ENHANCED VARIATIONAL ITERATION METHOD FOR SOLVING FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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DECLARATION

I hereby declare that this work is the product of my own research; undertaken under the supervision of Dr. Sirajo Lawan Bichi and has not been presented and will not be presented elsewhere for the award of a degree or certificate. All source have been duly acknowledged.

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CERTIFICATION

This is to certify that the research work for this dissertation and the subsequent write-up were carried out by Auwal Nura Shareef (SPS/15/MMT/00002) under my supervision.

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APPROVAL

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DEDICATION

I dedicate this research work to my dad, Alh Nura Shareef Indabawa and my mum, Malama Rabi Usman Sabo. May Allah (SWT) reward them abundantly (Amin).

ABSTRACT

This research considered problem of Linear/Nonlinear Fredholm integro differential equations (IDE). The n^{th} order Fredholm integro differential equation of second kind is first reduced to a system of 1^{st} order ordinary differential equation and integro differential equation and; Variational Iteration method was then used to derive iterative scheme to obtain approximate solution of the n^{th} order integro differential equation. Convergence analysis of the proposed method is established. Numerical examples are presented to test the efficiency of the proposed method. Comparison with other existing method are given to show the accuracy of the proposed method.

Contents

CHAPTER ONE

GENERAL INTRODUCTION

1.1 INTRODUCTION

In this chapter we give the basic background of the study, problem statement, aim and objectives, motivation of the study, scope of the study and definition of some basic terms.

1.2 BACKGROUND OF THE STUDY

Integro differential equation arise quite frequently as mathematical model in diverse disciplines. The origin of the study of integral and integro differential equation may be traced to the work of Abel, Lotka, Fredhom, Varhulst and Volterra on problems in mechanics, mathematical biology and economics [46, 67]. Integro differential equation on the other hand was first introduce by Volterra in 1990, the work of Volterra on the problem of competing species is of fundamental importance for the development of mathematical modelling of real world problems. From those beginning, the theory and application of integro differential equation have emerged as new areas of research. The application of IDE is an important subject within applied mathematics, physics and engineering, in particular, they are widely used in mechanics, geophysics, electricity and magnetism, hereditary phenomena in biology, quantum mechanics and mathematical economics.
Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro differential equations contain both integral and differential operators. The derivative

of the unknown functions may appear to any order.

An integro differential equation is a type of integral equation in which the unknown function u(x) contain an ordinary derivatives Wazwaz [84].

Nonlinear integro differential equation has the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b k(x,t)F(u(t))dt, \tag{1.2.1}$$

with initial condition $u^{(k)}(0) = b_k,\ \ 0 \leq k \leq (n-1)$

where $F(u(x))$ is a nonlinear function of $u(x)$, the kernel $k(x,t)$ and $f(x)$ is a given real valued function, λ is a constant, $u(x)$ is the unknown function to be determined and $u^{(n)}(x) = \frac{d^n u(x)}{dx^n}$.

In recent years, numerous work have been focussing on the development of more advanced and efficient method for solving IDE[67].

1.3 PROBLEM STATEMENT

There has recently been much attention devoted to the search for better and more efficient methods for solving IDE. In the literature, Linear and Nonlinear integro differential equation are solved by many methods such as Homotopy Perturbation, New Iterative Method and Adomian Decomposition Method. In this work, we considered Linear/Nonlinear Fredholm integro differential equation of the form:

$$y^{(n)}(x) + f(x)y(x) + \lambda \int_a^b k(x,t)y^{(q)}(t)y^{(m)}(t)dt = g(x), \tag{1.3.1}$$

subject to

$$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ y''(a) = \alpha_2,\ \cdots,\ y^{n-1}(a) = \alpha_{n-1},$$

where $\alpha_i,\ i = 0,\ 1,\ \cdots,\ n-1$ are real constants, m, q and n are integers with $q \leq m < n$. The functions f, g and k are given continuous functions and y is the solution to be determined.

1.4 AIM AND OBJECTIVES

Aim

The aim of this research is to develop an efficient method for approximating the solution of Fredholm integro differential equation (1.3.1) via Variational Iteration Method (VIM).

Objectives

The objectives includes:

1 To reduce the Fredholm integro differential equation (1.3.1) into a system of 1^{st} order ordinary differential equation and integro differential equation.

2 To obtain method of approximate solution of IDE (1.3.1) by solving the system of 1^{st} ordinary differential equation and integro differential equation obtain in (1) above via VIM.

3 To develop a Maple code for the proposed method for solving Fredholm integro differential equation (1.3.1).

4 To obtain Convergence analysis for the proposed method.

5 To test and compare the proposed method for solving Fredholm integro differential equation (1.3.1) with other existing method in the literature.

1.5 MOTIVATION OF THE STUDY

Finding the value of lagranges multiplier for any n^{th} order Fredholm integro differential equation requires alot of computational work when the order is higher as such we are motivated to present a generalised value of lagranges multiplier that can work for any n^{th} order Fredholm integro differential equation.

1.6 SCOPE OF THE STUDY

This work is concern with Linear/Nonlinear Fredholm integro differential equations of the second kind .

1.7 DEFINITIONS OF SOME BASIC TERMS

In order to have better understanding of this work, the definition of some basic terms needed in studying integro differential equations are given below.

Definition 1.7.1 *(Fredholm Integro Differential Equation): The Fredholm integro differential equation of the second kind appear in the form[85]*

$$y^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)y(t)dt, \tag{1.7.1}$$

where the limit of integration $a(x)$ and $b(x)$ are constant.

Definition 1.7.2 *(Volterra Integro Differential Equation): The Volterra integro differential equation of the second kind appear in the form[85]*

$$y^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)y(t)dt, \tag{1.7.2}$$

where the upper limit of integration $b(x)$ is variable.

Definition 1.7.3 *(Linearity of Integro Differential Equation): The integro differential equation of the form:*

$$y^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)y(t)dt, \tag{1.7.3}$$

is said to be linear if the exponent of the unknown function $y(x)$ under the integral sign is one and the equation does not contain Nonlinear functions of $y(x)$, otherwise, the equation is called Nonlinear. For example

$$y^{(\prime)}(x) = 1 - \frac{x}{3} + \int_0^x xty(t)dt,$$

is Linear equation, while

$$y^{('')}(x) = -\cos(x) - \frac{\Pi^2}{288}(x) + \frac{1}{72}\int_0^{\Pi} xty^2(t))dt,$$

is Nonlinear equation.

Definition 1.7.4 *(Homogeneity of Integro Differential Equation): If the function $f(x) = 0$ equation* $(1.6.2)$ *becomes*

$$y^{(n)}(x) = \lambda \int_{a(x)}^{b(x)} k(x,t)y(t)dt \tag{1.7.4}$$

$$y^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)y(t)dt, \tag{1.7.5}$$

they are called homogeneous integro differential equations respectively. Otherwise, they are non-homogeneous equations.

Definition 1.7.5 *If a functional $V[u(x)]$ which has a variation achieves a maximum or a minimum at $u = u_0(x)$, where $u(x)$ is an interior point of the domain of definition of the functional, then at $u = u_0(x)$[76]*

$$\delta v = 0 \tag{1.7.6}$$

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter reviews some related literature on numerical and analytic method used for solving Fredholm integro differential equations

2.2 REVIEW ON NUMERICAL AND ANALYTIC SOLUTION OF FREDHOLM INTEGRO DIFFERENTIAL EQUATIONS

Several numerical and analytical methods have been proposed by many authors to find a better and efficient solution method of integro differential equations. Lepik [43] use Harr Wavelet Method to solve Nonlinear integro differential equation of the form:

$$\alpha u'(x) + \beta u(x) = \int_0^t k(x,t,u(t),u'(t))dt + f(x), \tag{2.2.1}$$

subject to initial condition $u(0) = u_0$,
some numerical examples were provided to show the accuracy and efficiency of the method. Junfeng lu [36] solve two point boundary value problems using Variational Iteration Method, numerical results demonstrate that the method is promising and may overcome the difficulty arising in Adomian Decomposition Method. Variational Iteration Method which was proposed initially by HE[85], has been proved by many authors to be powerful mathematical tool for various kind of Nonlinear problems. Tatari and Dehghan

[76] studied convergence of VIM for solving second-order initial value problems. The efficiency of the approach was shown by applying the procedure on several interesting and important Models. Nadjafi and Tamamgar [60] describe VIM as a highly promising method for various classes of both Linear and Nonlinear IDE. In VIM solution is obtained in a series form, if the series solution obtained from VIM has a closed form, the method provides exact solution, otherwise, the solution is approximated to some degree of accuracy Wazwaz [84].

Abbasbandy and Shivanian [93] proposed an approach via Variational Iteration Method to find approximate solution for a Nonlinear Volterra integro differential equation of second kind of the form:

$$y'(t) = f(x) + \int_{t_0}^{x} k(x,t,y(t),y'(t))dt), \tag{2.2.2}$$

subject to initial condition $u(t_0) = u_0$ error evaluation of the method was presented and some numerical examples were provided to show the efficiency of the method . Batiha *et al.* [16] studied numerical solution of the Nonlinear IDE, ADM and VIM was applied to solve the IDE. Comparison was made between the ADM and VIM and result reveals that VIM is faster than ADM. Babolian *et al.* [17] employed Operational Matrix with Block Pulse Functions to solve Nonlinear Volterra-Fredholm integral and integro differential equation, error evaluation of the method was presented and numerical examples were provided to show the accuracy and efficiency of the method. Yildirim [89] used an application of HE's Variational Iteration Method to solve first order Nonlinear Fredholm integro differential equation of the second kind were some examples are given to illustrate the effectiveness of the method. khaleel [42] proposed Homotopy Perturbation Method to solve a special class of Nonlinear first order Fredholm integro differential eqaution of the second kind defined as:

$$u'(x) = f(x) + \lambda \int_{t_a}^{b} k(x,y)[u(y)]^q dy, \tag{2.2.3}$$

subject to initial condition $u(a) = \alpha$.

Some numerical examples were provided to show the accuracy and efficiency of the method. Borhanifar and Abazari [23] employed Differential Transform Method to solve a class of Nonlinear integro differential equation with derivative type kernel defined as:

$$u''(t) = f(t, u(t), u'(t), \int_{t_0}^{t} k(s, u(s), u'(s))ds),\ t\varepsilon[t_0, T], \tag{2.2.4}$$

subject to initial condition $u(t_0) = u_0$, $u'(t_0) = u_1$, the examples solved showed that the method is powerful in handling Nonlinear IDEs. (Biazar *et al* [22] describe the VIM to be of the useful techniques in solving numerous Linear and Nonlinear differential equations. For instance see HE and Wazwaz [30, 87]. An important advantage of the method is that it uses the initial condition only and does not require the specific transformations for Nonlinear terms as required by some existing techniques. Furthermore, VIM can be applied directly without linearization, discretization or perturbation.

Ordokhani and Davaei [62] studied Application of the Bernstein Polynomials for solving n^{th} order Nonlinear Fredholm integro differential equations of the form:

$$\sum_{j=0}^{n} p_j(x) y^{(j)}(x) = g(x) + \lambda \int_0^1 k(x,t)[y(t)]^p dt,\ \ 0 \le x, t \le 1. \tag{2.2.5}$$

with initial condition $y^k(0) = b_k,\ \ 0 \le k \le n-1$. Some numerical examples are given to show the accuracy of the method. Hemeda [29] proposed the general n^{th} order Linear and Nonlinear Fredholm integro differential equation of the second kind which was handled by applying the New Iterative Method defined as:

$$y^{(n)}(x) + f(x)y(x) + \int_a^b w(x,t) y^{(q)}(t) y^{(m)}(t) dt = g(x), \tag{2.2.6}$$

with initial condition $y(a) = \alpha_0$, $y'(a) = \alpha_1$, $y''(a) = \alpha_2$, $\cdots$, $y^{(n-1)}(a) = \alpha_{(n-1)}$. Comparison was made between the New Iterative Method and VIM.

Babolian *et al.*[18] obtained the numerical solution of Nonlinear integro differential equations via Direct Method using Triangular Functions defined as:

$$u'(x) = g(x) + \int_a^b k(x, t, u(t))dt.\ u(0) = \alpha \tag{2.2.7}$$

Numerical experiments were carried out to examine the accuracy of the proposed method. The technique of Modified Decomposition Method to approximate solution of system of

Linear Fredholm IDE were proposed by Rabbani [66]. Exact solution of the two test problems arising in many physical and biological models are calculated by using modified decomposition technique and procedure is quite efficient to determine the solution in a closed form. Behiry [20] employed Modified Differential Transform Method to handle Nonlinear IDE, the Nonlinear term was replaced by Adomian Polynomials and some numerical examples were given with different Nonlinearity. New Algorithm base on Harr Wavelet for the numerical solution of Nonlinear Fredholm and Volterra integral equation were proposed by Aziza and Islam [14].

Farshid and Fatemeh [27] produced an approach via Collocation Method to find an approximate solution of system of Fredholm integro differential equations with Fibonacci Polynomials of the form:

$$\sum_{n=0}^{m}\sum_{j=1}^{l} p_{ij}^{n}(x)y_{j}^{n}(x) = g_i(x) + \int_a^b \sum_{j=1}^{l} k_{ij}(x,t)y_j(t)dt, \tag{2.2.8}$$

where $i = 1,\ 2, \cdots,\ l.,\ 0 \leq a \leq x \leq b$. Some Numerical examples are provided to show the accuracy of the Method. Atabakan *et al.* [4] used Spectral Homotopy Analysis Method to solve Fredholm Integro Differential Equation of the form:

$$\sum_{j=0}^{2} a_j(x)y^j(x) = f(x) + \lambda \int_{t_{-1}}^{1} k(x,t)[y(t)]^r dt, \tag{2.2.9}$$

subject to initial condition $y(-1) = y'(1) = 0$,

error evaluation of the method was presented and numerical examples were provided to show the accuracy and efficiency of the method.

Oladotun and Ogunlaran [61] studied an approach via Cubic Spline Method to find the numerical solution for first order integro differential equation of the form:

$$y'(x) + f(x)y(x) + \int_a^b k(x,s)y(s))ds = g(s), \tag{2.2.10}$$

subject to initial condition $y(a) = y_0$. Numerical experiments were carried out to examine the accuracy of the proposed method. Waleed [79] combined Laplace Transform Method (LTM) with ADM to solve n^{th} order IDEs. Mohsen and El-Gamel[58] studied numerical solution of Nonlinear Fredholm IDE, Sinc Collocation Method was applied and the accuracy of the obtained solution reduces the computational difficulties of other traditional method.

Taiwo *et al.*[77] employed an efficient numerical collocation approximation method to obtain an approximate solution of Linear Fredholm integro differential difference equation with variable coefficients of the form:

$$\sum_{k=o}^{m} p_k y^k(x) + \sum_{r=o}^{n} q_r(x) y^r(x-\tau) = f(x) + \int_a^b k(x,t) y(t-\tau) dt = 0, \tag{2.2.11}$$

with mixed condition

$$\sum_{k=o}^{m-1} (a_{ik} y^k(a) + b_{ik} y^k(b) + c_{ik} y^k(c)) = \mu_i, \tag{2.2.12}$$

where $i = 0,\ 1,\ \cdots,\ m-1,\ a \le c \le b.$

Some numerical examples were provided to show the accuracy and efficiency of the method. Mosleh and Otadi [57] proposed a scheme base on iterative approach to obtain approximate solution of a class of Nonlinear Fuzzy Fredholm integral equation and IDE, result concerning the existence of solution of a class of Nonlinear Fuzzy Fredholm IDE were provided and some numerical examples were given to illustrate the effectiveness of the method. Az-Zobi [15] proposed Variational Iteration Method for solving system of conservation of laws with source terms, sufficient condition of convergence and error estimate for the proposed method are presented. Two examples are investigated to verify the effectiveness and reliability of the VIM. Pandey [63] studied numerical solution of Linear Fredholm integro differential equation using Non-Standard Finite Difference Method defined as:

$$y'(x) = f(x,y) + \int_a^b k(x,t) y(t)) dt,\ a \le x \le b \tag{2.2.13}$$

subject to initial condition $y(x) = \alpha$, numerical experiments were carried out to examine the accuracy of the proposed method. Combined Repreducing Kernel Method and Tailor series to approximate the solution of Nonlinear Volterra-Fredholm IDE were proposed by Alvandi and Paripour [11].

Maryam *et al.* [49] studied an approach via Two-Step Adomian Decomposition Method to solve second order integro differential equation define:

$$y''(x) = f(x) + \int_a^b k(x,t)(Lu(t) + N(u(t)))dt, \tag{2.2.14}$$

subject to the initial condition $y(0) = \alpha$, $y'(0) = \beta$, some numerical examples are provided to show the accuracy of the Method.

Some numerical examples were provided to show the accuracy and efficiency of the method. Aloko *et al.* [3] studied numerical solution of a special class of Nonlinear Fredholm integro differential equation and used Modified Variational Iteration Method to solve the problem. Some numerical examples were considered to show the accuracy and efficiency of the method. Al-Bar [9] proposed a scheme base on Variational Iteration Method for solving Nonlinear PDEs and convergence of VIM to Nonlinear PDEs were discussed.

2.3 BASIC IDEA OF VARIATIONAL ITERATION METHOD(VIM)

To clarify the basic of VIM, we consider the following general Nonlinear differential equation of the form[85]:

$$Lu(x) + Nu(x) = g(x), \tag{2.3.1}$$

where L is a Linear operator, N is a Nonlinear operator and $g(x)$ is a given continuous function.

According to Variational Iteration Method [85, 30], a correction functional for equation (2.3.1) can be constructed as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt, \tag{2.3.2}$$

where λ is a general Lagrange multiplier and the term $\tilde{u}$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. Now taking the variational derivative of (2.3.2) with respect to the independent variable u_n, we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt, \tag{2.3.3}$$

In order to identify the Lagrange multiplier, from (2.3.3) we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)(Lu_n(t))dt, \tag{2.3.4}$$

In general, the Lagrange multiplier λ, can be readily identified by imposing the stationary condition $\delta u_{n+1}(x) = 0$ on the correction functional (2.3.4). After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0, the successive approximations u_{n+1}, $n \geq 0$, of the solution u can be readily obtained.
Consequently, the solution of (2.3.1) is given by

$$u(x) = \lim_{n\to\infty} u_n(x).$$

CHAPTER THREE

METHODOLOGY

3.1 INTRODUCTION

In this Chapter, we derive an Iterative scheme for solving Fredholm integro differential equation and establish its convergence using Banach fixed point theorem.

3.2 DERIVATION OF THE METHOD FOR SOLVING FREDHOLM INTEGRO DIFFERENTIAL EQUATION

Consider the integro differential equation

$$y^{(n)}(x) + f(x)y(x) + \lambda \int_a^b k(x,t)y^{(q)}(t)y^{(m)}(t)dt = g(x), \qquad (3.2.1)$$

subject to

$$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ y''(a) = \alpha_2,\ \cdots,\ y^{n-1}(a) = \alpha_{n-1},$$

where $\alpha_i,\ i = 0,\ 1,\ \cdots,\ n-1$ are real constants, m, q and n are integers with $q \leq m < n$. The functions f, g and k are given continuous functions and y is the solution to be determined.

Using the transformation, let

$$y = y_1,\ \frac{dy}{dx} = y_2,\ \frac{d^2y}{dx^2} = y_3, \cdots,\ \frac{d^{(n-1)}y}{dx^{(n-1)}} = y_n, \qquad (3.2.2)$$

using equation (3.2.2), we can rewrite the integro differential equation (3.2.1) as the system of ordinary integro differential equation as follows

$$\begin{cases} \frac{dy_1}{dx} &= y_2, \\ \frac{dy_2}{dx} &= y_3, \\ \frac{dy_3}{dx} &= y_4, \\ \vdots & \\ \frac{dy_n}{dx} &= g(x) - f(x)y_1(x) - \lambda \int_a^b k(x,t) y_{q+1}(t) y_{m+1}(t) dt \end{cases} \tag{3.2.3}$$

subject to

$$y_1(a) = \alpha_0,\ y_2(a) = \alpha_1,\ y_3(a) = \alpha_2,\ \cdots,\ y_n(a) = \alpha_{n-1}$$

To illustrate the basic concepts of the VIM, we can rewrite equation (3.2.1) in a general Nonlinear operator form [65]:

$$Lu + Ru + N(u) = g(x) \tag{3.2.4}$$

where L and R are Linear bounded operators i.e., there exist a constant m_1, $m_2 \geq 0$ such that $\| Lu \| \leq m_1 \| u \|$, $\| Ru \| \leq m_2 \| u \|$. The Nonlinear term $N(u)$ is Lipschitz continuous with $| N(u) - N(v) | \leq m \mid u - v \mid$ for any constant $m \geq 0$.
Now, we construct the correction functional base on equation (3.2.1) which reads as:

$$y_p(x) = y_{p-1}(x) + \lambda \int_0^x [Lu_{p-1}(t) + R\tilde{u}_{p-1}(t) + N(\tilde{u}_{p-1}(t) - g(t)] dt, \tag{3.2.5}$$

where λ is a general lagrange multiplier, which can be identified optimally via integration by part, the subscript p denotes the number of iteration steps and $\tilde{u}_{p-1}$ is considered as a restricted variation i.e $\delta \tilde{u}_{p-1} = 0$.
Therefore, to solve the system (3.2.3), it requires constructing the two correction functional equations as:

$$y_{jp}(x) = y_{j(p-1)}(x) + \int_0^x \lambda(x,t) [y'_{j(p-1)}(t) - \tilde{y}_{(j+1)(p-1)}(t)] dt, \tag{3.2.6}$$

$j = 1, 2, \cdots, n-1$ and for $j = n$, we have

$$y_{np}(x) = y_{n(p-1)}(x) + \int_0^x \lambda(x,t)[y'_{n(p-1)}(t) - g(t) + f(t)\tilde{y}_{1(p-1)}(t) \\ + \int_a^b k(t,s)\tilde{y}_{(q+1)(p-1)}\tilde{y}_{(m+1)(p-1)}(s)ds]dt, \qquad (3.2.7)$$

Taking the variational derivative of equation (3.2.6) and (3.2.7) with respect to the independent variable $y_{j(p-1)}$ and imposing stationary condition i.e $\delta y_{jp} = 0,\ j = 1, 2, \cdots, n$ on the correction functional respectively, noting that $\delta\tilde{y}_{p-1} = 0$, we have

$$\begin{aligned} 0 &= \delta y_{jp}(x) \qquad (3.2.8) \\ &= \delta y_{j(p-1)}(x) + \delta\int_0^x \lambda_j(x,t)[(y'_{j(p-1)}(t) - \tilde{y}_{(j+1)(p-1)}(t)]dt \\ &= \delta y_{j(p-1)}(x) + \lambda_j(x,t)\delta y_{j(p-1)}(t)\mid_{t=x} - \int_0^x \frac{\partial\lambda_j(x,t)}{\partial t}\delta(y_{j(p-1)}(t)dt) \\ &= (1+\lambda_j(x,x))\delta y_{j(p-1)}(x) + \int_0^x \Big(\frac{-\partial\lambda_j(x,t)}{\partial t}\Big)\delta(y_{j(p-1)}(t))dt,\ j = 1, 2, \cdots, n-1 \end{aligned}$$

and for $j = n$

$$\begin{aligned} 0 &= \delta y_{np}(x) \qquad (3.2.9) \\ &= \delta y_{n(p-1)}(x) + \delta\int_0^x \lambda_n(x,t)[y'_{n(p-1)}(t) - g(t) + f(t)\tilde{y}_{1(p-1)}(t) \\ &\qquad + \int_a^b k(t,s)\tilde{y}_{(q+1)(p-1)}\tilde{y}_{(m+1)(p-1)}(s)ds]dt, \\ &= \delta y_{n(p-1)}(x) + \lambda_n(x,t)\delta y_{n(p-1)}(t)\mid_{t=x} - \int_0^x \frac{\partial\lambda_n(x,t)}{\partial t}\delta(y_{n(p-1)}(t)dt) \\ &= (1+\lambda_n(x,x))\delta y_{n(p-1)}(x) + \int_0^x \Big(\frac{-\partial\lambda_n(x,t)}{\partial t}\Big)\delta(y_{n(p-1)}(t))dt. \end{aligned}$$

For arbitrary δy_{jp}, $j = 1, 2, \cdots, n$, the following stationary condition are obtained;

$$-\frac{\partial\lambda_1(x,t)}{\partial t} = -\frac{\partial\lambda_2(x,t)}{\partial t} = \cdots = -\frac{\partial\lambda_n(x,t)}{\partial t} = 0$$

and

natural boundary condition : $1+\lambda_j(x,x) = 0,\ \ j = 1, 2, \cdots, n.$

The lagrange multiplier, therefore can be identified as:

$\lambda_j(x,t) = -1,\ \ j = 1, 2, \cdots, n$ and the iteration formula (3.2.6) and (3.2.7) can be written as:

$$y_{jp}(x) = y_{j(p-1)}(x) - \int_0^x [y'_{j(p-1)}(t) - y_{(j+1)(p-1)}(t)]dt, \; j = 1, \; 2, \; \cdots, \; n-1 \quad (3.2.10)$$

and for $j = n$

$$y_{np}(x) = y_{n(p-1)}(x) - \int_0^x [y'_{j(p-1)}(t) - g(t) + f(t) y_{1(p-1)}(t) + \int_a^b k(t,s) y_{(q+1)(p-1)} y_{(m+1)(p-1)}(s) ds] dt. \quad (3.2.11)$$

Beginning with

$y_{10}(x) = \alpha_0, \; y_{20}(x) = \alpha_1, \; y_{30}(x) = \alpha_2, \; \cdots, \; y_{n0}(x) = \alpha_{n-1}$, by the iteration formula (3.2.10) and (3.2.11), we can obtain the approximate solution of equation (3.2.1)

Now the algorithm of the proposed method is as follows:

Algorithm (EVIM)

STEP 1: Given y_o, y_{ext} and ε (where ε is the tolerance).

STEP 2: Compute the recursive relation using (3.2.11) for $j = 1$

STEP 3: Compute the recursive relation using (3.2.10) and (3.2.11) for $j = 2, \; 3, \; \cdots n$

STEP 4: Compute

$$y_{apr} = y_{j(p-1)}(x) - \int_0^x [y'_{j(p-1)}(t) - y_{(j+1)(p-1)}(t)]dt$$

STEP 5: If $| \; y_{apr} - y_{ext} \; | < \varepsilon$ then go to STEP 6 else $j = 1$ and go to step 2

STEP 6: Print $y_{apr}(x)$ as the approximate of the exact solution.

3.2.1 Convergence analysis of the iterative scheme

In this section we present the convergence of the iteration formula (3.2.10) and (3.2.11). Let us write equation (3.2.10) and (3.2.11) in the operator form as follows:

$$u_p = A[u_{p-1}], \quad (3.2.12)$$

where the operator A takes the following form:

$$A[u] = -\int_0^x [Lu + Ru + N(u)]dt, \tag{3.2.13}$$

Theorem 3.2.1 *(Banach fixed point theorem) Assume that X is a Banach space and* $A : X \to X$ *is a Nonlinear mapping, and suppose that*

$$\| A[u] - A[v] \| \leq \gamma \| u - v \|, \tag{3.2.14}$$

for each $u, v \in X$,

for some constant $\gamma < 1$. *Then A has a unique fixed point. Furthermore, the sequence*

$$u_p = A[u_{p-1}], \tag{3.2.15}$$

with an arbitrary choice of $u_0 \in X$, *converges to the fixed point of A and*

$$\| u_p - u_q \| \leq \frac{\gamma^q}{1-\gamma} \| u_1 - u_0 \| . \tag{3.2.16}$$

Proof of Theorem 3.2.1

We want prove that the sequence u_p is a cauchy sequence in this Banach space,

$$\begin{aligned}
\| u_p - u_q \| &= \max_{t\in[a,b]} \| u_p - u_q \|, \\
&= \max_{t\in[a,b]} \mid -\int_0^x [L(u_{p-1}-u_{q-1}) + R(u_{p-1}-u_{q-1}) \\
&\qquad\qquad + N(u_{p-1}) - N(u_{q-1})]dt \mid \\
&\leq \max_{t\in[a,b]} \int_0^x [\mid L(u_{p-1}-u_{q-1}) \mid + \mid R(u_{p-1}-u_{q-1}) \mid \\
&\qquad\qquad + \mid N(u_{p-1}) - N(u_{q-1}) \mid]dt \\
&\leq \max_{t\in[a,b]} \int_0^x [(m_1+m_2+m)(u_{p-1}-u_{q-1})]dt \\
&\leq \gamma \| u_{p-1} - u_{q-1} \|
\end{aligned} \tag{3.2.17}$$

let $p = q+1$ then

$$\| u_{q+1} - u_q \| \leq \gamma \| u_q - u_{q-1} \| \leq \gamma^2 \| u_{q-1} - u_{q-2} \| \leq ... \leq \gamma^q \| u_1 - u_0 \|$$

From the triangle inequality we have

$$\begin{aligned}
\| u_p - u_q \| &\leq \| u_{q+1} - u_q \| + \| u_{q+2} - u_{q+1} \| + ... + \| u_p - u_{p-1} \| \\
&\leq [\gamma^q + \gamma^{q+1} + \cdots + \gamma^{p-1}] \| u_1 - u_0 \| \\
&\leq \gamma^q [1 + \gamma + \gamma^2 + ... + \gamma^{p-q-1}] \| u_1 - u_0 \| \\
&\leq \frac{\gamma^q}{1-\gamma} [1 - \gamma^{p-q-1}] \| u_1 - u_0 \|
\end{aligned}$$

Since $0 < \gamma < 1$ so, $(1 - \gamma^{p-q-1}) < 1$ then:

$$\| u_p - u_q \| \leq \frac{\gamma^q}{1-\gamma} \| u_1 - u_0 \| .$$

The proof is complete.

*

CHAPTER FOUR

NUMERICAL RESULT AND DISCUSSIONS

4.1 INTRODUCTION

4.2 NUMERICAL EXAMPLES

In this section we present some numerical examples that uses the iterative scheme after reducing the Fredholm integro differential equations to a system of 1^{st} order ordinary differential equation and integro differential equation. The computations associated with the examples were performed using MAPLE and result obtained by the proposed scheme is compared with some other existing method in the literature as shown in the given tables.

Example 4.2.1 Consider the first-order nonlinear integro-differential equation of second kind:

$$y'(x) = \frac{5}{4} - \frac{x^2}{2} + \int_0^1 (x^2 - t)y^2(t)dt, \qquad y(0) = 0$$

the exact solution of the above equation is $y(x) = x$.

We solve the equation by using the method in chapter 3. Starting with the initial approximation $y_{10}(x) = 0$, by the iteration formula (3.2.10) and (3.2.11) we obtain the following result:

$$
\begin{aligned}
y_{11}(x) &= 1.250000000x - 0.1666666667x^3, \\
y_{12}(x) &= 0.9253472222x - 0.0195105820x^3, \\
&\vdots \\
y_{15}(x) &= 1.016558220x - 0.06210624728x^3, \\
&\vdots \\
y_{110}(x) &= 1.013294240x - 0.06058336078x^3,
\end{aligned}
\tag{4.2.1}
$$

we used Maple to compute the absolute error at different values of x as shown in the table below:

Table 4.1: The exact and approximate solution of example 4.2.1

p	x	Exact solution	$EVIM$	Absolute Error
	0	0	0	0
	0.2	0.2	0.2028147940	$2.81479400E-3$
	0.4	0.4	0.4026484882	$2.64848820E-3$
5	0.6	0.6	0.5965199826	$3.48001740E-3$
	0.8	0.8	0.7814481774	$1.85518226E-2$
	1	1.0	0.9544519727	$4.55480273E-2$
	0	0	0	0
	0.2	0.2	0.2021741811	$2.17418110E-3$
	0.4	0.4	0.4014403609	$1.44036090E-3$
10	0.6	0.6	0.5948905381	$5.10946190E-3$
	0.8	0.8	0.7796167113	$2.03832887E-2$
	1	1.0	0.9527108792	$4.72891208E-2$

Table 4.1 shows the exact solution of problem in example 4.2.1 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation of Fredholm IDEs of second kind.

Example 4.2.2 Consider the first-order integro-differential equation of second kind:

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, \qquad y(0) = 0$$

the exact solution of the above equation is $y(x) = xe^x$.

We solve the equation by using the method in chapter 3. Starting with the initial approximation $y_{10}(x) = 0$, by the iteration formula (3.2.10) and (3.2.11) we obtain the following result:

$$\begin{aligned} y_{11}(x) &= xe^x - \frac{1}{2}x^2, \\ y_{12}(x) &= xe^x - \frac{1}{12}x^2, \\ &\vdots \\ y_{15}(x) &= xe^x - \frac{1}{2592}x^2, \\ &\vdots \\ y_{110}(x) &= xe^x - \frac{1}{20155392}x^2, \end{aligned} \tag{4.2.2}$$

we used Maple to compute the absolute error at different values of x as shown in the table below:

Table 4.2: The exact and approximate solution of example 4.2.2

p	x	Exact solution	$EVIM$	Absolute Error
	0	0	0	0
	0.2	0.2442805516	0.2442651195	$1.54321E-5$
	0.4	0.5967298792	0.5966681508	$6.17284E-5$
5	0.6	1.093271280	1.093132391	$1.38889E-4$
	0.8	1.7804327420	1.7801858280	$2.46914E-4$
	1	2.7182818280	2.7178960260	$3.85802E-3$
	0	0	0	0
	0.2	0.2442805516	0.2442805496	$2.0E-10$
	0.4	0.5967298792	0.5967298713	$7.9E-9$
10	0.6	1.093271280	1.093271262	$1.8E-8$
	0.8	1.780432742	1.780432710	$3.2E-8$
	1	2.718281828	2.7182817780	$5.0E-8$

Table 4.2 shows the exact solution of problem in example 4.2.2 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation of Fredholm IDEs of second kind. It can be seen from the the table that the error decreases with increase in the number of terms p which shows that the solution is approaching the exact solution.

Table 4.3: Comparison between $EVIM$ and BPM

x_i	$EVIM$ $n=5$	BPM $n=32$
0.1	$1.5937E-3$	$1.19E-2$
0.2	$2.8148E-3$	$2.20E-2$
0.3	$3.2906E-3$	$3.22E-2$
0.4	$2.6485E-3$	$4.14E-2$
0.5	$5.1583E-3$	$4.93E-2$
0.6	$3.4800E-3$	$5.57E-2$
0.7	$9.7117E-3$	$6.01E-2$
0.8	$1.8552E-2$	$6.23E-2$
0.9	$3.0373E-2$	$6.19E-2$

Table 4.3 compares the error obtained by $EVIM$ and Bernstein Polynomial Method in example 4.2.1. The numerical result indicated that the proposed method i.e $EVIM$ with less number of iterations gives a better result compared to that obtained by BPM [3]. It is clear that the accuracy of the result obtained by the proposed method is very effective.

Table 4.4: Comparison between $EVIM$ and DTM

x_i	$EVIM$ $n=10$	DTM $n=10$
0.1	$5.0E-10$	$1.0E-2$
0.2	$2.0E-9$	$2.8E-2$
0.3	$4.5E-9$	$5.1E-2$
0.4	$7.9E-9$	$7.6E-2$
0.5	$1.2E-8$	$9.7E-2$
0.6	$1.8E-8$	$1.1E-1$
0.7	$2.4E-8$	$1.0E-1$
0.8	$3.2E-8$	$6.9E-2$
0.9	$4.0E-8$	$1.6E-1$

Table 4.4 compares the error obtained by $EVIM$ and Differential Transform Method(DTM) in example 4.2.2. The numerical result indicated that the proposed method i.e $EVIM$ subject to the same number of iteration gives a better result to that obtained by DTM [93].

Example 4.2.3 Consider the second-order nonlinear integro-differential equation of second kind:

$$y''(x) = 2 - \frac{x}{2} + \int_0^1 xty(t)y'(t)dt, \qquad y(0) = 0, y'(0) = 0$$

the exact solution of the above equation is $y(x) = x^2$.
We solve the equation by using the method in chapter 3. Starting with the initial approximation $y_{10}(x) = 0$, $y_{20}(x) = 0$, by the iteration formula (3.2.10) and (3.2.11) we obtain the following result:

$$\begin{aligned}
y_{11}(x) &= 0, \\
y_{12}(x) &= x^2 - 0.08333333333x^3, \\
&\vdots \\
y_{15}(x) &= x^2 - 0.006794142237x^3, \\
&\vdots \\
y_{110}(x) &= x^2 - 0.00001667612000x^3, \\
&\vdots \\
y_{115}(x) &= x^2 - 5.170000010^{-8}x^3,
\end{aligned} \tag{4.2.3}$$

we used Maple to compute the absolute error at different values of x as shown in the table below:

Table 4.5: The exact and approximate solution of example 4.2.3

p	x	Exact solution	$EVIM$	Absolute Error
	0	0	0	0
	0.2	0.04	0.0399456469	$5.435314E-5$
	0.4	0.16	0.1595651749	$4.348251E-4$
5	0.6	0.36	0.3585324653	$1.467535E-3$
	0.8	0.64	0.6365213992	$3.478601E-3$
	1	1.0	0.9932058578	$6.794142E-3$
	0	0	0	0
	0.2	0.04	0.0399998666	$1.3341E-7$
	0.4	0.16	0.1599989327	$1.0673E-6$
10	0.6	0.36	0.3599963980	$3.6020E-6$
	0.8	0.64	0.6399914618	$8.5382E-6$
	1	1.0	0.9999833239	$1.6676E-5$
	0	0	0	0
	0.2	0.04	0.0399999844	$1.56E-8$
	0.4	0.16	0.1599998750	$1.25E-7$
15	0.6	0.36	0.3599995781	$4.22E-7$
	0.8	0.64	0.6399990000	$1.00E-6$
	1	1.0	0.9999980469	$1.95E-6$

Table 4.5 shows the exact solution of problem in example 4.2.3 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation of Fredholm IEs of second kind. It can be seen from the the table that the error decreases with increase in the number of terms p which shows that the solution is approaching the exact solution.

Example 4.2.4 Consider the second-order integro-differential equation of second kind:

$$y''(x) = e^x - x + \int_0^1 xty(t)dt, \qquad y(0) = 1, y'(0) = 1$$

the exact solution of the above equation is $y(x) = e^x$.

We solve the equation by using the method in chapter 3. Starting with the initial approximation $y_{10}(x) = 1$, $y_{20}(x) = 1$, by the iteration formula (3.2.10) and (3.2.11) we obtain the following result:

$$\begin{aligned}
y_{11}(x) &= 1 + x, \\
y_{12}(x) &= e^x - \frac{1}{12}x^3, \\
&\vdots \\
y_{15}(x) &= e^x - \frac{1}{1080}x^3, \\
&\vdots \\
y_{110}(x) &= e^x - \frac{1}{9720000}x^3, \\
&\vdots \\
y_{115}(x) &= e^x - \frac{1}{26244000000}x^3,
\end{aligned} \tag{4.2.4}$$

we used Maple to compute the absolute error at different values of x as shown in the table below:

Table 4.6: The exact and approximate solution of example 4.2.4

p	x	Exact solution	$EVIM$	Absolute Error
5	0	1	1	0
	0.2	1.221402758	1.22139535	$7.4070E-6$
	0.4	1.491824698	1.4917654	$5.9259E-5$
	0.6	1.822118800	1.82191880	$2.0000E-4$
	0.8	2.225540928	2.22506685	$4.7407E-4$
	1	2.718281828	2.71735590	$9.2592E-4$
10	0	1	1	0
	0.2	1.22140276	1.22140276	$1.0E-9$
	0.4	1.491824698	1.49182469	$7.0E-9$
	0.6	1.82211880	1.82211878	$2.2E-8$
	0.8	2.22554093	2.22554088	$5.3E-8$
	1	2.71828182	2.71828173	$1.0E-7$
15	0	1	1	0
	0.2	1.22140276	1.22140276	0
	0.4	1.491824698	1.491824698	0
	0.6	1.82211880	1.82211880	0
	0.8	2.22554093	2.22554093	0
	1	2.71828183	2.71828183	0

Table 4.6 shows the exact solution of problem in example 4.2.4 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation of Fredholm IDEs of second kind. It can be seen from the the table that the error decreases with increase of the number of terms p which shows that the solution is approaching the exact solution.

Table 4.7: Comparison between $EVIM$ and HPM

x_i	$EVIM$ $n = 10$	HPM $n = 10$
0.1	0	$1.E-9$
0.2	$1.0E-9$	$8.0E-8$
0.3	$3.0E-9$	$2.80E-8$
0.4	$7.0E-9$	$6.60E-8$
0.5	$1.3E-8$	$1.29E-7$
0.6	$2.2E-8$	$2.22E-7$
0.7	$3.5E-8$	$3.53E-7$
0.8	$5.3E-8$	$5.27E-7$
0.9	$7.5E-8$	$7.50E-7$

Table 4.7 compares the error obtained by $EVIM$ and Homotopy Perturbation Method(HPM) in example 4.2.4. The numerical result indicated that the proposed method i.e $EVIM$ gives a better result to that obtained by HPM [3] with the same number of iteration. It is clear that the accuracy of the result obtained by the proposed method is efficient.

Example 4.2.5 Consider the third-order integro-differential equation of second kind:

$$y'''(x) = \sin(x) - x + \int_0^{\Pi/2} xty'(t)dt, \qquad y(0) = 1, y'(0) =, y'' = -1$$

the exact solution of the above equation is $y(x) = \cos(x)$.

we solve the equation by using the method in chapter 3. Starting with the initial approximation $y_{10}(x) = y_{20}(x) = y_{30}(x) = 1$, by the iteration formula (3.2.10) and (3.2.11) we obtain the following result:

$$
\begin{aligned}
y_{11}(x) &= 1, \\
y_{12}(x) &= 1-\frac{1}{2}x^2, \\
&\vdots \\
y_{15}(x) &= cos(x)-\frac{1}{24}x^4, \\
&\vdots \\
y_{110}(x) &= cos(x)-0.06581044108x^4,
\end{aligned}
\tag{4.2.5}
$$

we used Maple to compute the absolute error at different values of x as shown in the table below:

Table 4.8: The exact and approximate solution of example 4.2.5

p	x	Exact solution	$EVIM$	Absolute Error
5	0	1	1	0
	0.2	0.9800665778	0.9799119931	$1.5458E-4$
	0.4	0.9210609940	0.9185876388	$2.4733E-3$
	0.6	0.8253356149	0.8253356149	$1.2521E-2$
	0.8	0.6967067093	0.6967067093	$3.9574E-2$
	1	0.5403023059	0.4436868682	$9.6615E-2$
10	0	1	1	0
	0.2	0.9800665778	1.22140276	0
	0.4	0.9210609940	1.49182469	0
	0.6	0.8253356149	1.82211879	$1.0E-9$
	0.8	0.6967067093	2.22554092	$4.0E-9$
	1	0.5403023059	2.71828182	$1.0E-8$

Table 4.8 shows the exact solution of problem in example 4.2.5 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation of Fredholm IEs of second kind. It can be seen from the

table that the error decreases with increase in the number of terms p which shows that the solution is approaching the exact solution.

Table 4.9: Comparison between $EVIM$ and $MVIM$

x_i	$EVIM\ n=3$	$MVIM\ n=3$
0.1	$4.1667E-6$	$4.1659E-6$
0.2	$6.6667E-5$	$6.6570E-5$
0.3	$3.3750E-4$	$3.3585E-4$
0.4	$1.0667E-3$	$1.0543E-3$
0.5	$2.6041E-3$	$2.5452E-3$
0.6	$5.4000E-3$	$5.1886E-3$
0.7	$1.0004E-2$	$9.3823E-3$
0.8	$1.7067E-2$	$1.5483E-2$
0.9	$2.7338E-2$	$2.3726E-2$

Table 4.9 compares the error obtained by $EVIM$ and Modified Variational Iteration Method(MVIM) [3] in example 4.2.5. The absolute error shows slight difference due to the approximate identification of the multiplier. The more accurate the identification of the multiplier, the faster the approximation converge to the exact solutions. The error obtained indicated that the proposed method is efficient. *

Table 4.10: Comparison between $EVIM$ and DTM

x_i	$EVIM\ n=5$	$MVIM\ n=5$
0.1	$1.5937E-3$	$3.8205E-3$
0.2	$2.8148E-3$	$7.5340E-3$
0.3	$3.2906E-3$	$1.1034E-3$
0.4	$2.6485E-3$	$1.4213E-3$
0.5	$5.1583E-3$	$1.6964E-3$
0.6	$3.4800E-3$	$1.9180E-3$
0.7	$9.7117E-3$	$2.0755E-3$
0.8	$1.8552E-2$	$2.1582E-3$
0.9	$3.0373E-2$	$2.1553E-3$

Table 4.10 compares the error obtained by $EVIM$ and Modified Variational Iteration Method [3] in example 4.2.1. The numerical result indicated that the proposed method i.e $EVIM$ subject to the same number of iteration gives a better result while the $MVIM$ will converge faster than the proposed method.

*

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 INTRODUCTION

This chapter contains summary, conclusion and recommendation

5.2 SUMMARY

In this work, we were able to derive an iterative scheme for solving Linear/Nonlinear Fredholm integro differential equations of the second kind by reducing the n^{th}-order Fredholm integro differential equations to a system of 1^{st} order integro differential equations. Chapter one of this thesis, gives detail background of Fredholm integro-differential equations of the second kind. Beside that, statement of the research problem, aim and objectives of the research, motivation of the study, its scope and limitations and definition of some basic terms are given for good understanding of the work. Chapter two reviewed some literatures related to integro differential equations. The aim of the chapter is to give a detail overview of how integro differential equations are solved, and investigate the improvements in the area. Chapter three discussed the derivation of the iterative scheme for solving integro differential equations. In this chapter,the convergence theorem of the derived method was established. Chapter four discussed the numerical result where five problems of integro differential equations were solved to show the efficiency of the method. For each problem solved in chapter four, tables of absolute errors were given to compare our method with other known methods such as BPM, DTM, $MVIM$ and HPM. The results obtained showed that our method is a good tool for approximating the solution of Linear/Nonlinear Fredholm integro differential equations. These shows that the proposed method is very effective and simple. Moreover, The derived iterative scheme reduced the difficulties arising when solving higher order Fredholm integro differential equations due to the fact that the scheme is using only one lagrange multiplier derived for 1^{th}-order integro differential equations to solve higher-other Fredholm integro differential equations instead of finding the lagrange multiplier for each n^{th}-order Fredholm integro differential equations.

5.3 CONCLUSION

In this work, an iterative scheme has been successfully applied to find the approximate solution of n^{th}-order Linear/Nonlinear Fredholm integro differential equations. The presented examples shows that the result of the proposed method agreed well with the exact solution, Moreover the computation associated with these examples were performed using Maple.

5.4 RECOMMENDATION

We recommend for extension of the proposed scheme used for solving Fredholm integro differential equations to solve Volterra integro differential equations of the second kind.

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APPENDIX A

Maple code for Example (4.2.1)

$Restart$;

$> k{:=}1;\ a_1 := -1$

$>$ for i from 1 to k do; $b[1,1] = a_i$; od;

$> b[1,1] := x \to$ -1

$> b_{1,1}(x)$

$> g := x \to evalf(-\frac{5}{4} + \frac{x^2}{2})$

$> f := t \to 0$

$> w := (t,r) \to (t^2 - r)$

$> y[0,1] := x \to 0$

$> y_{0,1}(x)$

$>$ for i from 1 to k do; $y[i,1](x) = y[i-1,1](x) + \int_0^x b[1,1](x).((y[i-1,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i-1,1](r)dr)dt$; od;

$> y[1,1] := x \to y_{1,1}(x)$

$> y_{1,1}(x)$

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i,1](r)dr)dt$; od;

$> y[2,1] := x \to y_{2,1}(x)$

$> y_{2,1}(x)$

$>$ for i from 1 to k do; $y[i+2,1](x) = y[i+1,1](x) + \int_0^x b[1,1](x).((y[i+1,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+1,1](r)dr)dt$; od;

$\vdots$

$> y[4,1] := x \to y_{4,1}(x)$

$> y_{4,1}(x)$

$>$ for i from 1 to k do; $y[i+4,1](x) = y[i+3,1](x) + \int_0^x b[1,1](x).((y[i+3,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+3,1](r)dr)dt$; od;

$\vdots$

$> y[9,1] := x \to y_{9,1}(x)$

$> y_{9,1}(x)$

$>$ for i from 1 to k do; $y[i+9,1](x) = y[i+8,1](x) + \int_0^x b[1,1](x).((y[i+8,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+8,1](r)dr)dt$; od;

APPENDIX B

Maple code for Example (4.2.2)

$Restart$;

$> k$:=1; $a_1 := -1$

$>$ for i from 1 to k do; $b[1,1] = a_i$; od;

$> b[1,1] := x \to$ -1

$> b_{1,1}(x)$

$> g := x \to -xe^x - e^x + x$

$> f := t \to 0$

$> w := (t,r) \to t$

$> y[0,1] := x \to 0$

$> y_{0,1}(x)$

$>$ for i from 1 to k do; $y[i,1](x) = y[i-1,1](x) + \int_0^x b[1,1](x).((y[i-1,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i-1,1](r)dr)dt$; od;

$> y[1,1] := x \to y_{1,1}(x)$

$> y_{1,1}(x)$

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i,1](r)dr)dt$; od;

$> y[2,1] := x \to y_{2,1}(x)$

$> y_{2,1}(x)$

$>$ for i from 1 to k do; $y[i+2,1](x) = y[i+1,1](x) + \int_0^x b[1,1](x).((y[i+1,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+1,1](r)dr)dt$; od;

$\vdots$

$> y[4,1] := x \to y_{4,1}(x)$

$> y_{4,1}(x)$

$>$ for i from 1 to k do; $y[i+4,1](x) = y[i+3,1](x) + \int_0^x b[1,1](x).((y[i+3,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+3,1](r)dr)dt$; od;

$\vdots$

$> y[9,1] := x \to y_{9,1}(x)$

$> y_{9,1}(x)$

$>$ for i from 1 to k do; $y[i+9,1](x) = y[i+8,1](x) + \int_0^x b[1,1](x).((y[i+8,1])'(t) + g(t) + f(t) - \int_0^1 w(t,r).y[i+8,1](r)dr)dt$; od;

APPENDIX C

Maple code for Example (4.2.3)

$Restart$;

$> k{:}{=}1; a_1 := -1$

$>$ for i from 1 to k do; $b[1,1] = a_i$; od;

$> b[1,1] := x \rightarrow$ -1

$> b_{1,1}(x)$

$> g := x \rightarrow -2 + \frac{x}{2}$

$> f := t \rightarrow 0$

$> w := (t,r) \rightarrow t$

$> y[0,1] := x \rightarrow 0$

$> y_{0,1}(x)$

$> y[0,2] := x \rightarrow 0$

$> y_{0,2}(x)$

$>$ for i from 1 to k do; $y[i,1](x) = y[i-1,1](x) + \int_0^x b[1,1](x).((y[i-1,1])'(t) - y[i-1,2](t))dt$; od;

$>$ for i from 1 to k do; $y[i,2](x) = y[i-1,2](x) + \int_0^x b[1,1](x).((y[i-1,2])'(t) + g(t) + f(t) - \int_0^1 w(t,r).(y[i-1,1])(r).(y[i-1,2])(r)dr)dt$; od;

$> y[1,2] := x \rightarrow y_{1,2}(x)$

$> y_{1,2}(x)$

$> y[1,1] := x \rightarrow y_{1,1}(x)$

$> y_{1,1}(x)$

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) - (y[i,2](t)))dt$; od;

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) - y[i,2](t))dt$;

od;

> for i from 1 to k do; $y[i+1,2](x) = y[i,2](x) + \int_0^x b[1,1](x).((y[i,2])'(t) + g(t) + f(t) - \int_0^1 w(t,r).(y[i,1])(r).(y[i,2])(r)dr)dt$; od;

> $y[2,1] := x \to y_{2,1}(x)$

> $y_{2,1}(x)$

> $y[2,2] := x \to y_{2,2}(x)$

> $y_{2,2}(x)$

> for i from 1 to k do; $y[i+2,1](x) = y[i+1,1](x) + \int_0^x b[1,1](x).((y[i+1,1])'(t) - (y[i+1,2](t)))dt$; od;

$\vdots$

> $y[4,1] := x \to y_{4,1}(x)$

> $y_{4,1}(x)$

> $y[4,2] := x \to y_{4,2}(x)$

> $y_{4,2}(x)$

> for i from 1 to k do; $y[i+4,1](x) = y[i+3,1](x) + \int_0^x b[1,1](x).((y[i+3,1])'(t) - y[i+3,2](t))dt$; od;

$\vdots$

> $y[9,1] := x \to y_{9,1}(x)$

> $y_{9,1}(x)$

> $y[9,2] := x \to y_{9,2}(x)$

> $y_{9,2}(x)$

> for i from 1 to k do; $y[i+9,1](x) = y[i+8,1](x) + \int_0^x b[1,1](x).((y[i+8,1])'(t) - y[i+3,1](t))dt$; od;

APPENDIX D

Maple code for Example (4.2.4)

$Restart$;

$> k$:=1; $a_1 := -1$

$>$ for i from 1 to k do; $b[1,1] = a_i$; od;

$> b[1,1] := x \rightarrow$ -1

$> b_{1,1}(x)$

$> g := x \rightarrow -2 + x/2$

$> f := t \rightarrow 0$

$> w := (t,r) \rightarrow t$

$> y[0,1] := x \rightarrow 0$

$> y_{0,1}(x)$

$> y[0,2] := x \rightarrow 0$

$> y_{0,2}(x)$

$>$ for i from 1 to k do; $y[i,1](x) = y[i-1,1](x) + \int_0^x b[1,1](x).((y[i-1,1])'(t) - y[i-1,2](t))dt$; od;

$>$ for i from 1 to k do; $y[i,2](x) = y[i-1,2](x) + \int_0^x b[1,1](x).((y[i-1,2])'(t) + g(t) + f(t) - \int_0^1 w(t,r).(y[i-1,1])(r).(y[i-1,2])(r)dr)dt$; od;

$> y[1,2] := x \rightarrow y_{1,2}(x)$

$> y_{1,2}(x)$

$> y[1,1] := x \rightarrow y_{1,1}(x)$

$> y_{1,1}(x)$

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) - (y[i,2](t)))dt$; od;

$>$ for i from 1 to k do; $y[i+1,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) - y[i,2](t))dt$;

od;

$>$ for i from 1 to k do; $y[i+1,2](x) = y[i,2](x) + \int_0^x b[1,1](x).((y[i,2])'(t) + g(t) + f(t) - \int_0^1 w(t,r).(y[i,1])(r).(y[i,2])(r)dr)dt$; od;

$> y[2,1] := x \to y_{2,1}(x)$

$> y_{2,1}(x)$

$> y[2,2] := x \to y_{2,2}(x)$

$> y_{2,2}(x)$

$>$ for i from 1 to k do; $y[i+2,1](x) = y[i+1,1](x) + \int_0^x b[1,1](x).((y[i+1,1])'(t) - (y[i+1,2](t)))dt$; od;

$\vdots$

$> y[4,1] := x \to y_{4,1}(x)$

$> y_{4,1}(x)$

$> y[4,2] := x \to y_{4,2}(x)$

$> y_{4,2}(x)$

$>$ for i from 1 to k do; $y[i+4,1](x) = y[i+3,1](x) + \int_0^x b[1,1](x).((y[i+3,1])'(t) - y[i+3,2](t))dt$; od;

$\vdots$

$> y[9,1] := x \to y_{9,1}(x)$

$> y_{9,1}(x)$

$> y[9,2] := x \to y_{9,2}(x)$

$> y_{9,2}(x)$

$>$ for i from 1 to k do; $y[i+9,1](x) = y[i+8,1](x) + \int_0^x b[1,1](x).((y[i+8,1])'(t) - y[i+3,1](t))dt$; od;

APPENDIX E

Maple code for Example (4.2.5)

$Restart$;

> k:=1; $a_1 := -1$

> for i from 1 to k do; $b[1,1] = a_i$; od;

> $b[1,1] := x \to$ -1

> $b_{1,1}(x)$

> $g := x \to -\sin(x) + x$

> $f := t \to 0$

> $w := (t,r) \to t.r$

> $y[0,1] := x \to 1$

> $y_{1,1}(x)$

> $y[0,2] := x \to 0$

> $y_{0,2}(x)$

> $y[0,3] := x \to$ -1

> $y_{0,3}(x)$

> for i from 1 to k do; $y[i,1](x) = y[i-1,1](x) + \int_0^x b[1,1](x).((y[i-1,1])'(t) - y[i-1,2](t))dt$; od;

> for i from 1 to k do; $y[i,2](x) = y[i-1,2](x) + \int_0^x b[1,1](x).((y[i-1,2])'(t) - y[i-1,3](t))dt$; od;

> $y[1,2] := x \to y_{1,2}(x)$

> $y_{1,2}(x)$

> $y[1,1] := x \to y_{1,1}(x)$

> $y_{1,1}(x)$

> for i from 1 to k do; $y[2,1](x) = y[i,1](x) + \int_0^x b[1,1](x).((y[i,1])'(t) - y[i,2](t))dt$; od;

⋮

> $y[4,1] := x \to y_{4,1}(x)$

> $y_{4,1}(x)$

> $y[4,2] := x \to y_{4,2}(x)$

> $y_{4,2}(x)$

> for i from 1 to k do; $y[i+4,1](x) = y[i+3,1](x) + \int_0^x b[1,1](x).((y[i+3,1])'(t) - (y[i+3,2](t)))dt$; od;

$\vdots$

> $y[9,1] := x \to y_{9,1}(x)$

> $y_{9,1}(x)$

> $y[9,2] := x \to y_{9,2}(x)$

> $y_{9,2}(x)$

> for i from 1 to k do; $y[i+9,1](x) = y[i+8,1](x) + \int_0^x b[1,1](x).((y[i+8,1])'(t) - y[i+8,2](t))dt$; od;