; \tilde{x}\} \tag{3.2.2}$$

called the generator of the dynamics. Hence we have the following proposition.

Proposition 3.2.2

The generator $\mathcal{L}_{X,\Lambda}$ of the finite volume stochastic dynamics for spin system defined by equation (3.2.2), annihilates the identity map, and is a *-invariant, dissipative, bounded map on $\mathcal{M}_0$, such that, the extension of the map onto the Hilbert space $L_2(\mathcal{M}_0)$ is symmetric with respect to the scalar product on $L_2(\mathcal{M}_0)$.

Formally, we outline the properties as follows:

(i) $\mathcal{L}_{X,\Lambda}\,(1) = 0$

(ii) $\mathcal{L}_{X,\Lambda}\,(\tilde{x})^{*} = \big(\mathcal{L}_{X,\Lambda}\,(\tilde{x})\big)^{*}$

(iii) $\mathcal{L}_{X,\Lambda}\big(\,(\tilde{x})^{*}(\tilde{x})\big) - \mathcal{L}_{X,\Lambda}\,(\tilde{x})^{*}\,\tilde{x} - \tilde{x}^{*}\mathcal{L}_{X,\Lambda}\,(\tilde{x}) \geq 0$

(vi) $\langle \mathcal{L}_{X,\Lambda}(\tilde{x}), \; \tilde{y} \rangle = \langle \tilde{x} \, , \mathcal{L}_{X,\Lambda}(\tilde{y}) \rangle$

Proof:

(i)

$$\mathcal{L}_{X,\Lambda}\,(1) = \mathcal{L}_{X,\Lambda}\,(1) \; = E_{X,\Lambda}(1\,) - \tfrac{1}{2}\{E_{X,\Lambda}(1), 1\}$$

$$= E_{X,\Lambda}(1\,) - \tfrac{1}{2}\big(E_{X,\Lambda}(1).\,1 + 1.\,E_{X,\Lambda}(1)\big)$$

$$= E_{X,\Lambda}(1\,) - E_{X,\Lambda}(1) = 0$$

(ii)

$$\left(\mathcal{L}_{X,\Lambda}\,(\tilde{x})\right)^* = \left(E_{X,\Lambda}(\tilde{x}) - \tfrac{1}{2}\{E_{X,\Lambda}(1),(\tilde{x})\}\right)^*$$

$$= \left(E_{X,\Lambda}(\tilde{x}) - (\tilde{x})\right)^*$$

$$= E_{X,\Lambda}(\tilde{x})^* - (\tilde{x})^*$$

$$= \mathcal{L}_{X,\Lambda}\,(\tilde{x})^*$$

(iii)

Note that we have

$$\mathcal{L}_{X,\Lambda}[\,(\tilde{x})^*(\tilde{x})] = E_{X,\Lambda}\big(\,(\tilde{x})^*(\tilde{x})\big) - \tilde{x}^*\tilde{x}$$

$$\mathcal{L}_{X,\Lambda}[\,(\tilde{x})^*](\tilde{x}) = [\,E_{X,\Lambda}(\tilde{x})^*]\tilde{x} - \tilde{x}^*\tilde{x}$$

thus we have,

$$\mathcal{L}_{X,\Lambda}[\,(\tilde{x})^*(\tilde{x})] - \mathcal{L}_{X,\Lambda}[\,(\tilde{x})^*]\tilde{x} \quad -\tilde{x}^{\,*}\mathcal{L}_{X,\Lambda}(\tilde{x})$$

$$= \big(\,E_{X,\Lambda}[\,\tilde{x}^*\tilde{x}] - \tilde{x}^*\tilde{x}\,\big) - \big([\,E_{X,\Lambda}(\tilde{x})^*]\tilde{x} - \tilde{x}^*\tilde{x}\big) - \tilde{x}^*[\,E_{X,\Lambda}(\tilde{x}) - \tilde{x}]$$

$$= E_{X,\Lambda}[\,\tilde{x}^*\tilde{x}] - \tilde{x}^*\tilde{x} - [\,E_{X,\Lambda}(\tilde{x})^*]\tilde{x} + \tilde{x}^*\tilde{x} - \tilde{x}^*[\,E_{X,\Lambda}(\tilde{x})] + \tilde{x}^*\tilde{x}$$

the second term and fourth term cancelled out, and we have,

$$= E_{X,\Lambda}[\,\tilde{x}^*\tilde{x}] - [\,E_{X,\Lambda}(\tilde{x})^*]\tilde{x} - \tilde{x}^*[\,E_{X,\Lambda}(\tilde{x})] + \tilde{x}^*\tilde{x} \qquad (**)$$

adding the zero term $E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x}) - E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x}) = 0$ into equation (**)

$$E_{X,\Lambda}[\,(\tilde{x})^*(\tilde{x})] - E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x}) + E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x}) - \tilde{x}^*[\,E_{X,\Lambda}(\tilde{x})] - [\,E_{X,\Lambda}(\tilde{x})^*]\tilde{x}$$
$$+ \tilde{x}^*\tilde{x} = \left(E_{X,\Lambda}[\,(\tilde{x})^*(\tilde{x})] - E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x})\right) + \left|\,E_{X,\Lambda}(\tilde{x}) - \tilde{x}\right|^2$$

by the Kadison -Schwarz inequality the term in the bracket is nonnegative, that is,

$$E_{X,\Lambda}[\,\tilde{x}^*\tilde{x}] - E_{X,\Lambda}(\tilde{x})^*\,E_{X,\Lambda}(\tilde{x}) \geq 0$$

and also the term $\left| E_{X,\Lambda}(\tilde{x}) - \tilde{x} \right|^2$ is also nonnegative,

hence, $\mathcal{L}_{X,\Lambda}(\tilde{x}^*\tilde{x}) - \mathcal{L}_{X,\Lambda}[\tilde{x}^*]\tilde{x} - \tilde{x}^*\mathcal{L}_{X,\Lambda}(\tilde{x})$

$$= E_{X,\Lambda}[(\tilde{x})^*(\tilde{x})] - E_{X,\Lambda}(\tilde{x})^* E_{X,\Lambda}(\tilde{x}) + \left| E_{X,\Lambda}(\tilde{x}) - \tilde{x} \right|^2 \geq 0$$

this implies that $\mathcal{L}_{X,\Lambda}(\tilde{x}^*\tilde{x}) - \mathcal{L}_{X,\Lambda}(\tilde{x}^*)\tilde{x} - \tilde{x}^*\mathcal{L}_{X,\Lambda}(\tilde{x}) \geq 0$

(iv)

$$\langle \mathcal{L}_{X,\Lambda}(\tilde{x}), \tilde{y} \rangle = \langle \left(E_{X,\Lambda}(\tilde{x}) - \tilde{x} \right), \tilde{y} \rangle = \langle E_{X,\Lambda}(\tilde{x}), \tilde{y} \rangle - \langle \tilde{x}, \tilde{y} \rangle$$

$$= \langle \tilde{x}, E_{X,\Lambda}(\tilde{y}) \rangle - \langle \tilde{x}, \tilde{y} \rangle$$

$$= \langle \tilde{x}, E_{X,\Lambda}(\tilde{y}) - \tilde{y} \rangle = \langle \tilde{x}, \mathcal{L}_{X,\Lambda}(\tilde{y}) \rangle$$

Consider a bounded symmetric Markov elementary generator $\mathcal{L}_{X+\boldsymbol{j}}$, where $X + \boldsymbol{j}$ is a translate of the set X by a vector $\boldsymbol{j} \in \mathbb{Z}^d$. We define the generator of the quantum dynamical semigroup for a finite volume as a self- adjoint operator $\mathcal{L}^{X,\Lambda} = \sum_{j \in \Lambda} \mathcal{L}_{\Lambda, X+\boldsymbol{j}}$ on $\mathcal{M}_0$, with $X \subseteq \Lambda$ a finite set of $\mathbb{Z}^d$. Let $P_t^{X,\Lambda} = e^{t\mathcal{L}^{X,\Lambda}}$ be the corresponding finite volume dynamics.

Proposition 3.2.3

The finite volume stochastic dynamics for spin system defined by $P_t^{X,\Lambda} = e^{t\mathcal{L}^{X,\Lambda}}$ with generator $\mathcal{L}^{X,\Lambda} = \sum_{\boldsymbol{j} \in \Lambda} \mathcal{L}_{\Lambda, X+\boldsymbol{j}}$ satisfying $\sum_{\boldsymbol{j} \in \Lambda} \left\| \mathcal{L}_{\Lambda, X+\boldsymbol{j}} \right\| < \infty$, is a positive, unit-preserving map on $\mathcal{M}_0$ such that the extended map is $L_2 -$ symmetric with respect to the inner product and contractive with respect to the $L_p(\mathcal{M}_0)$ norm.

We outline formally these properties as follows;

(i) $P_t^{X,\Lambda}(1) = 1$

(ii) $\langle P_t^{X,\Lambda}(\tilde{x}), \tilde{y} \rangle = \langle \tilde{x}, P_t^{X,\Lambda}(\tilde{y}) \rangle$, $\tilde{x}, \tilde{y} \in L_2(\mathcal{M}_0)$

(iii) $\left\| P_t^{X,\Lambda} \tilde{x} \right\|_{L_p(\mathcal{M}_0)} \leq \|\tilde{x}\|_{L_p(\mathcal{M}_0)}$, $\tilde{x} \in L_p(\mathcal{M}_0)$

(iv) $\varphi_\Lambda \left(P_t^{X,\Lambda}(\tilde{x}) \right) = \varphi_\Lambda(\tilde{x})$, $\tilde{x} \in \mathcal{M}_0$.

Proof:

(i) Using the Taylor expansion of the exponential we write the dynamics as follows.

$$P_t^{X,\Lambda}(1) = e^{t\mathcal{L}^{X,\Lambda}}(1)$$

$$P_t^{X,\Lambda}(1) = 1 + t\mathcal{L}^{X,\Lambda}(1) + \frac{t^2\mathcal{L}^{X,\Lambda}\left(\mathcal{L}^{X,\Lambda}(1)\right)}{2} + \cdots$$

since $\mathcal{L}^{X,\Lambda}(1) = 0$, we have all the remaining terms to be zero

hence , $P_t^{X,\Lambda}(1) = 1$

(ii) Now $\mathcal{L}^{X,\Lambda}\left(\int_0^t ds P_s^{X,\Lambda}\right) = P_t^{X,\Lambda} - P_0^{X,\Lambda}$

$$\langle (P_t^{X,\Lambda} - P_0^{X,\Lambda})\tilde{x}, \tilde{y}\rangle = \langle \mathcal{L}^{X,\Lambda}\left(\int_0^t ds P_s^{X,\Lambda}\right)\tilde{x}, \tilde{y}\rangle$$

$$= \langle \tilde{x}, \mathcal{L}^{X,\Lambda}\left(\int_0^t ds P_s^{X,\Lambda}\right)^* \tilde{y}\rangle$$

$$= \langle \tilde{x}, \mathcal{L}^{X,\Lambda}\left(\int_0^t ds P_s^{X,\Lambda}\right) \tilde{y}\rangle$$

$$\langle (P_t^{X,\Lambda} - P_0^{X,\Lambda})\tilde{x}, \tilde{y}\rangle = \langle \tilde{x}, (P_t^{X,\Lambda} - P_0^{X,\Lambda})\tilde{y}\rangle$$

hence , $\langle P_t^{X,\Lambda}(\tilde{x}), \tilde{y}\rangle = \langle \tilde{x}, P_t^{X,\Lambda}(\tilde{y})\rangle$

(iii) For contractivity of the semi-group, we have from Olkiewicz and Zegarlinski (1999) the following definition of the tangential functional. If $q \in (1,\infty)$, then for any $\tilde{x} \in L_q^+(\mathcal{M}_0)$, there exists a unique $\phi_p(\tilde{x}) \in L_p^+(\mathcal{M}_0)$ with $\frac{1}{p} + \frac{1}{q} = 1$ defined by

$$\phi_p(\tilde{x}) = \frac{\tilde{h}^{-\frac{1}{2p}}\left(\tilde{h}^{\frac{1}{2q}}\tilde{x}\,\tilde{h}^{\frac{1}{2q}}\right)^{\frac{q}{p}}\tilde{h}^{-\frac{1}{2p}}}{\|\tilde{x}\|_q^{q-2}}$$, and $\phi_p(0) = 0$ for $\tilde{x} = 0$,

with the following properties;

(a) $\|\tilde{x}\|_{L_q(\mathcal{M}_0)}^2 = \langle \phi_p(\tilde{x}), \tilde{x}\rangle$, $\tilde{x} \in L_q^+(\mathcal{M}_0)$, and $\langle . \, , \, . \rangle$ is the duality pairing

(b) $\left\|\phi_p(\tilde{x})\right\|_{L_p(\mathcal{M}_0)} = \|\tilde{x}\|_{L_q(\mathcal{M}_0)}$

(c) $\phi_p\left(|\tilde{x}|_q\right) = \left|\phi_p(\tilde{x})\right|_p$, where $|\tilde{x}|_q = \tilde{h}^{-\frac{1}{2q}}\left|\tilde{h}^{\frac{1}{2q}}\tilde{x}\tilde{h}^{\frac{1}{2q}}\right|\tilde{h}^{-\frac{1}{2q}}$

Proof

From Olkiewicz and Zegarlinski (1999) the contractivity of $P_t^{X,\Lambda}$ for arbitrary $p \in (1,\infty)$ follows from the interpolation theorem of Riesz-Thorin, because $L_\infty(\mathcal{M}_0)$ is dense in each $L_P^+(\mathcal{M}_0)$, hence the extension of $P_t^{X,\Lambda}$ onto L_p is positive and this extension is L_2 symmetric. Now let $\tilde{x}, \tilde{y} \in L_p(\mathcal{M}_0)$, such that $\phi_q(\tilde{y}) = \phi_q(\tilde{x})$, we have from property (a)

$$\|\tilde{y}\|^2_{L_p(\mathcal{M}_0)} = \langle \phi_q(\tilde{y}), \tilde{y} \rangle$$

$$\|\tilde{y}\|^2_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{y})\right\|_{L_q(\mathcal{M}_0)} \|\tilde{y}\|_{L_p(\mathcal{M}_0)}$$

$$\|\tilde{y}\|_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{y})\right\|_{L_q(\mathcal{M}_0)}$$

$$\|\tilde{y}\|_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{x})\right\|_{L_q(\mathcal{M}_0)}$$

replacing $\tilde{y}$ with $P_t^{X,\Lambda}(\tilde{x})$ and using property (b), we have,

$$\left\|P_t^{X,\Lambda}(\tilde{x})\right\|_{L_p(\mathcal{M}_0)} \leq \left\|\phi_q(\tilde{x})\right\|_{L_q(\mathcal{M}_0)} = \|\tilde{x}\|_{L_p(\mathcal{M}_0)}$$

$$\left\|P_t^{X,\Lambda}(\tilde{x})\right\|_{L_p(\mathcal{M}_0)} \leq \|\tilde{x}\|_{L_p(\mathcal{M}_0)}.$$

(iv)

$$\varphi\left(P_t^{X,\Lambda}(\tilde{x})\right) = \langle 1, P_t^{X,\Lambda}(\tilde{x})\rangle = \langle P_t^{X,\Lambda}(1), (\tilde{x})\rangle = \langle 1, \tilde{x}\rangle = \varphi_\Lambda(\tilde{x})$$

CHAPTER 4

INFINITE VOLUME QUANTUM STOCHASTIC DYNAMICS

4.0 Introduction

In this chapter we construct the infinite volume stochastic dynamics for spin system directly as the limit of the finite volume stochastic dynamics for spin system and we show that it has an exponential decay to equilibrium and is strongly ergodic, taking into consideration the problem of convergence. We proceed as follows: Let the closure of a pre-Markov elementary generator defined an elementary generator $\mathcal{L}_{X+j}(\tilde{x}) = E_{X+j}(\tilde{x}) - \tilde{x}$, where $X \subset \Lambda$ is a finite set and E_{X+j} is a 2-positive unit preserving map such that $E_{X+j}(\mathcal{M}_\Lambda) \subseteq \mathcal{M}_{\Lambda^c+\mathrm{j}}$.

We defined a finite volume generator $\mathcal{L}^{X,\Lambda}$ as follows $\mathcal{L}^{X,\Lambda} = \sum_{\boldsymbol{j}\in\Lambda} \mathcal{L}_{\Lambda,X+\boldsymbol{j}}$, such that, $\sum_{\boldsymbol{j}\in\Lambda}\left\|\mathcal{L}_{\Lambda,X+\boldsymbol{j}}\right\| < \infty$. The generator $\mathcal{L}^{X,\Lambda}$ is a well defined bounded operator on all the algebra $\mathcal{M}_0$. We define also an infinite volume generator $\mathcal{L}^X$ formally by the same formula with $\Lambda \equiv \mathbb{Z}^d$ that is , $\mathcal{L}^X = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \mathcal{L}_{X+\boldsymbol{j}}$ such that $\|\mathcal{L}^X\| < \infty$.

For this to be defined on a large domain, we will require that the elementary generator $\mathcal{L}_{X+\boldsymbol{j}}$ satisfy the following regularity property (Majewski and Zegarlinski,1996). We start with the following definitions.

Definition 4.1.1

The discrete gradient $\partial_{\boldsymbol{j}}\, \widetilde{x}$ is defined by $\partial_{\boldsymbol{j}}\, \widetilde{x} = \widetilde{x} - Tr_{\boldsymbol{j}}\widetilde{x}$, for a vector $\boldsymbol{j} \in \mathbb{Z}^d$.

This defines a seminorm $|||.|||$ on $\mathcal{M}_0$ given by $|||\widetilde{x}||| \equiv \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\left\|\partial_{\boldsymbol{j}}\, \widetilde{x}\right\|$.

Let the set of operators in $\mathcal{M}_0$ with finite seminorm $|||.|||$ be denoted by $\mathcal{M}_1$, that is, the set $\mathcal{M}_1 = \{\widetilde{x}\colon\ \widetilde{x} \in \mathcal{M}_0\, , |||\widetilde{x}||| < \infty\ \}$.

Definition 4.1.2

For any $\widetilde{x} \in \mathcal{M}_1$ an elementary operator $\mathcal{L}_{X+\boldsymbol{j}}$ is called **regular** if there is a positive constant $b_{\boldsymbol{jk}}$ with $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$ such that $\left\|\mathcal{L}_{X+\boldsymbol{j}}\widetilde{x}\right\| \le \sum_k b_{\boldsymbol{jk}}\left\|\partial_{\boldsymbol{j}}\, \widetilde{x}\right\|$ and $b_{\boldsymbol{jk}} \in [0,\infty)$, such that $sup_{\boldsymbol{j}} \sum_{\boldsymbol{k}} b_{\boldsymbol{jk}} < \infty$.

Definition 4.1.3

The elementary generators $\mathcal{L}_{X+\boldsymbol{j}},\ \boldsymbol{j} \in \mathbb{Z}^d,$ satisfy the condition

$$\left\|\left[\partial_k, \mathcal{L}_{X+\boldsymbol{j}}\right]\widetilde{x}\right\| \le \sum_{\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+j}\, \|\partial_{\boldsymbol{l}}\widetilde{x}\|$$

if there is a positive constant $a_{\boldsymbol{kl}}^{X+\boldsymbol{j}}$ $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^d$ such that

(i) $\frac{1}{|X|}\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+\boldsymbol{j}} < \infty$

(ii) $\sum_{\boldsymbol{j}:X+\boldsymbol{j}\not\ni\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+\boldsymbol{j}} \le \lambda|X| < \infty$

for any $\widetilde{x} \in \mathcal{M}_1$, $\lambda \in (0,1)$ and $|X|$ is the cardinality of the finite set X.

4.1 Infinite Volume Quantum Stochastic Dynamics for Spin System

In this section we show that the infinite volume stochastic dynamics exists and has exponential decay to equilibrium and is strongly ergodic .

Theorem 4.1

Suppose the elementary generator $\mathcal{L}_{X+j}$, $j \in \mathbb{Z}^d$ is regular and satisfies the condition $\left\|\left[\partial_k, \mathcal{L}_{X+\boldsymbol{j}}\right]\tilde{x}\right\| \le \sum_{\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+j} \|\partial_{\boldsymbol{l}}\tilde{x}\|$ with $\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+j} < \infty$. If $\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}} < 1$, for $0 < s < t$, where $t > 0$ is fixed. Then the stochastic dynamics P_t^{X,Λ_n} sequence is Cauchy in the norm topology for the sets of increasing bounded regions Λ_n satisfying $\Lambda_{n+1} \supset \Lambda_n$ and $\bigcup\Lambda_n \equiv \mathbb{Z}^d$. The limit exists as $\Lambda_n \to \infty$ and defines an infinite volume quantum stochastic dynamics P_t^X on $\mathcal{M}_0$.

Proof :

For $\Lambda_i \in \mathcal{F},\ i = 1,2$ and $\tilde{x} \in \mathcal{M}_1$,

we have

$$\frac{d}{ds}\left(P_s^{\Lambda_2}(\tilde{x}) - P_s^{\Lambda_1}(\tilde{x})\right) = \frac{d}{ds}P_s^{\Lambda_2}(\tilde{x}) - \frac{d}{ds}P_s^{\Lambda_1}(\tilde{x})$$

$$= \mathcal{L}_2 P_s^{\Lambda_2}(\tilde{x}) - \mathcal{L}_1 P_s^{\Lambda_1}(\tilde{x})$$

$$= \mathcal{L}_2 P_s^{\Lambda_2}(\tilde{x}) - \mathcal{L}_2 P_s^{\Lambda_1}(\tilde{x}) + \mathcal{L}_2 P_s^{\Lambda_1}(\tilde{x}) - \mathcal{L}_1 P_s^{\Lambda_1}(\tilde{x})$$

$$= \mathcal{L}_2 P_s^{\Lambda_2}(\tilde{x}) - \mathcal{L}_2 P_s^{\Lambda_1}(\tilde{x}) + (\mathcal{L}_2 - \mathcal{L}_1) P_s^{\Lambda_1}(\tilde{x})$$

hence

$$\frac{d}{ds}P_{t-s}^{\Lambda_2}\left(P_s^{\Lambda_2}(\tilde{x}) - P_s^{\Lambda_1}(\tilde{x})\right) = \frac{d}{ds}P_{t-s}^{\Lambda_2}P_s^{\Lambda_2}(\tilde{x}) - \frac{d}{ds}P_{t-s}^{\Lambda_2}P_s^{\Lambda_1}(\tilde{x})$$

$$= -\mathcal{L}_2 P_{t-s}^{\Lambda_2}P_s^{\Lambda_2}(\tilde{x}) + P_{t-s}^{\Lambda_2}\mathcal{L}_2 P_s^{\Lambda_1}(\tilde{x}) + \mathcal{L}_2 P_{t-s}^{\Lambda_2}P_s^{\Lambda_1}(\tilde{x}) - P_{t-s}^{\Lambda_2}\mathcal{L}_1 P_s^{\Lambda_1}(\tilde{x})$$

the first and the third term cancel out, we have

$$\frac{d}{ds}P_{t-s}^{\Lambda_2}\left(P_s^{\Lambda_2}(\tilde{x}) - P_s^{\Lambda_1}(\tilde{x})\right) = P_{t-s}^{\Lambda_2}(\mathcal{L}_2 - \mathcal{L}_1)P_s^{\Lambda_1}(\tilde{x})$$

To control the convergence of the sequence, we study the difference of consecutive elements acting on local elements as follows

$$\int_0^t \frac{d}{ds}P_{t-s}^{\Lambda_2}\left(P_s^{\Lambda_2}(\tilde{x}) - P_s^{\Lambda_1}(\tilde{x})\right) = \int_0^t ds P_{t-s}^{\Lambda_2}(\mathcal{L}_2 - \mathcal{L}_1)\, P_s^{\Lambda_1}(\tilde{x})$$

$$P_t^{\Lambda_2}(\tilde{x}) - P_t^{\Lambda_1}(\tilde{x}) = \int_0^t ds P_{t-s}^{\Lambda_2}(\mathcal{L}_2 - \mathcal{L}_1)\, P_s^{\Lambda_1}(\tilde{x})$$

and taking the norm on both sides

$$\left\|P_t^{\Lambda_2}(\tilde{x}) - P_t^{\Lambda_1}(\tilde{x})\right\| = \left\|\int_0^t ds P_{t-s}^{\Lambda_2}(\mathcal{L}_2 - \mathcal{L}_1)\, P_s^{\Lambda_1}(\tilde{x})\right\| .$$

Using the contractivity property of the dynamics on the right hand side we have

$$\left\|P_t^{\Lambda_2}(\tilde{x}) - P_t^{\Lambda_1}(\tilde{x})\right\| \le \int_0^t ds \left\|(\mathcal{L}_2 - \mathcal{L}_1)P_s^{\Lambda_1}(\tilde{x})\right\| \qquad (4.1)$$

We study carefully the expression $(\mathcal{L}_2 - \mathcal{L}_1)P_s^{\Lambda_1}(\tilde{x})$. The difference of two elementary Markov generators is also an elementary Markov generator. It is sufficient to study the expression $\mathcal{L}_{X+\boldsymbol{j}}P_s^{\Lambda_1}(\tilde{x})$. By regularity assumption we have

$$\left\|\mathcal{L}_{X+\boldsymbol{j}}P_s^{\Lambda_1}(\tilde{x})\right\| \leq \sum_k b_{\boldsymbol{j}k}\left\|\partial_{\boldsymbol{j}}P_s^{\Lambda_1}(\tilde{x})\right\|$$

we study the term $\partial_j P_s^{\Lambda_1} f_\Lambda$ using the differential form

$$\frac{d}{d\tilde{s}}\left(\partial_{\boldsymbol{j}}P_s^{\Lambda_1}(\tilde{x})\right) = \partial_{\boldsymbol{j}}\frac{d}{d\tilde{s}}P_{\tilde{s}}^{\Lambda_1}(\tilde{x}) = \partial_{\boldsymbol{j}}\mathcal{L}_1 P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

we have the following

$$\frac{d}{d\tilde{s}}P_{s-\tilde{s}}^{\Lambda_1}\left(\partial_j P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right) = -\mathcal{L}_1 P_{s-\tilde{s}}^{\Lambda_1}\partial_{\boldsymbol{j}}P_{\tilde{s}}^{\Lambda_1}(\tilde{x}) + P_{s-\tilde{s}}^{\Lambda_1}\partial_{\boldsymbol{j}}\mathcal{L}_1 P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$= P_{s-\tilde{s}}^{\Lambda_1}\partial_{\boldsymbol{j}}\mathcal{L}_1 P_{\tilde{s}}^{\Lambda_1}(\tilde{x}) - P_{s-\tilde{s}}^{\Lambda_1}\mathcal{L}_1\partial_j P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$= P_{s-\tilde{s}}^{\Lambda_1}(\partial_j\mathcal{L}_1 - \mathcal{L}_1\partial_j)P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$\frac{d}{d\tilde{s}}P_{s-\tilde{s}}^{\Lambda_1}\left(\partial_j P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right) = P_{s-\tilde{s}}^{\Lambda_1}[\partial_j, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

hence,

$$\frac{d}{d\tilde{s}}P_{s-\tilde{s}}^{\Lambda_1}\left(\partial_k P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right) = P_{s-\tilde{s}}^{\Lambda_1}[\partial_k, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

integrating, and using contractivity property of the Markov semi-group we have the following.

$$\int_0^s\frac{d}{d\tilde{s}}P_{s-\tilde{s}}^{\Lambda_1}\left(\partial_k P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right)d\tilde{s} = \int_0^s d\tilde{s}\,P_{s-\tilde{s}}^{\Lambda_1}[\partial_k, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$\int_0^s\frac{d}{d\tilde{s}}P_{s-\tilde{s}}^{\Lambda_1}\left(\partial_k P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right)d\tilde{s} = \int_0^s d\tilde{s}\,P_{s-\tilde{s}}^{\Lambda_1}[\partial_k, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$P_0^{\Lambda_1}\left(\partial_k P_s^{\Lambda_1}(\tilde{x})\right) - P_s^{\Lambda_1}\left(\partial_k P_0^{\Lambda_1}(\tilde{x})\right) = \int_0^s d\tilde{s}\,P_{s-\tilde{s}}^{\Lambda_1}[\partial_k, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$\partial_k P_s^{\Lambda_1}(\tilde{x}) = P_s^{\Lambda_1}(\partial_k(\tilde{x})) + \int_0^s d\tilde{s}\,P_{s-\tilde{s}}^{\Lambda_1}[\partial_k, \mathcal{L}_1]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})$$

$$\left\|\partial_k P_s^{\Lambda_1}(\tilde{x})\right\| \leq \|\partial_k(\tilde{x})\| + \sum_{i\in\Lambda_1}\int_0^s d\tilde{s}\left\|[\partial_k, \mathcal{L}_{X+i}]P_{\tilde{s}}^{\Lambda_1}(\tilde{x})\right\| \qquad (4.2)$$

from the condition

$$\left\|[\partial_k, \mathcal{L}_{X+j}]P_s^{\Lambda_1}(\tilde{x})\right\| \le \sum_{\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+\boldsymbol{j}} \left\|\partial_{\boldsymbol{l}} P_s^{\Lambda_1}(\tilde{x})\right\|,$$

the right hand side becomes bounded by

$$\left\|\partial_k P_s^{\Lambda_1}(\tilde{x})\right\| \le \|\partial_k(\tilde{x})\| + \int_0^s d\tilde{s}\ \sum_{i\in\Lambda_1} \left(\sum_{\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+\boldsymbol{j}} \left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\|\right).$$

Since we have

$$\sup_{\boldsymbol{k}\,\in\,\mathbb{Z}^d} \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+j} = \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{kl} < \infty,$$

thus,

$$\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le \|\partial_{\boldsymbol{k}}(\tilde{x})\| + \int_0^s d\tilde{s}\ \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}} \left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\|$$

therefore, $\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| - \int_0^s d\tilde{s}\ \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{kl} \left\|\partial_k P_s^{\Lambda_1}(\tilde{x})\right\| \le \|\partial_k(\tilde{x})\|$

$$\left(1 - \sum_{k,l\in\mathbb{Z}^d} a_{kl} \int_0^s d\tilde{s}\right)\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le \|\partial_k(\tilde{x})\|$$

$$\left(1 - \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}\right)\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le \|\partial_k(\tilde{x})\|.$$

Since $\left(1 - \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}\right)$ is positive, we can write

$$\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le \left(1 - \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}\right)^{-1} \|\partial_k(\tilde{x})\|$$

from the relation $\frac{1}{(1-x)} \le (1-x) \le e^x.$

We have that, $\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le \left(1 - \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}\right)^{-1} \|\partial_k(\tilde{x})\| \le e^{\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}} \|\partial_k(\tilde{x})\|$

thus, (4.2) becomes bounded by, $\left\|\partial_{\boldsymbol{k}} P_s^{\Lambda_1}(\tilde{x})\right\| \le e^{\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}} \|\partial_k(\tilde{x})\|$

and therefore, (4.1) is bounded by,

$$\left\|P_t^{\Lambda_2}(\tilde{x}) - P_t^{\Lambda_1}(\tilde{x})\right\| \le \int_0^t ds \left\|(\mathcal{L}_2 - \mathcal{L}_1)P_s^{\Lambda_1}(\tilde{x})\right\| \le t \sum_{\boldsymbol{k}} b_{\boldsymbol{jk}}\, e^{\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} s a_{\boldsymbol{kl}}} \|\partial_k(\tilde{x})\|$$

hence we have the following bound

$$\left\|P_t^{\Lambda_2}(\tilde{x}) - P_t^{\Lambda_1}(\tilde{x})\right\| \le t \sum_{\boldsymbol{j}\in\Lambda_{2\setminus\Lambda_1}} \sum_{\boldsymbol{k},\boldsymbol{l}} b_{\boldsymbol{jk}}\, e^{s a_{\boldsymbol{kl}}} \|\partial_{\boldsymbol{k}}(\tilde{x})\|$$

this holds for any $\Lambda_2 \subseteq \mathbb{Z}^d$ containing a set Λ_1 .

The summability properties of the matrices $b_{\boldsymbol{jk}}$, $\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}$ on the right hand and norm continuity of the semigroup, lead us to conclude that the sequence of semigroups on $\mathcal{M}_1$ is Cauchy. This limit exists and is given by $P_t^X = e^{t\mathcal{L}^X}$.

We therefore have that $\left\|P_t^{\Lambda_n}(\tilde{x}) - P_t^X(\tilde{x})\right\| \to 0$ as $|\Lambda_n| \to \infty$,

Theorem 4.2

If the condition $\sum_{\boldsymbol{j}:X+\boldsymbol{j}\not\ni\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^d} a_{\boldsymbol{kl}}^{X+\boldsymbol{j}} < \infty$ is satisfied and $P_t^{X,k}$ is a Markov semigroup with generator $\mathcal{L}^{X,k} = \mathcal{L}^X - \sum_{j:X+j\ni k} \mathcal{L}_{X+j}$, for $\boldsymbol{k} \in X + j$, then the infinite volume quantum stochastic dynamics P_t^X is strongly ergodic, that is,

$$|||P_t^X(\tilde{x})||| \le e^{-(1-\lambda)|X|t}|||\tilde{x}||| , \quad \text{with } \lambda \in (0,1).$$

Proof:

Let P_t^X denote the semigroup corresponding to the generator $\mathcal{L}^X = \sum_{J\in\mathbb{Z}^d} \mathcal{L}_{X+\boldsymbol{j}}$,

where $\mathcal{L}_{X+\boldsymbol{j}}((\tilde{x})) = E_{X+\boldsymbol{j}}((\tilde{x})) - (\tilde{x})$.We note that $E_{X+j}(\mathcal{M}_\Lambda) \subseteq \mathcal{M}_{\Lambda^c+\mathrm{j}}$, and

$\partial_k \mathcal{L}_{X+\boldsymbol{j}}(\tilde{x}) = \partial_{\boldsymbol{k}}\big(E_{X+\boldsymbol{j}}(\tilde{x}) - (\tilde{x})\big) = -\partial_{\boldsymbol{k}}(\tilde{x})$, for $\boldsymbol{k} \in X + \boldsymbol{j}$. To show the exponential decay in the triple bar we need to study the term $\|\partial_k P_s^X(\tilde{x})\|$ for all $j \in \mathbb{Z}^d$. For $P_t^{X,k}$ with the corresponding generator $\mathcal{L}^{X,k} = \mathcal{L}^X - \sum_{\boldsymbol{j}:X+\boldsymbol{j}\ni k} \mathcal{L}_{X+\boldsymbol{j}}$, $\boldsymbol{k} \in X + \boldsymbol{j}$. Let $s \in [0,t)$, we have, $\frac{d}{ds} P_{t-s}^{X,k}(\partial_k P_s^X(\tilde{x})) = P_{t-s}^{X,k}[\partial_k, \mathcal{L}^{X,k}]P_s^X(\tilde{x})$,

multiplying both sides by $e^{s|X|}$, where $|X|$ is the cardinality of the finite set $X \subset \Lambda$,

$$\frac{d}{ds} e^{s|X|}P_{t-s}^{X,k}\, \partial_k P_s^X(\tilde{x}) = e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x}) .$$

Integrating this equation from 0 to t,

$$\int_0^t \frac{d}{ds}\left(e^{s|X|}\, P_{t-s}^{X,k}\, \partial_k P_s^X(\tilde{x})\right) ds = \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x})$$

$$e^{t|X|}\, P_0^{X,k} \partial_{\boldsymbol{k}} P_t^X(\tilde{x}) - e^{0|X|}\, P_t^{X,k} \partial_{\boldsymbol{k}} P_0^X(\tilde{x}) = \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_{\boldsymbol{k}}, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x})$$

$$e^{t|X|}\,\partial_{\boldsymbol{k}} P_t^X(\tilde{x}) \;-\; P_t^{X,k}\partial_{\boldsymbol{k}}(\tilde{x}) \;=\; \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_{\boldsymbol{k}}\,, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x})$$

$$e^{t|X|}\,\partial_k P_t^X(\tilde{x}) \;=\; P_t^{X,k}\partial_{\boldsymbol{k}}(\tilde{x}) \;+\; \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_{\boldsymbol{k}}\,, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x})$$

multiplying both sides by $e^{-t|X|}$.

$$\partial_{\boldsymbol{k}} P_t^X(\tilde{x}) \;=\; e^{-t|X|} P_t^{X,k}\partial_{\boldsymbol{k}}(\tilde{x}) \;+\; \int_0^t ds\, e^{-(t-s)|X|}\, P_{t-s}^{X,k}\,[\partial_{\boldsymbol{k}}\,, \mathcal{L}^{X,\boldsymbol{k}}]\, P_s^X(\tilde{x}).$$

Using contraction property of the Markov semigroup $P_t^{X,k}$ we have,

$$\|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\| \leq\; e^{-t|X|}\|\partial_{\boldsymbol{k}}(\tilde{x})\| + \left\| \int_0^t ds\, e^{-(t-s)|X|}\; [\partial_{\boldsymbol{k}}\,, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x}) \right\|$$

$$\|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\| \leq\; e^{-t|X|}\|\partial_{\boldsymbol{k}}(\tilde{x})\| + \int_0^t ds\, e^{-(t-s)|X|} \|\, [\partial_k\,, \mathcal{L}^{X,k}]\, P_s^X(\tilde{x})\|$$

we note that by definition $\mathcal{L}^{X,k} \equiv \sum_{j\,:X+j\not\ni k} \mathcal{L}_{X+j}$.

thus $\|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\| \leq\; e^{-t|X|}\|\partial_{\boldsymbol{k}}(\tilde{x})\| + \int_0^t ds\, e^{-(t-s)|X|} \sum_{\boldsymbol{j}:X+\boldsymbol{j}\not\ni k} \left\| \,[\partial_k\,,\; \mathcal{L}_{X+\boldsymbol{j}}]\, P_s^X(\tilde{x}) \right\|$

since from definition 4.1.3 condition (ii) we have,

$$\sum_{\boldsymbol{j}:X+\boldsymbol{j}\not\ni k} \left\| \,[\partial_{\boldsymbol{k}}\,, \mathcal{L}_{X+\boldsymbol{j}}] P_t^X(\tilde{x}) \right\| \leq\; \sum_{\boldsymbol{j}:X+\boldsymbol{j}\not\ni k}\; \sum_{\boldsymbol{l}\in\mathbb{Z}^d} a_{kl}^{X+j}\, \|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\|$$

where $\sup_{l\,\in\,\mathbb{Z}^d}\; \sum_k\; \sum_{j\,:X+j\not\ni k} a_{kl}^{X+j} \leq \lambda|X| \,< \infty$

$$\|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\| \leq\; e^{-t|X|}\|\partial_{\boldsymbol{k}}(\tilde{x})\| + \int_0^t ds\, e^{-(t-s)|X|} \sum_{j\,:X+j\not\ni k}\; \sum_{l\in\mathbb{Z}^d} a_{kl}^{X+j}\, \|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\|$$

$$\|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\| \leq\; e^{-t|X|}\|\partial_{\boldsymbol{k}}(\tilde{x})\| + \lambda|X| \int_0^t ds\, e^{-(t-s)|X|}\, \|\partial_{\boldsymbol{k}} P_t^X(\tilde{x})\|$$

thus summing the inequalities over $\boldsymbol{k} \in \mathbb{Z}^d$ we have ,

$$|||P_t^X(\tilde{x})||| \leq\; e^{-|X|t}|||\tilde{x}||| + \lambda|X| \int_0^t ds\, e^{-(t-s)|X|}\, |||P_t^X(\tilde{x})|||$$

solving the inequality we have,

$$|||P_t^X(\tilde{x})||| - \lambda|X|\, e^{-|X|t} \int_0^t ds\, e^{|X|s}\, |||P_t^X(\tilde{x})||| \;\leq\; e^{-t|X|}|||\tilde{x}|||$$

multiplying by $e^{|X|t}$ and factoring $|||P_t^X(\tilde{x})|||$ we have,

$$\left(e^{|X|t} - \lambda|X| \int_0^t e^{|X|s} ds \right) |||P_t^X(\tilde{x})||| \;\leq\; |||\tilde{x}|||$$

hence writing $e^{|X|t} = 1 + |X| \int_0^t ds e^{s|X|}$ we have,

$$\left(1+|X|\int_0^t dse^{|X|s}-\lambda|X|\int_0^t ds\, e^{|X|s}\right)|||P_t^X(\tilde{x})|||\ \le\ |||\tilde{x}|||$$

collecting the terms in the bracket, we have,

$$\left(1+(1-\lambda)|X|\int_0^t ds\, e^{|X|s}\right)|||P_t^X(\tilde{x})|||\ \le\ |||\tilde{x}|||$$

we observed that for $\lambda \in (0,1)$ we have the relation $\int_0^t dse^{(1-\lambda)|X|s} \le \int_0^t dse^{|X|s}$

hence, $\left(1+(1-\lambda)|X|\int_0^t ds\, e^{(1-\lambda)|X|s}\right) \le \left(1+(1-\lambda)|X|\int_0^t ds\, e^{|X|s}\right)$

we have $\left(1+(1-\lambda)|X|\int_0^t ds\, e^{(1-\lambda)|X|s}\right)|||P_t^X(\tilde{x})|||\ \le\ |||\tilde{x}|||$

now since $\quad e^{(1-\lambda)|X|t} = 1+(1-\lambda)|X|\int_0^t ds\, e^{(1-\lambda)|X|s}$

we have, $\quad e^{(1-\lambda)|X|t}\,|||P_t^X(\tilde{x})|||\ \le\ |||\tilde{x}|||$

$$|||P_t^X(\tilde{x})|||\ \le\ e^{-(1-\lambda)|X|t}|||\tilde{x}|||$$

Theorem 4.3

The semi-group $(P_t^X)_{t\ge 0}$ is strongly ergodic in the sense that there is a unique $(P_t^X)_{t\ge 0}$ - invariant locally normal state φ_Λ for which we have

$$|||P_t^X(\tilde{x})-\varphi_\Lambda(\tilde{x})||| \le 2e^{-(1-\lambda)|X|t}|||\tilde{x}|||\,.$$

Proof

To show the strong ergodicity property of the dynamics P_t^X, we have the following formulation. We note that by the weak compactness of the space of state on $\mathcal{M}_\Lambda$ and the fact that the dynamics P_t^X has a Feller property, the set of invariant states with respect to the dynamics is non-empty. Let φ_Λ be such an invariant locally normal state,

then $\quad |||P_t^X(\tilde{x})-\varphi_\Lambda(\tilde{x})||| = \|P_t^X(\tilde{x})-\varphi_\Lambda(P_t^X(\tilde{x}))\|$

now we consider the tensor product algebra of $\mathcal{M}_\Lambda$ by itself, and, from (Takesaki,1979), we have the completely positive map $\theta : \mathcal{M}_\Lambda \otimes \mathcal{M}_\Lambda \longrightarrow \mathcal{M}_\Lambda$

such that if $\tilde{x}_{\Lambda_1}, \tilde{x}_{\Lambda_2} \in \mathcal{M}_\Lambda$ we have, $\quad \theta\left(\tilde{x}_{\Lambda_1} \otimes \tilde{x}_{\Lambda_2}\right) = \varphi_\Lambda(\tilde{x}_{\Lambda_1})\tilde{x}_{\Lambda_2}$.

We note that $\quad \theta(P_t^X(\tilde{x}) \otimes I) = \varphi_\Lambda(P_t^X(\tilde{x}))I = \varphi_\Lambda(P_t^X(\tilde{x}))$

$$\theta(I \otimes P_t^X(\tilde{x})) = \varphi_\Lambda(I)P_t^X(\tilde{x}) = P_t^X(\tilde{x})\,,$$

since $\varphi_\Lambda \circ P_t^X = \varphi_\Lambda$ we have,

$$\|P_t^X(\tilde{x}) - \varphi_\Lambda(\tilde{x})\| = \|P_t^X(\tilde{x}) - \varphi_\Lambda(P_t^X(\tilde{x}))\|$$

$$= \|\theta(I \otimes P_t^X(\tilde{x})) - \theta(P_t^X(\tilde{x}) \otimes I)\|$$

$$\leq \|\theta(I \otimes P_t^X(\tilde{x}) - P_t^X(\tilde{x}) \otimes I)\|$$

$$\leq \|I \otimes P_t^X(\tilde{x}) - P_t^X(\tilde{x}) \otimes I\|.$$

With this formulation in mind, we may express $P_t^X(\tilde{x})$ as follows:

$$P_t^X(\tilde{x}) = P_t^X(\tilde{x}) + Tr_{j_1}P_t^X(\tilde{x}) - Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_2}P_t^X(\tilde{x}) - Tr_{j_2}P_t^X(\tilde{x}) + Tr_{j_3}P_t^X(\tilde{x})$$

$$- Tr_{j_3}P_t^X(\tilde{x}) + Tr_{j_4}P_t^X(\tilde{x}) - Tr_{j_4}P_t^X(\tilde{x}) + \ldots + Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}P_t^X(\tilde{x})$$

thus let $\{j_n\}_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ be a sequence with lexicographic ordering such that for each $j_i \in \Lambda_i$ and $\Lambda_{i-1} \subset \Lambda_i$ we have $j_{i-1} \leq j_i$. Since the partial traces $Tr_{j_{i-1}}$ and Tr_{j_i} are projections with the ordering $Tr_{j_1} \leq Tr_{j_2}$ we have the relation

$$Tr_{j_i}Tr_{j_{i-1}} = Tr_{j_{i-1}}Tr_{j_i} = Tr_{j_{i-1}},$$

hence we rewrite the zero terms as follows

$$P_t^X(\tilde{x}) = P_t^X(\tilde{x}) + Tr_{j_1}P_t^X(\tilde{x}) - Tr_{j_1}Tr_{j_2}P_t^X(\tilde{x}) + Tr_{j_2}P_t^X(\tilde{x}) - Tr_{j_2}Tr_{j_3}P_t^X(\tilde{x})$$

$$+Tr_{j_3}P_t^X(\tilde{x}) - Tr_{j_3}Tr_{j_4}P_t^X(\tilde{x}) \quad \ldots \quad +Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}Tr_{j_{n+1}}P_t^X(\tilde{x}).$$

$$P_t^X(\tilde{x}) = P_t^X(\tilde{x}) - Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_1}P_t^X(\tilde{x}) + Tr_{j_1}(P_t^X(\tilde{x}) - Tr_{j_2}P_t^X(\tilde{x}))$$

$$+ Tr_{j_2}\left(P_t^X(\tilde{x}) - Tr_{j_3}P_t^X(\tilde{x})\right) + Tr_{j_3}\left(P_t^X(\tilde{x}) - Tr_{j_4}P_t^X(\tilde{x})\right)$$

$$+Tr_{j_4}(P_t^X(\tilde{x}) - Tr_{j_5}P_t^X(\tilde{x})) \ldots \quad +Tr_{j_n}(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x}))$$

hence from $\partial_{j_1}P_t^X(\tilde{x}) = P_t^X(\tilde{x}) - Tr_{j_1}P_t^X(\tilde{x})$,

we have $P_t^X(\tilde{x}) = \partial_{j_1}P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1,j_2,j_3,\ldots,j_n\}}\left(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})\right)$

note that the summation is finite because ,

$$Tr_{j_n}(P_t^X(\tilde{x}) - Tr_{j_{n+1}}P_t^X(\tilde{x})) = Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}Tr_{j_{n+1}}P_t^X(\tilde{x})$$

$$= Tr_{j_n}P_t^X(\tilde{x}) - Tr_{j_n}P_t^X(\tilde{x}) = 0 \quad \text{for } n \in \mathbb{N}.$$

Thus we have

$$\sum_{n\in\mathbb{N}} \left\| Tr_{\{j_1,j_2,j_3,\ldots,j_n\}} \left(P_t^X(\tilde{x}) - Tr_{j_{n+1}} P_t^X(\tilde{x})\right)\right\| = \sum_{n\in\mathbb{N}} \left\| Tr_{\{j_1\ldots j_n\}} \partial_{j_{n+1}} P_t^X(\tilde{x})\right\| < \infty$$

therefore we can write $I \otimes P_t^X(\tilde{x}) = \partial_{j_1} P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}} \partial_{j_{n+1}} P_t^X(\tilde{x})$

and also we have $P_t^X(\tilde{x}) \otimes I = \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}} \partial_{j_{n+1}} P_t^X(\tilde{x}) - \partial_{j_1} P_t^X(\tilde{x})$

Hence $\|P_t^X(\tilde{x}) - \varphi_\Lambda(\tilde{x})\| \leq \left\|\left(I \otimes P_t^X(\tilde{x})\right) - (P_t^X(\tilde{x}) \otimes I)\right\|$

$$\leq \left\|\left(\partial_{j_1} P_t^X(\tilde{x}) + \sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}} \partial_{j_{n+1}} P_t^X(\tilde{x})\right) - \left(\sum_{n\in\mathbb{N}} Tr_{\{j_1\ldots j_n\}} \partial_{j_{n+1}} P_t^X(\tilde{x}) - \partial_{j_1} P_t^X(\tilde{x})\right)\right\|$$

$$\leq 2\left\|\partial_{j_1} P_t^X(\tilde{x})\right\| = 2|\|P_t^X(\tilde{x})\||$$

therefore we have $\|P_t^X(\tilde{x}) - \varphi_\Lambda(\tilde{x})\| \leq 2|\|P_t^X(\tilde{x})\|| \leq 2\, e^{-(1-\lambda)|X|t} |\|\tilde{x}\||$

CHAPTER FIVE

QUANTUM ENTANGLEMENT OF TWO HARMONIC OSCILLATORS

5.0 Introduction

Entangled quantum states are characterized by non local correlations that cannot be described by classical mechanics. Such correlations play an important role in quantum information science (Its etal., 2008). Quantum information is indispensable for the description and performance of tasks such as teleportation, super dense coding, quantum cryptography and quantum computation (Nielsen and Chuang, 2000). It is therefore essential to be able to generate, detect and quantify entanglement. In its full generality this is still an open problem (Its et al, 2008). Entanglement has also given new insight for understanding many physical phenomenon like super-radiance and superconductivity. In particular, understanding the role of entanglement in the existing methods of simulation of quantum spins systems allowed for significant improvement of the method, as well as understanding their limitations (Horodecki et al, 2007).

In this chapter, we use the Lindblad theory of open quantum systems to derive the equation of motion and the Simon-Peres type equation in terms of the variance and covariance of the coordinates q_x, q_y and momenta p_x, p_y operators, of two harmonic oscillators interacting with an environment.

5.1 The Lindbladian Operator

Within the theory of quantum open systems, we consider the bounded Lindblad-type generator of a dynamical semigroup Φ_t defined by,

$$\mathcal{L}(f) = \sum_j \; 2(V_j\, f\, V_j^*) - \left(V_j^* V_j\right) f - f\left(V_j^* V_j\right), \qquad f \in \mathfrak{H}$$

where V_j , V_j^* are operators defined on a Hilbert space $\mathfrak{H}$, are called the jump operators. Physically these operators represent the interaction of the open system with the environment and can be chosen freely.

A simple condition imposed to the operators V_j , V_j^* is that they are functions of the basic observables of the one dimensional quantum mechanical system p, q (with $[p, q] = -[q, p] \subset iI$, where I is the identity operator on $\mathfrak{H}$ and $[p, p] = [q, q] = 0$, here we assume that $\hbar = 1$) . This condition allow the obtained model to be exactly solvable. A precise version of this condition is that the linear space spanned by the noncommutative polynomials in p, q are invariant under the action of the completely dissipative mapping $\mathcal{L}$. This condition implies that V_j are at most the first degree polynomials in p, q (Isar, etal., 1994).We assume that for two Harmonic oscillators defined by the canonical observables of coordinates q_x, q_y and momenta p_x, p_y,the operator V_j generate four linearly independent operators. This is because the operators q_x, q_y ,p_x, p_y give a basis (Sandulescu and Scutaru,1987) which we define as follows,

$$V_j = a_{jx}\, p_x + a_{jy}\, p_y + b_{jx}\, q_x + b_{jy}\, q_y \quad \text{and} \quad V_j^* = a_{jx}^*\, p_x + a_{jy}^*\, p_y + b_{jx}^*\, q_x + b_{jy}^*\, q_y,$$

$j = 1,2,3,4$ Where a_{jx}, b_{jx} , a_{jy} , b_{jy} are complex numbers and a_{jx}^* a_{jy}^* , b_{jx}^*, b_{jy}^* their corresponding complex conjugates. Now for $j = 1,2,3,4$ we derive the coordinate form for the Lindbladian operator by substituting the following equations

$$\begin{aligned} V_j^* V_j f = {} & (a_{jx}^* a_{jx}) p_x^2 f + (a_{jy}^* a_{jy}) p_y^2 f + (b_{jx}^* b_{jx}) q_x^2 f + (b_{jy}^* b_{jy}) q_y^2 f + (a_{jx}^* b_{jx}) p_x q_x f \\ & + (b_{jx}^* a_{jx}) q_x p_x f + (a_{jy}^* b_{jy}) p_y q_y f + (b_{jy}^* a_{jy}) q_y p_y f + (a_{jx}^* b_{jy}) p_x q_y f + (b_{jy}^* a_{jx}) q_y p_x f \\ & + (b_{jx}^* a_{jy}) q_x p_y f + (a_{jy}^* b_{jx}) p_y q_x f + (a_{jx}^* a_{jy}) p_x p_y f + (a_{jy}^* a_{jx}) p_y p_x f + (b_{jx}^* b_{jy}) q_x q_y f \\ & + (b_{jy}^* b_{jx}) q_y q_x f. \end{aligned}$$

$$\begin{aligned} f V_j^* V_j = {} & (a_{jx}^* a_{jx}) f p_x^2 + (a_{jy}^* a_{jy}) f p_y^2 + (b_{jx}^* b_{jx}) f q_x^2 + (b_{jy}^* b_{jy}) f q_y^2 + (a_{jx}^* b_{jx}) f p_x q_x \\ & + (b_{jx}^* a_{jx}) f q_x p_x + (a_{jy}^* b_{jy}) f p_y q_y + (b_{jy}^* a_{jy}) f q_y p_y + (a_{jx}^* b_{jy}) f p_x q_y + (b_{jy}^* a_{jx}) f q_y p_x \\ & + (b_{jx}^* a_{jy}) f q_x p_y + (a_{jy}^* b_{jx}) f p_y q_x + (a_{jx}^* a_{jy}) f p_x p_y + (a_{jy}^* a_{jx}) f p_y p_x + (b_{jx}^* b_{jy}) f q_x q_y \\ & + (b_{jy}^* b_{jx}) f q_y q_x . \end{aligned}$$

$$V_j^* f V_j = (a_{jx}^* a_{jx}) p_x f p_x + (a_{jy}^* a_{jy}) p_y f p_y + (b_{jx}^* b_{jx}) q_x f q_x + (b_{jy}^* b_{jy}) q_y f q_y$$

$$+(a_{jx}^* b_{jx})p_x f q_x + (b_{jx}^* a_{jx})q_x f p_x + (a_{jy}^* b_{jy})p_y f q_y + (b_{jy}^* a_{jy})q_y f p_y + (a_{jx}^* b_{jy})p_x f q_y$$

$$+(b_{jy}^* a_{jx})q_y f p_x + (b_{jx}^* a_{jy})q_x f p_y + (a_{jy}^* b_{jx})p_y f q_x + (a_{jx}^* a_{jy})p_x f p_y + (a_{jy}^* a_{jx})p_y f p_x$$

$$+(b_{jx}^* b_{jy})q_x f q_y + (b_{jy}^* b_{jx})q_y f q_x .$$

into the operator

$$\mathcal{L}(f) = \sum_j^4 2V_j f V_j^* - V_j^* V_j f - f V_j^* V_j$$

For $j = 1,2,3,4$

$$\mathcal{L}(f) = (a_x^* a_x)(2p_x f p_x - p_x^2 f - f p_x^2) + (a_y^* a_y)(2p_y f p_y - p_y^2 f - f p_y^2)$$

$$+(b_x^* b_x)(2q_x f q_x - q_x^2 f - f q_x^2) + (b_y^* b_y)(2q_y f q_y - q_y^2 f - f q_y^2)$$

$$+(a_x^* b_x)(2p_x f q_x - p_x q_x f - f p_x q_x) + (b_x^* a_x)(2q_x f p_x - q_x p_x f - f q_x p_x)$$

$$+(a_y^* b_y)(2p_y f q_y - p_y q_y f - f p_y q_y) + (b_y^* a_y)(2q_y f p_y - q_y p_y f - f q_y p_y)$$

$$+(a_x^* b_y)(2p_x f q_y - p_x q_y f - f p_x q_y) + (b_y^* a_x)(2q_y f p_x - q_y p_x f - f q_y p_x)$$

$$+(b_x^* a_y)(2q_x f p_y - q_x p_y f - f q_x p_y) + (a_y^* b_x)(2p_y f q_x - p_y q_x f - f p_y q_x)$$

$$+(a_x^* a_y)(2p_x f p_y - p_x p_y f - f p_x p_y) + (a_y^* a_x)(2p_y f p_x - p_y p_x f - f p_y p_x)$$

$$+(b_x^* b_y)(2q_x f q_y - q_x q_y f - f q_x q_y) + (b_y^* b_x)(2q_y f q_x - q_y q_x f - f q_y q_x)$$

We have

$$\mathcal{L}(f) = (a_x^* a_x)(p_x[f, p_x] - [f, p_x]p_x) + (a_y^* a_y)(p_y[f, p_y] - [f, p_y]p_y)$$

$$+(b_x^* b_x)(q_x[f, q_x] - [f, q_x]q_x) + (b_y^* b_y)(q_y[f, q_y] - [f, q_y]q_y)$$

$$+(a_x^* b_x)(p_x[f, q_x] - [f, p_x]q_x) + (b_x^* a_x)(q_x[f, p_x] - [f, q_x]p_x)$$

$$+(a_y^* b_y)(p_y[f, q_y] - [f, p_y]q_y) + (b_y^* a_y)(q_y[f, p_y] - [f, q_y]p_y)$$

$$+(a_x^* b_y)(p_x[f, q_y] - [f, p_x]q_y) + (b_y^* a_x)(q_y[f, p_x] - [f, q_y]p_x)$$

$$+\left(b_x^* a_y\right)\left(q_x[f,p_y]-[f,q_x]p_y\right)+\left(a_y^* b_x\right)(p_y[f,q_x]-[f,p_y]q_x)$$

$$+\left(a_x^* a_y\right)(p_x[f,p_y]-[f,p_x]p_y)+\left(a_y^* a_x\right)(p_y[f,p_x]-[f,p_y]p_x)$$

$$+\left(b_x^* b_y\right)(q_x[f,q_y]-[f,q_x]q_y)+\left(b_y^* b_x\right)(q_y[f,q_x]-[f,q_y]q_x)$$

hence, the fact that $\mathcal{L}$ is a generator of a dynamical semigroup implies the positivity of the following matrix whose entries are,

$$\begin{pmatrix} (a_{jx}^* a_{jx}) & (a_{jx}^* b_{jx}) & (a_{jx}^* a_{jy}) & (a_{jx}^* b_{jy}) \\ (b_{jx}^* a_{jx}) & (b_{jx}^* b_{jx}) & (b_{jx}^* a_{jy}) & (b_{jx}^* b_{jy}) \\ (a_{jy}^* a_{jx}) & (a_{jy}^* b_{jx}) & (a_{jy}^* a_{jy}) & (a_{jy}^* b_{jy}) \\ (b_{jy}^* a_{jx}) & (b_{jy}^* b_{jx}) & (b_{jy}^* a_{jy}) & (b_{jy}^* b_{jy}) \end{pmatrix}$$

$j=1,2,3,4$. For simplicity we use the following notations,

$$D_{q_x q_y}=Re\left(a_{jx}^* a_{jy}\right)= D_{q_y q_x}=Re\left(a_{jy}^* a_{jx}\right)$$

$$D_{p_x p_y}=Re\left(b_{jx}^* b_{jy}\right)= D_{p_x p_y}=Re\left(b_{jy}^* b_{jx}\right),$$

$$D_{p_x q_y}=-Re\left(b_{jx}^* a_{jy}\right)=-Re\left(a_{jy}^* b_{jx}\right)=D_{q_y p_x}$$

$$D_{p_y q_x}=-Re\left(b_{jy}^* a_{jx}\right)=-Re\left(a_{jx}^* b_{jy}\right)=D_{q_x p_y}$$

$$D_{p_x q_x}=-Re\left(b_{jx}^* a_{jx}\right)=-Re\left(a_{jx}^* b_{jx}\right)=D_{q_x p_x}$$

$$D_{p_y q_y}=-Re\left(b_{jy}^* a_{jy}\right)=-Re\left(a_{jy}^* b_{jy}\right)=D_{q_y p_y}$$

$$D_{q_x q_x}=(a_{jx}^* a_{jx})=D_{q_y q_y}=(a_{jy}^* a_{jy})$$

$$D_{p_x p_x}=(b_{jx}^* b_{jx})=D_{p_y p_y}=(b_{jy}^* b_{jy})$$

For the imaginary part we use the following

$$\lambda=-im(a_{jx}^* b_{jx})=-im(a_{jy}^* b_{jy})=-im\left(a_{jx}^* b_{jy}\right)=-im\left(a_{jy}^* b_{jx}\right)$$

$$\lambda=im\left(b_{jy}^* a_{jy}\right)=im(b_{jx}^* a_{jx})=im\left(b_{jx}^* a_{jy}\right)=im\left(b_{jy}^* a_{jx}\right)$$

Hence for simplicity the matrix takes the following form

$$\begin{pmatrix} D_{q_xq_x} & -D_{p_xq_x}-i\lambda & D_{q_xq_y} & -D_{q_xp_y} \\ -D_{p_xq_x}+i\lambda & D_{p_xp_x} & -D_{q_yp_x} & D_{p_xp_y} \\ D_{q_xq_y} & -D_{q_yp_x} & D_{q_yq_y} & -D_{q_yp_y}-i\lambda \\ -D_{q_xp_y} & D_{p_xp_y} & -D_{q_yp_y}+i\lambda & D_{p_yp_y} \end{pmatrix},$$

where D and λ are real quantities.

The matrix can be conveniently written as $\begin{pmatrix} C_1 & C_3 \\ C_3^T & C_2 \end{pmatrix}$ in terms of a 2×2 matrices with

$C_1 = C_1^T$, $C_2 = C_2^T$ the constants satisfy the condition $D_{qq} > 0$, $D_{pp} > 0$, $D_{pq} > 0$ such that for example, we write one of the conditions obtained from the positivity of the matrix $D_{qq}\ D_{pp} - {D_{pq}}^2 > \lambda$, this inequality and the corresponding ones derived from the matrix are constraints imposed on the fact that Φ_t is a dynamical semigroup.

hence we have,

$$\begin{aligned}\mathcal{L}(f) = {} & D_{q_xq_x}(p_x[f,p_x]-[f,p_x]p_x) + D_{q_yq_y}(p_y[f,p_y]-[f,p_y]p_y) \\ & +D_{p_xp_x}(q_x[f,q_x]-[f,q_x]q_x) + D_{p_yp_y}(q_y[f,q_y]-[f,q_y]q_y) \\ & -D_{p_xq_x}-i\lambda(p_x[f,q_x]-[f,p_x]q_x) - D_{p_xq_x}+i\lambda(q_x[f,p_x]-[f,q_x]p_x) \\ & -D_{q_yp_y}-i\lambda(p_y[f,q_y]-[f,p_y]q_y) - D_{q_yp_y}+i\lambda(q_y[f,p_y]-[f,q_y]p_y) \\ & -D_{q_xp_y}(p_x[f,q_y]-[f,p_x]q_y) - D_{q_xp_y}(q_y[f,p_x]-[f,q_y]p_x) \\ & -D_{q_yp_x}\left(q_x[f,p_y]-[f,q_x]p_y\right) - D_{q_yp_x}(p_y[f,q_x]-[f,p_y]q_x) \\ & +D_{q_xq_y}(p_x[f,p_y]-[f,p_x]p_y) + D_{q_xq_y}(p_y[f,p_x]-[f,p_y]p_x) \\ & +D_{p_xp_y}(q_x[f,q_y]-[f,q_x]q_y) + D_{p_xp_y}(q_y[f,q_x]-[f,q_y]q_x)\end{aligned}$$

The Lindbladian operator in terms of the coordinates and momenta is then given by,

$$\begin{aligned}\mathcal{L}(f) = {} & -i\lambda(p_x[f,q_x]+[f,q_x]p_x) + i\lambda([f,p_x]q_x + q_x[f,p_x]) \\ & -i\lambda\left(p_y[f,q_y]+[f,q_y]p_y\right) + i\lambda([f,p_y]q_y + q_y[f,p_y]) \\ & + D_{q_xq_x}([p_x,[f,p_x]]) + D_{q_yq_y}([p_y,[f,p_y]]) + D_{p_xp_x}\left([q_x,[f,q_x]]\right) \\ & +D_{p_yp_y}([q_y,[f,q_y]]) - D_{p_xq_x}([p_x,[f,q_x]] + [q_x,[f,p_x]]) \\ & -D_{q_yp_y}\left([p_y,[f,q_y]] + [q_y,[f,p_y]]\right) - D_{q_xp_y}([p_x,[f,q_y]] + [q_y,[f,p_x]])\end{aligned}$$

$$-D_{q_y p_x}\left([q_x,[f,p_y]]+[p_y,[f,q_x]]\right)+D_{q_x q_y}\left([p_x[f,p_y]]+[p_y,[f,p_x]]\right)$$

$$+D_{p_x p_y}\left([q_x,[f,q_y]]+[q_y,[f,q_x]]\right)$$

5.2 Time- dependent Equations For p^2, q^2 and pq.

To get the dependence on time for the variance and covariance of the coordinate and momentum we consider the following equation,

$$\mathcal{L}(f) = -i\lambda(p[f,q] + [f,q]p) + i\lambda([f,p]q + q[f,p]) - i\lambda(p[f,q] + [f,q]p)$$
$$+i\lambda([f,p]q + q[f,p])) \quad +2\, D_{qq}\,([p,[f,p]]) + 2\, D_{pp}\,([q,[f,q]])$$
$$-2D_{pq}\,([p,[f,q]] + [q,[f,p]]) \;\; -2D_{qp}\,([p,[f,q]] + [q,[f,p]])$$
$$+D_{qq}\,([p,[f,p]] + [p,[f,p]]) \;\; +D_{pp}\,([q,[f,q]] + [q,[f,q]])$$

We substitute p^2 in our equation, and noting that the following commutation relation holds $[p^2,q] = p[p,q] + [p,q]p$ $\quad [p^2,pq] = p(p[p,q] + [p,q]p)$, and $\quad [p^2,p] = 0$

we have

$$\mathcal{L}(p^2) = -i\lambda(p[p^2,q] + [p^2,q]p) + i\lambda([p^2,p]q + q[p^2,p]) - i\lambda(p[p^2,q] + [p^2,q]p)$$
$$+i\lambda([p^2,p]q + q[p^2,p])) \quad +2\, D_{qq}\,([p,[p^2,p]]) + 2\, D_{pp}\,([q,[p^2,q]])$$
$$-2D_{pq}\,([p,[p^2,q]] + [q,[p^2,p]]) \;\; -2D_{qp}\,([p,[p^2,q]] + [q,[p^2,p]])$$
$$+D_{qq}\,([p,[p^2,p]] + [p,[p^2,p]]) \;\; +D_{pp}\,([q,[p^2,q]] + [q,[p^2,q]])$$

$$\mathcal{L}(p^2) = -2i\lambda(p[p^2,q]) + 2i\lambda(q[p^2,p]) - 2i\lambda[p^2,q]p)$$
$$+4\, D_{pp}\,([q,[p^2,q]]) - 2D_{pq}\,([p,[p^2,q]]) \;\; -2D_{qp}\,([p,[p^2,q]])$$
$$\mathcal{L}(p^2) = -2i\lambda(p(p[p,q] + [p,q]p) - 2i\lambda(p[p,q] + [p,q]p)p)$$
$$+4\, D_{pp}\,([q,(p[p,q] + [p,q]p)]) - 2D_{pq}\,([p,(p[p,q] + [p,q]p)])$$
$$-2D_{qp}\,([p,(p[p,q] + [p,q]p)])$$
$$\mathcal{L}(p^2) = -2i\lambda(p(pi + ip) - 2i\lambda(pi + ip)p + 4\, D_{pp}\,([q,(pi + ip)]) - 2D_{pq}\,([p,(pi + ip)])$$
$$-2D_{qp}\,([p,(pi + ip)])$$

hence we have $\qquad \mathcal{L}(p^2) = 8\lambda p^2 \; -8\, D_{pp}$

For q^2 we note that that the following commutation relation holds $[q^2,p] = q[q,p] + [q,p]q$, $\quad [q^2,pq] = (q[q,p] + [q,p]q)q \quad [q^2,q] = 0$

we have,

$$\mathcal{L}(q^2) = -i\lambda(p[q^2,q] + [q^2,q]p) + i\lambda([q^2,p]q + q[q^2,p]) - i\lambda(p[q^2,q] + [q^2,q]p)$$
$$+i\lambda([q^2,p]q + q[q^2,p])) \quad +2\, D_{qq}([p,[q^2,p]]) + 2\, D_{pp}([q,[q^2,q]])$$
$$-2D_{pq}([p,[q^2,q]] + [q,[q^2,p]]) \ -2D_{qp}([p,[q^2,q]] + [q,[q^2,p]])$$
$$+D_{qq}([p,[q^2,p]] + [p,[q^2,p]]) \ +D_{pp}([q,[q^2,q]] + [q,[q^2,q]])$$

$$\mathcal{L}(q^2) = i\lambda([q^2,p]q + q[q^2,p]) + i\lambda([q^2,p]q + q[q^2,p])) \quad +2\, D_{qq}([p,[q^2,p]])$$
$$-2D_{pq}([q,[q^2,p]]) \ -2D_{qp}([q,[q^2,p]])$$
$$+D_{qq}([p,[q^2,p]] + [p,[q^2,p]])$$

$$\mathcal{L}(q^2) = \ i\lambda((q[q,p] + [q,p]q\,)q + q(q[q,p] + [q,p]q\,))$$
$$+i\lambda((q[q,p] + [q,p]q\,)q + q(q[q,p] + [q,p]q\,))$$
$$+2\, D_{qq}([p,(q[q,p] + [q,p]q\,)]) -2D_{pq}([q,(q[q,p] + [q,p]q\,)])$$
$$-2D_{qp}([q,(q[q,p] + [q,p]q\,)])$$
$$+D_{qq}([p,(q[q,p] + [q,p]q\,)] \ +[p,(q[q,p] + [q,p]q\,)])$$

$$\mathcal{L}(q^2) = i\lambda((qi + iq\,)q + q(qi + iq\,)) + i\lambda((qi + iq\,)q + q(qi + iq\,))$$
$$+2\, D_{qq}([p,(qi + iq\,)]) \ -2D_{pq}([q,(qi + iq\,)]) -2D_{qp}([q,(qi + iq\,)])$$
$$+D_{qq}([p,(qi + iq\,)] + [p,(qi + iq\,)])$$

$$\mathcal{L}(q^2) = i\lambda(4iq^2) \ +i\lambda(4iq^2) \quad +4i\, D_{qq}([p,q\,]) \ -4iD_{pq}([q,q\,]) \ -4iD_{qp}([q,q\,])$$
$$+2D_{qq}([p,2iq\,])$$

hence we have $\mathcal{L}(q^2) = -8\lambda q^2 - 8\, D_{qq}$.

For $pq + qp$ we note that the relation $[pq, p] = p[q, p]$, $[pq, q] = [p, q]q$, holds. Hence we have,

$$\mathcal{L}(f) = -i\lambda(p[f,q] + [f,q]p) + i\lambda([f,p]q + q[f,p]) - i\lambda(p[f,q] + [f,q]p)$$

$$+i\lambda([f,p]q + q[f,p])) \ +2\,D_{qq}([p,[f,p]]) + 2\,D_{pp}([q,[f,q]])$$

$$-2D_{pq}([p,[f,q]] + [q,[f,p]]) \ -2D_{qp}([p,[f,q]] + [q,[f,p]])$$

$$+D_{qq}([p,[f,p]] + [p,[f,p]]) \ +D_{pp}([q,[f,q]] + [q,[f,q]])$$

$$\mathcal{L}(pq + qp) = -i\lambda(p[(pq + qp), q] + [(pq + qp), q]p))$$

$$+i\lambda([(pq + qp), p]q + q[(pq + qp), p]$$

$$-i\lambda(p[(pq + qp), q] + [(pq + qp), q]p)$$

$$+i\lambda([(pq + qp), p]q + q[(pq + qp), p]))$$

$$+2\,D_{qq}([p,[(pq + qp), p]]) + 2\,D_{pp}([q,[(pq + qp), q]])$$

$$-2D_{pq}([p,[(pq + qp), q]] + [q,[(pq + qp), p]])$$

$$-2D_{qp}([p,[(pq + qp), q]] + [q,[(pq + qp), p]])$$

$$+D_{qq}([p,[(pq + qp), p]] + [p,[(pq + qp), p]])$$

$$+D_{pp}([q,[(pq + qp), q]] + [q,[(pq + qp), q]])$$

$$\mathcal{L}(pq + qp) = -i\lambda(p([p,q]q\ ) + ([p,q]q)p)) + i\lambda((p[q,p] + [q,p]p)q$$

$$+q(p[q,p] + [q,p]p)) - i\lambda(p([p,q]q) + ([p,q]q\ )p) + i\lambda(p[q,p] + [q,p]p)q$$

$$+q(p[q,p] + [q,p]p)) \ +2\,D_{qq}([p, p[q,p] + [q,p]p]) + 2\,D_{pp}([q,[p,q]q\ ])$$

$$-2D_{pq}([p,[p,q]q] + [q, p[q,p] + [q,p]p]) - 2D_{qp}([p,[p,q]q\ ] + [q, p[q,p] + [q,p]p])$$
$$+D_{qq}([p, p[q,p]] + [p, p[q,p] + [q,p]p]) \ +D_{pp}([q,[p,q]q\ ] + [q,[p,q]q\ ])$$

$$\mathcal{L}(pq + qp) = -i\lambda(ipq + iqp\ ) + i\lambda(2ipq + 2iqp)$$

$$-i\lambda(ipq + iqp\ ) \ +i\lambda(2ipq + 2iqp))$$

$$+2\,D_{qq}\,([p,2ip]) + 2\,D_{pp}\,([q,iq\,])$$

$$-2D_{pq}\,([p,iq] + [q,2ip])\ \ -2D_{qp}\,([p,iq\,] + [q,2ip])$$

$$+D_{qq}\,([p,pi] + [p,2ip])\ +D_{pp}\,([q,iq\,] + [q,iq\,])$$

$$\mathcal{L}(pq+qp) = \lambda pq + \lambda qp - 2\lambda pq - 2\lambda qp + \lambda pq + \lambda qp\ \ -2\lambda pq - 2\lambda qp$$

$$+2\,D_{qq}\,([p,2ip]) + 2\,D_{pp}\,([q,iq\,])$$

$$-2D_{pq}\,([p,iq] + [q,2ip])\ \ -2D_{qp}\,([p,iq\,] + [q,2ip])$$

$$+D_{qq}\,([p,pi] + [p,2ip])\ +D_{pp}\,([q,iq\,] + [q,iq\,])$$

$$\mathcal{L}(pq+qp) =\ -2\lambda pq - 2\lambda qp - 2D_{pq}\,([p,iq] + [q,2ip])\ \ -2D_{qp}\,([p,iq\,] + [q,2ip])$$

$$\mathcal{L}(pq+qp) =\ -2\lambda(pq+qp) - 2D_{pq}\,(i[p,q] + 2i[q,p]) - 2D_{qp}\,(i[p,q\,] + 2i[q,p])$$

$$\mathcal{L}(pq+qp) =\ -2\lambda(pq+qp) - 6D_{pq}\ -6D_{qp}$$

Now since $D_{pq} = D_{qp}$ we have the equation as

$$\mathcal{L}(pq+qp) =\ -2\lambda(pq+qp) - 12D_{pq}$$

Hence our three equations are the following

$$\mathcal{L}(p^2) = 8\lambda p^2\ \ -8\,D_{pp}$$

$$\mathcal{L}(q^2) = -8\lambda q^2\ -8\,D_{qq}$$

$$\mathcal{L}(pq+qp) =\ -2\lambda(pq+qp) - 12D_{pq}$$

From (Isar, etal., 1994) the following notations for the variance and covariance, in terms of the observables p^2, q^2 and pq is given by,

$$\sigma_{qq}(t) = Tr\big(\rho\Phi_t(q^2)\big) - Tr\big(\rho\Phi_t(q)\big)^2$$

$$\sigma_{pp}(t) = Tr\big(\rho\Phi_t(p^2)\big) - Tr\big(\rho\Phi_t(p)\big)^2$$

$$\sigma_{pq}(t) = \tfrac{1}{2}Tr\big(\rho\Phi_t(pq+qp)\big)$$

where Tr is the trace, ρ is a density matrix and Φ_t a dynamical semigroup. An important consequence of the solvability condition is that $(d/dt)\,\sigma_{qq}(t)$, $(d/dt)\,\sigma_{pp}(t)$,

$(d/dt)\,\sigma_{pq}(t)$ are functions of $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ Now since $\mathcal{L}$ is the generator of the dynamical semigroup Φ_t we have the differential equations in the following form

$$\frac{d\sigma_{pp}(t)}{dt} = Tr\big(\rho\mathcal{L}(\Phi_t(p^2)\big)$$

$$\frac{d\sigma_{qq}(t)}{dt} = Tr\big(\rho\mathcal{L}(\Phi_t(q^2)\big)$$

$$\frac{d\sigma_{pq}(t)}{dt} = \frac{1}{2}Tr\big(\rho\mathcal{L}(\Phi_t(pq+qp)\big)$$

The equation of motion for the variance and covariance is

$$\frac{d\sigma_{pp}(t)}{dt} = 8\lambda Tr\big(\rho\mathcal{L}(\Phi_t(p^2)\big) - 8D_{pp}$$

$$\frac{d\sigma_{qq}(t)}{dt} = -8\lambda Tr(\rho\mathcal{L}(\Phi_t(q^2)) - 8D_{qq}$$

$$\frac{d\sigma_{pq}(t)}{dt} = -2\lambda Tr(\rho\mathcal{L}(\Phi_t(qp+pq)) - 12D_{pq}$$

hence we have, the equations in the following form

$$\frac{d\sigma_{pp}(t)}{dt} = 8\lambda\sigma_{pp}(t) - 8D_{pp}$$

$$\frac{d\sigma_{qq}(t)}{dt} = -8\lambda\sigma_{qq}(t) - 8D_{qq}$$

$$\frac{d\sigma_{pq}(t)}{dt} = -2\lambda\sigma_{pq}(t) + 12D_{pq}$$

We solve the three differential equations above by using the simple connection between their asymptotic values and the diffusion coefficients given by

$X(\infty) = -R^{-1}D$ in (Sandulescu and Scutaru,1987).

The above equation in matrix form is, $\frac{dX(t)}{dt} = RX(t) + D$

where

$$\frac{dX(t)}{dt} = \begin{pmatrix} \frac{d\sigma_{pp}(t)}{dt} \\ \frac{d\sigma_{qq}(t)}{dt} \\ \frac{d\sigma_{pq}(t)}{dt} \end{pmatrix},\ R = \begin{pmatrix} 8\lambda & 0 & 0 \\ 0 & -8\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix},\ X(t) = \begin{pmatrix} \sigma_{pp}(t) \\ \sigma_{qq}(t) \\ \sigma_{pq}(t) \end{pmatrix}, D = \begin{pmatrix} -8D_{pp} \\ -8D_{qq} \\ 12D_{pq} \end{pmatrix},$$

that is $\begin{pmatrix} \frac{d\sigma_{pp}(t)}{dt} \\ \frac{d\sigma_{qq}(t)}{dt} \\ \frac{d\sigma_{pq}(t)}{dt} \end{pmatrix} = \begin{pmatrix} 8\lambda & 0 & 0 \\ 0 & -8\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \begin{pmatrix} \sigma_{pp}(t) \\ \sigma_{qq}(t) \\ \sigma_{pq}(t) \end{pmatrix} + \begin{pmatrix} -8D_{pp} \\ -8D_{qq} \\ 12D_{pq} \end{pmatrix}$,

we now use the relation $X(\infty) = -R^{-1}D$ to find the asymptotic values ,hence we

have, $R^{-1} = \frac{1}{DetR}[R_{adj}]$, $DetR = (8)(16)\lambda^3$,

$R_{adj} = \begin{pmatrix} 16\lambda^2 & 0 & 0 \\ 0 & -16\lambda^2 & 0 \\ 0 & 0 & 64\lambda^2 \end{pmatrix}$ $and \ \ R^{-1} = \frac{1}{(8)(16)\lambda^3} \begin{pmatrix} 16\lambda^2 & 0 & 0 \\ 0 & -16\lambda^2 & 0 \\ 0 & 0 & 64\lambda^2 \end{pmatrix}$

from the simple relation , we have the asymptotic equations given by ,

$X(\infty) = \frac{1}{16\lambda^3} \begin{pmatrix} 2\lambda^2 & 0 & 0 \\ 0 & -2\lambda^2 & 0 \\ 0 & 0 & 8\lambda^2 \end{pmatrix} \begin{pmatrix} -8D_{pp} \\ -8D_{qq} \\ 12D_{pq} \end{pmatrix}$, where $X(\infty) = \begin{pmatrix} \sigma_{pp}(\infty) \\ \sigma_{qq}(\infty) \\ \sigma_{pq}(\infty) \end{pmatrix}$

and we have the following asymptotic values for the variance and covariance

$\sigma_{pp}(\infty) = \frac{16\lambda^2 D_{pp}}{16\lambda^3}$, $\sigma_{qq}(\infty) = \frac{16\lambda^2 D_{qq}}{16\lambda^3}$ and $\sigma_{pq}(\infty) = \frac{8\lambda^2 D_{pq}}{16\lambda^3}$

we have on simplifying

$\sigma_{pp}(\infty) = \frac{D_{pp}}{\lambda}$, $\sigma_{qq}(\infty) = \frac{D_{qq}}{\lambda}$ and $\sigma_{pq}(\infty) = \frac{D_{pq}}{2\lambda}$

5.3 Asymptotic Entanglement

Here we discuss the entanglement of two independent oscillators interacting with the environment in the long time regime, $t \to \infty$, $u$sing the Peres –Simon formula for bipartite system (Isar,2008).

$$S \equiv detA\, detB - \frac{1}{4}(detA + detB) + \left(\frac{1}{4} - |detC|\right)^2 - Tr(AJCJBJC^TJ).$$

where the matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is called the symplectic matrix and A, B, C are sub matrices of the 4×4 real, symmetric and positive matrix called the covariance matrix with block structure given in (Isar,2008) , by $\sigma(t) \equiv \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$. This decomposition has a direct physical interpretation: the elements containing the diagonal contributions of $\sigma(t)$ represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in C represent environment generated couplings between the two ,initial independent oscillators. (Isar,2008).

Proposition 5.3.1

Suppose the sub-matrices A and B are equal and symmetric then Peres-Simon type equation in terms of the variance and covariance of the coordinates and momenta operators is given by

$$S = \left(\sigma_{q_xq_x}\sigma_{p_xp_x} - \sigma^2_{q_xp_{x,}}\right)^2 - \frac{1}{2}\left(\sigma_{q_xq_x}\sigma_{p_xp_x} - \sigma^2_{q_xp_{x,}}\right)$$

$$+\left[\left(\sigma_{q_xp_y}\sigma_{p_xp_y} - \sigma^2_{q_yp_x}\right)^2 - \frac{1}{2}\left(\sigma_{q_xp_y}\sigma_{p_xq_y} - \sigma^2_{q_yp_x}\right) + \left(\frac{1}{4}\right)^2\right]$$

$$-\left((\sigma^2_{q_xp_x} + \sigma^2_{q_xp_x})(\sigma^2_{q_xq_y} + \sigma^2_{q_xp_x})\right) + \left((\sigma^2_{q_xp_x} + \sigma^2_{q_xp_x})(\sigma^2_{q_xp_y} + \sigma^2_{q_xp_2})\right)$$

Proof

To derive the Peres-Simon type equation for the entanglement of a bipartite system, we assumed that the sub matrices A and B are equal and symmetric, then the Peres-Simon formula becomes $S = detA^2 - \frac{1}{2}detA + \left(\frac{1}{4} - |detC|\right)^2 - Tr(AJCJ)^2$. we define the matrices A, C as follows,

$$A = \begin{pmatrix} \sigma_{q_x q_x} & \sigma_{q_x p_y} \\ \sigma_{q_x p_x} & \sigma_{p_x p_x} \end{pmatrix} \; and \qquad C = \begin{pmatrix} \sigma_{q_x q_y} & \sigma_{q_x p_y} \\ \sigma_{q_y p_x} & \sigma_{p_x p_y} \end{pmatrix}$$

hence we have

$$detA^2 - \frac{1}{2}detA = \left(\sigma_{q_x q_x}\sigma_{p_x p_x} - \sigma^2_{q_x p_{x,}}\right)^2 - \frac{1}{2}\left(\sigma_{q_x q_x}\sigma_{p_x p_x} - \sigma^2_{q_x p_{x,}}\right)$$

$$\left(\frac{1}{4} - |detC|\right)^2 = \left[\left(\sigma_{q_x p_y}\sigma_{p_x p_y} - \sigma^2_{q_y p_x}\right)^2 - \frac{1}{2}\left(\sigma_{q_x p_y}\sigma_{p_x q_y} - \sigma^2_{q_y p_x}\right) + \left(\frac{1}{4}\right)^2\right]$$

and

$$tr(AJCJ)^2 = \left((\sigma^2_{q_x p_x} + \sigma^2_{q_x p_x})(\sigma^2_{q_x q_y} + \sigma^2_{q_x p_x})\right) + \left((\sigma^2_{q_x p_x} + \sigma^2_{q_x p_x})(\sigma^2_{q_x p_y} + \sigma^2_{q_x p_2})\right)$$

this gives the Peres-Simon type equation in terms of the variance and covariance of the coordinates and momenta operators.

$$S = \left(\sigma_{q_x q_x}\sigma_{p_x p_x} - \sigma^2_{q_x p_{x,}}\right)^2 - \frac{1}{2}\left(\sigma_{q_x q_x}\sigma_{p_x p_x} - \sigma^2_{q_x p_{x,}}\right)$$

$$+\left[\left(\sigma_{q_x p_y}\sigma_{p_x p_y} - \sigma^2_{q_y p_x}\right)^2 - \frac{1}{2}\left(\sigma_{q_x p_y}\sigma_{p_x q_y} - \sigma^2_{q_y p_x}\right) + \left(\frac{1}{4}\right)^2\right]$$

$$-\left((\sigma^2_{q_x p_x} + \sigma^2_{q_x p_x})(\sigma^2_{q_x q_y} + \sigma^2_{q_x p_x})\right) + \left((\sigma^2_{q_x p_x} + \sigma^2_{q_x p_x})(\sigma^2_{q_x p_y} + \sigma^2_{q_x p_2})\right).$$

The Peres-Simon type equation is then use to investigate in the long-time regime whether or not the two harmonic oscillators are entangled, by using the Lindbladian operator derived in section 5.1 hence we substitute $\mathcal{L}(p_x p_y)$, $\mathcal{L}(q_x q_y)$, $\mathcal{L}(p_x q_y + q_x p_y)$ and by direct calculation we can evaluate the equations of motions as we did in section 5.2 on solving the differential equation in the long term we get the entries for matrix A and entries for matrix C.

From (Isar, 2008) we know that if $S \leq 0$ then the bipartite systems are entangled in the long-time regime.

CHAPTER SIX

SUMMARY AND CONCLUSION

6.1 Summary

In the study of stochastic dynamics on spin algebra, we made use of the technique and argument developed by Zegarlinski and Majewski. We have been able to establish the existence of an infinite volume stochastic dynamics having an exponential decay to equilibrium and is strongly ergodic.

Quantum entanglement represent the physical resource in quantum information science. Here we have considered the asymptotic entanglement of an open quantum system based on completely positive dynamical semigroups (Isar, 2007).

6.2 Conclusion

The techniques of noncommutative L_p- spaces in the construction and analysis of quantum stochastic dynamics has been shown to be useful, especially if the underlying configuration space is infinite dimensional.

One of the important problems in the theory of quantum entanglement is the question of separability of states. We have shown the plausibility of investigating such questions, within the Lindblad theory of open quantum system.

REFERENCES:

1. Accardi, L. and Cecchini, C. (1982). Conditional Expectation in von Neumann algebra and a Theorem of Takesaki. *Journal Functional Analysis*, 45, pp. 245-273.
2. Ahmed, N.U. (1991). *Semigroup theory with Applications to Systems and Control*, Longman group UK.
3. Attal, S., Joyce A., Pillet, C.A. (Eds.) (2006). *Open Quantum Systems vol. I and vol. II*, Springer-Verlag, Berlin Heidelberg.
4. Attal, S. And Joyce, A. (2006). *The Langevin Equation for a Quantum Heat Bath*. Preprint.
5. Averson, W. (1976). *An Invitation to C*-algebra*. Springer-Verlag, Berlin.
6. Bratteli, O. and Robinson, D.W. (1979). *Operator Algebra and Quantum Statistical Mechanics*. Springer-verlag New York-Heidelberg-Berlin Vol. 1.
7. Christensen, E. (1978). Generators of Semigroups of Completely Positive Maps. *Communications in Mathematical Physics*, vol.62, pp. 167-171**.**
8. Cipriani, F., Fagnola, F. and Lindsay, J.M. (2000). Spectral Analysis and Feller property for quantum Ornstein-Uhlenbeck Semigroups. *Communications in Mathematical Physics*, 210, pp. 85-105.
9. Davies, E.B. and Lindsay, J.M. (1992). Non Commutative Symmetric Markov Semigroups. *Mathematische Zeitschrift*, 210, pp. 379-411.
10. Ekhaguere, G.O.S. (1978). Markov fields in Noncommutative Probability Theory on W* algebras. *Journal of Mathematical Physics*, vol.20.
11. Evan, D.E. and Hanchen-Olsen, H. (1979). Generators of Positive Semigroups, *Journal Functional Analysis*, vol.32, pp. 207-212.

12. Evan, D.E. (1977). Irreducible Quantum Dynamical Semigroup. *Communication in Mathematical Physics*, vol. 54, pp. 293-297.

13. Evans D.E.(1976). Positive Linear Maps on Operator Algebra. *Communication in Mathematical Physics*, vol.48, pp. 15-22.

14. Fannes, M., Nachtergaele, B., and Werner, R.F. (1992). Finitely Correlated States in Quantum Spin Chains. *Communication in Mathematical Physics*, vol.144, pp. 443-490.

15. Frigerio, A. (1977). Quantum dynamical Semigroups and approach to equilibrium. *Letters in Mathematical Physics*, vol.2, pp. 79-87.

16. Frigerio, A. (1978). Stationary state of Quantum Dynamical semigroup. *Communication in Mathematical Physics*, vol. 63, pp. 269-276.

17. Golodez, V.Y. (1972). Conditional Expectation and modular automorphism of von Neumann algebras. *Functional Analysis and its Application*, vol.6, pp. 68-69.

18. Haagerup, U. (1975). Normal Weights on W*-algebra. *Journal Functional Analysis*, vol. 19, pp. 302-317.

19. Horodecki, R. Horodecki,P. Horodecki,M. and Horodecki,K.(2007). Quantum Entanglement. ArXiv : 070225v2 (quant-ph).

20. Isar, A. (2007). Entanglement in Open Quantum Systems. *Romanian Reports in Physics*, vol. 59, pp. 1103-1110.

21. Isar, A. (2008). Asymptotic Entanglement in Open Quantum Systems. *Romanian Reports in Physics*, vol. 69, pp. 1113-1120.

22. Isar, A., Sandulescu, A., Scutaru, H., Stefanescu, E. And Scheid, W. (1994). Open Quantum Systems. *International Journal of Modern Physics*, E3,635.

23. Its, A.R., Mazzadri, F. And Mo, M.Y. (2008). *Entanglement Entropy in Quantum Spin Chain with finite range interaction*, ariv.0708.0161v2(math-ph).

24. Kadison, V.R., and Ringrose, J.R. (1983). *Fundamentals of The Theory of Operator Algebras, Vol. 1 and II*, Academic Press, New York.

25. Kosaki, H. (1984). Application of the complex interpolation method to a von Neumann algebra(Non-commutative L_p). *Journal of Functional Analysis*, vol.56, pp. 29-78.

26. Li, M., Fei, S., and Wang, Z. (2008). *Separability and Entanglement of Quantum states based on the Covariance Matrices*, arXiv: 0805.1632v1 (quant-ph).

27. Lindblad, G. (1976). On the generators of quantum dynamical semigroups. *Communication in Mathematical Physics*, vol. 48, pp. 119-130.

28. Majewski, A.W. and Zegarlinski, B. (1995). Quantum Stochastic Dynamics 1: Spin System on a Lattice. *Mathematical Physics Electronic Journal vol.1*.

29. Majewski, A.W. and Zegarlinski, B. (1996). On Quantum Stochastic Dynamics and Non commutative L_p Spaces. *Letters Mathematical Physics*, vol. 36, pp. 337-349.

30. Majewski, A.W. and Zegarlinski, B. (1996). On Quantum L_p Spaces Technique. *Acta Physical Polonica (B)* ,vol.l27.

31. Nelson, E. (1974). Notes on Noncommutative Integration. *Journal of Functional*, vol. 15, pp. 103-116.

32. Nielsen, M.A. and Chuang, I.L. (2000). *Quantum computation and Quantum information*, Cambridge university press, Cambridge.

33. Olkiewicz, R. and Zegarlinski, B. (1999). Hypercontractivity in Noncommutative L_p-spaces. *Journal Functional Analysis,* vol.161, pp. 246-285**.**

34. Parthasarathy, K. R. (2004). On the Maximal dimension of a completely entangled subspace of finite quantum systems. *Proceedings Indian Academy of Science,* vol. 144**.**

35. Peres, A. (1996). Separability Criterion for density matrices. *Physical Review Letters,* vol**.77,** No81**.**

36. Robinson, D.W. (1982). *Basic theory of one parameter semigroups*, Center for mathematical analysis, Australian National University press.

37. Robinson, D. And Ruelle, D. (1967). Mean Entropy of States in Classical Statistical Mechanics. *Communication in Mathematical Physics*, vol. 9 pp. 288-300.

38. Ruelle, D. (1970). *Equilibrium States of Infinite System in Statistical Mechanics*, International Congress of Mathematician conference, Nice.

39. Ruelle, D., (1969). *Statistical Mechanics Rigorous Results*, Benjamin, New York .

40. Sakai, S. (1971). *C*-algebras and W*-algebras,* Springer -Verlag, Berlin.

41. Sandulescu, A., Scutaru, H. (1987). Open Quantum Systems and the Damping of Collective Modes in Deep Inelastic Collisions. *Annals of Physics,* vol.173, pp. 277-317.

42. Segal, I.E. (1953). A Noncommutative Extension of Abstract Integration. *Annals of Maths*, vol. 57, pp. 401-457.

43. Stinespring, W.F. (1959). Positive functions on C*-algebras. *Proceedings America Mathematical Society*, vol.6, pp. 211-216.

44. Sunders, V. S. (1987). *An Invitation to von Neumann algebras*, Springer-Verlag, Berlin.

45. Takesaki, M. (1970). *Tomita Theory of Modular Hilbert Algebras*, Lecture Notes in Mathematics Vol. 128, Springer -Verlag, Berlin.

46. Takesaki, M. (1972). Conditional Expectation in von Neumann algebras, *Journal Functional Analysis*, vol. 9, pp. 306-321.

47. Takesaki, M. (1979). *Theory of operator algebras 1*, Springer-Verlag Berlin.

48. Takesaki, M. (1970). *States and Automorphism Groups of operator Algebras,* International Congress of Mathematician Conference, Nice.

49. Terp, M. (1981). L_p-spaces Associated with von Neumann algebras. *Kobenhavns Universitet*,Mathematisk Institut, Rapport no3.

50. Tikhonov, O.E. (1982). L_p spaces with respect to a weight on a von Neumann. *Izvestiya vuz, matematika*, vol 26, No8, pp. 76-78.

51. Tomiyama, J. (1957). On the Projection of Norm one in W*-algebras. *Proceedings Japan Academy*, vol.33, pp. 608-612.

52. Tomiyama, J. (1958). On the Projection of Norm one in W*-algebras II. *Tohoku Mathematics Journal*, vol.10, pp. 204-209.

53. Trunov, N. V. (1982). On the Theory of Normal weights on von Neumann algebras, *Izvestiya Vuz, Mathmatika***,** vol. 26, pp. 61-70.

54. Trunov, N. V. (1979). On a non-Commutative analogue of the L_P Space. *Izvestiya Vuz, Soviet Mathematika*, vol. 23, pp. 69-77.

55. Umegaki, H. (1954). Conditional Expectation in an operator algebras. *Tohoku Mathematics Journal*, vol. 6, pp. 177-181.

56. Umegaki, H. (1956). Conditional Expectation in an operator algebras II. *Tohoku Mathematics Journal*, vol.8, pp. 86-100**.**

57. Yeadon, F.J. (1975). Non-commutative L_p spaces. *Mathematical Proceedings Cambridge Philosophical Society*, vol. 77, pp. 91-102**.**

58. Zolotarev, A.A. (1982). L_p- Spaces with respect to state on a von Neumann algebra. *Izvestiya Vuz , Soviet Mathematika*, vol.26, pp. 36-43.