CONTINUOUS BLOCK – PREDICTOR BLOCK - CORRECTOR METHODS FOR THE SOLUTION OF HIGHER ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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(PhD/MA/08/0381)

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By

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A Thesis submitted to the Department of Mathematics, School of Pure and Applied Sciences, Modibbo Adama University of Technology, Yola, Adamawa State in fulfillment of the requirements for the award of the degree of Doctor of Philosophy in Mathematics.

DECEMBER, 2014

DECLARATION

I hereby declare that this thesis was written by me and it is a record of my own research work. It has not been presented before in any previous application for a higher degree. All references cited have been duly acknowledged.

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Udo, Mfon Okon Date

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DEDICATION

This thesis is dedicated to God Almighty for His faithfulness and mercies towards me.

APPROVAL PAGE

This thesis entitled "CONTINUOUS BLOCK – PREDICTOR BLOCK - CORRECTOR METHODS FOR THE SOLUTION OF HIGHER ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS" meets the regulations governing the award of Doctor of Philosophy of the Madibbo Adama University of Technology, Yola and is approved for its contribution to knowledge and literary presentation.

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ABSTRACT

The technique of collocating and interpolating the differential system and approximate solution respectively to generate predictor and corrector methods which were implemented in block form is adopted. A total of nine new methods of the same family but different applications were developed. Six of these methods were for second order ordinary differential equations (ODEs) while third order ODEs had three methods. The least, in terms of the parameters considered in this research involved four collocation points while the highest was six collocation points. All methods developed were zero stable and consistent, hence, convergent and they allowed for the evaluation at non overlapping intervals. It was found that the number of interpolation points used within a specific step length in developing a method is not significantly advantageous, rather it increased the number of variables involved in the evaluation of such a method. Sampled numerical examples of second and third order initial value problems were chosen to test the performance of our methods. Our methods performed comparatively in terms of level of accuracies than existing ones. We make bold to recommend that the block- predictor block- corrector method be used in the quest for solutions to second and third order initial value problems of ODEs.

KEYWORDS: Block-predictor block-corrector, higher order, k-step method, Continuous linear multistep methods, power series, Convergence.

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CHAPTER ONE
INTRODUCTION

1.1 General Introduction

In the 18th century, great scientists like Isaac Newton and Gottfried Leibniz introduced calculus which subsequently led to the development of the theory of differential equations. Since then, there is no doubt that the concept of differential equation has been of immense theoretical and practical application to research. The theory of differential equation can be found in various forms which includes modeling theories of electrical circuit (the relationship between the input voltage I(t) at time (t) and the output voltage O(t) at time (t) for the circuit), mechanical systems (the force exerted by the spring of a mass is proportional to the displacement from equilibrium), chemical system (quantity of chlorine placed in water is expected to dissolve completely over time (t)) and biological system (determination of calories intake for a cat on diet). It is now possible for general complex problems that are modeled by semi-discretized partial differential equations to be solved by the use of any of the newly developed methods for solving ordinary differential equations.

It is available in literature that not all ordinary differential equations have exact analytical solution. Consequently scholars have developed various numerical methods for solving ordinary differential equations with the aim of providing approximate solutions. In this class are: the forward and backward methods by Euler, linear multistep method by Adams and Bashforth. Thus, instead of determining the solution $y(x)$ of the non-linear initial value problem we now find a sequence $\{y_n\}$ which satisfies the differential equation.

According to Lambert (1973) and Fatunla (1988), the convergence and stability of the approximate solutions are veritable tools in establishing the validity of such a method. Convergence is the necessary and sufficient condition which any acceptable linear multistep method must satisfy.

The highest order we can expect from a k-step method is $2k$ if the method is implicit, and $2k - 1$ if it is explicit. This implies that both the implicit and explicit methods have limitation to the order of performance. However, these maximal orders cannot in general be attained without violating the conditions of zero stability as found in Fatunla (1988).

1.2 Statement of the Problem

Differential equations require solutions. Scholars have developed methods of varying magnitude to solve differential equations. One step and multi-step methods are the most popular approaches used in recent times. The latter provides for better performance in terms of speed, accuracy, convergence and stability when used to evaluate general initial value problems.

Methods like predictor-corrector, block and block-predictor corrector have been proposed. The solutions provided by these methods are yet to be optimal, thus the need to seek higher performing methods in the quest to finding numerical solutions to problems of general initial value problems.

1.3 Aim and Objectives

The aim of this research work is to develop k -step block-predictor block-corrector methods for the solution of second and third order initial value ordinary differential equations. The objectives are to:

(i) develop continuous linear multistep methods using power series as approximate solution.

(ii) implement both the predictor and corrector methods developed in block method.

(iii) analyze the basic properties of both the block-predictor and the block-corrector methods by considering the order, zero stability, consistency and convergence of the new methods.

(iv) write a computer code to implement the developed methods.

(v) demonstrate the efficiency of the methods on some second and third order initial value problems of ordinary differential equations using numerical examples.

1.4 **Justification of the Study**

Conventionally, higher order ordinary differential equations are solved by the method of reduction. This method has been reported to increase the dimension of the resulting differential equations, hence writing the computer code is difficult since it requires a special way to incorporate the subroutine to supply the starting values. This led to longer computer time and wastage of human effort (Adesanya *et al.*, 2012a; Awoyemi and Idowu, 2005c; Awoyemi *et al.*, 2011; Fatunla, 1983; Jator and Li, 2009). In order to avoid some of the setbacks of the method of reduction, scholars have developed what came to be known as direct methods in the form of linear multistep method which can be either implicit or explicit. Implicit linear multistep method which has a greater stability condition are implemented in predictor-corrector mode (Awoyemi, 2003). The major setback of this method is that the predictors are in reducing order of accuracy, hence this has a great effect on the result generated (Adesanya *et al.*,2013a). Block method was later proposed to correct some of the setbacks of predictor-corrector method. Despite the success of this method, its major setback is that the interpolation points cannot exceed the order of the differential equation; hence all the interpolation points cannot be exhausted resulting in a method of lower order being developed (Adetola *et al.*, 2012 and Adesanya *et al.*, 2013a). In order to correct the setback of block method, scholars developed a constant order predictor-corrector method. This method formed a bridge between the predictor-corrector method and block method (see James *et al*., 2013; Adetola *et al*., 2012 and Adesanya *et al*., 2013b). The major setback of constant order predictor corrector method is that the results are generated at an overlapping interval, which affects the accuracy of the method and the nature of the model cannot be determined at the selected grid points (Adesanya *et al*., 2012a).

Consequent upon these setbacks, we seek for a method that would be effective in terms of cost of implementation, ease of development and flexible in usage. We combined the desired qualities of predictor - corrector method (Awoyemi, 2001; Yahaya and Badmus, 2009; Kayode and Awoyemi, 2010; and Jator and Li, 2009), constant order predictor corrector (block predictor-corrector) (Adesanya *et al*., 2013b) and block methods (Omar and Suleiman, 2003; Zarina *et al*., 2009 and Adesanya, 2011) in developing a new method called block-predictor block-corrector method for the solution of higher order ordinary differential equations.

1.5 **Scope of the Study**

In carrying out this research work we had to limit ourselves to some conditions which included:

(i) consideration of the function f as continuous within the region,
$\Re = \{(x,y): a \leq x \leq b, -\infty < y < \infty, a\ and\ b\ finite\}$, hence the research did not consider situations where f is not continuous.

(ii) interest was limited to second and third order ordinary differential equations and cases where the order are higher than three were not considered.

(iii) partial differential equations were not considered.

(iv) considered power series as our basis function.

(v) all methods developed were those restricted to the concept of the conventional linear multistep method. Hybrid cases were not considered.

(vi) all methods developed were implicit.

(vii) gave consideration to general ordinary differential equations and no consideration was given to cases of stiff and oscillatory problems.

(viii) solutions were sought for only initial value problems and not for boundary value problems.

1.6 Definition of Basic Terms of Differential Equation

Equations describe the relations between the dependent and independent variables.

An equal sign "=" is required in every equation.

Definition 1.6.1: Differential Equations

Equations that involve dependent variables and their derivatives with respect to the independent variables are called differential equations.

Definition 1.6.2: Ordinary Differential Equation

Differential equations that involve only one independent variable are called ordinary differential equations.

Definition 1.6.3: Partial Differential equation

Differential equations that involve two or more independent variables are called partial differential equations.

Definition 1.6.4: Order of Differential Equation

The order of a differential equation is the highest derivative that appears in the differential equation.

Definition 1.6.5: Degree of Differential Equation

The degree of a differential equation is the power of the highest derivative term.

Definition 1.6.6: Initial Condition

Constraints that are specified at the initial point, generally time point, are called initial condition.

Definition 1.6.7: Boundary Condition

Constraints that are specified at the boundary point, generally time point, are called boundary condition.

A differential equation is called linear if there are no multiplications among dependent variables and their derivatives, otherwise it is said to be non-linear.

1.7 Initial Value Problems of Ordinary Differential Equations

The general $_{nth}$ order ordinary differential equation is given by

$$y'' = f(x, y, y', \dots, y^{n-1}) \quad (1.1)$$

where n is the order of the differential equation, f is a continuous function, x is an independent variable while $y(x), y^{'}(x), \dots, y^{n}(x)$ are dependent variables.

Problems with specified initial conditions are called initial value problems (IVP) and stated in the form

$$y^{n} = f(x, y, y', \ldots, y^{n-1}); y^{s}(a) = \eta_s, s = 0(1)n-1 \qquad (1.2)$$

However problems with specified boundary conditions are called boundary value problems (BVP).

In this research, we will consider only initial value problems of ordinary differential equations where $n = 2,3$.

Definition 1.7.1: Solution of Differential Equations (Lambert, 1973)

A function or a set of functions is a solution of a differential equation if the derivatives that appear in the differential equation exist on a certain domain and the differential equation is satisfied for all the values of the independent variables in that domain.

In literature we have it that higher order ordinary differential equation can be solved by either:

(i) method of reduction to system of first order ordinary differential equation after which a numerical solution is obtained.

(ii) by a direct method which does not resort to the reduction method.

1.8 **Existence and Uniqueness of a Solution**

The following theorem states the condition of $f(x, y)$ which guarantees the existence of a unique solution of initial value problem of the form (1.2).

Theorem 1.8.1: Uniqueness of Solution (Lambert, 1973)

Let $f(x,y)$ be defined and continuous for all points (x,y) in the region D defined by $a \leq x \leq b, -\infty < y < \infty, a \text{ and } b$ finite, and let there exist a constant L such that for every x, y, y^* such that (x,y) and (x, y^*) are both in D,

$$\left| f(x,y) - f(x,y^*) \right| \leq L \left| y - y^* \right|.$$

Then, if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1.2) where $y(x)$ is continuous and differentiable for all (x,y) in D. This requirement is known as a Lipschitz condition and the constant L as a Lipschitz constant. In this research f was considered to be Lipschitz continuous in the region of definition D of the $x - y$ plane, such that if the partial derivative of f with respect to y are continuous and bounded in D, then the Lipschitz constant L of the system is given by

$$L = \sup_{(x,y)\in D} \left\| \frac{\partial f(x,y)}{\partial y} \right\|$$

for $x \in [a, b]$, the eigenvalues of $\frac{\partial f}{\partial y}$ given as λ_j is such that $\left|R_e \lambda_j\right| = L$ and $\max_j \left|R_e \lambda_j\right| > 0,\ j = 1, 2, 3, \ldots$ This condition holds if and only if D is convex otherwise it will not. A vital condition for stiff problems is that $L >> 0$. The system of initial value problems of ordinary differential equation with this property is a system with large Lipschitz constant which is referred to as stiff problem.

The proof of Theorem 1.8.1 is found in Henrici, (1962); Ince, (1956) and Pontryagin, (1962).

Theorem 1.8.2:

Let

$$y^n = f(x, y, y', \ldots, y^{(n-1)}),\ y^k(x_0) = c_k,\ k = 0(1)n-1;$$
$$n \ and \ f \ are\ scalars.$$

Let D be the region defined by the inequalities

$$x_0 \le x \le x_0 + a, \left| s_j - c_j \right| \le b,\ \ j = 0, 1, \ldots, n-1 \quad (a > 0, b > 0)$$

Suppose that $f(x, c_0, \ldots, c_{n-1}) > 0; f(x, s_0, \ldots, s_{n-1}) > 0$ is defined in $\Re$ and in addition

(i) f is non-negative and non-decreasing in each of $x, s_0, \ldots, s_{n-1}$ in $\Re = \left\{ \left(x, s_j, c_j\right) \backslash x_0 \le x \le x_0 + a, \left| s_j - c_j \right| \le b,\ \ j = 0, 1, \ldots, n-1, (a > 0, b > 0). \right\}$

(ii) $f(x, c_0, \ldots, c_{n-1}) > 0$ for $x_0 \le x \le x_0 + a$

(iii) $c_k \ge 0, k = 1(1)n - 1$. Then the initial value problem (1.2) has a unique solution in $\Re$. The proof of this theorem is found in Wend (1969).

1.9 **Definition of Terms**

There are some basic concepts and principles that are relevant to the development of numerical integration method of ordinary differential equation. These concepts and their significance in the development of a linear multistep method will now be considered

1.9.1 ***Linear Multistep Method***

Definition 1.9.1: Linear Multistep Method (Fatunla, 1988)

Let y_n be an approximation to the theoretical solution at x_n to (1.2), that is, to $y(x_n)$, and let $f_n = f(x_n, y_n)$. If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, j = 0, 1..k,$ we call it a linear multi-step method of step number k, or a linear k-step method. Thus generally, we have a linear multi-step as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^n \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.3}$$

where h is the step size, α_j and β_j are constants, $\alpha_k \neq 0$ and $f_{n+J} = f(x_{n+j,} y_{n+j}, y'_{n+j}, \dots, y_{n+j}^{n-1})$,
$j = 0(1)k;\ \alpha_k, \beta_k \neq 0, h_i = x_{i+1} - x_i, i = 0(1)n - 1$.To remove arbitrariness, we assume throughout that $\alpha_k = 1$.

1.9.1.1 *Implicit and Explicit Methods*

A linear multistep method of the form (1.3) is said to be implicit if $\beta_k \neq 0$ or explicit, otherwise.

If explicit, the value of y_{n+k} can be determined directly in terms of y_{n+j} and f_{n+j}, $j = 0(1)k - 1$. Explicit methods are self starting but if implicit, it requires determination of initial values for $y_{n+k}, y'_{n+k}, \dots, y_{n+k}^{n-1}$ in terms of $f_{n+k} = f(x_{n+k,} y_{n+k}, y'_{n+k}, \dots, y_{n+k}^{n-1})$.

Various methods could be adopted to generate these additional starting values. These initial starting estimates for $y_{n+j}, j = 0(1)k - 1$ are then adopted for the calculation of

$$y_{n+k} = \sum_{j=0}^{r} \alpha_j y_{n+j} + h^n \sum_{j=0}^{s} \beta_j f_{n+j} \tag{1.4}$$

where h is the step length, the initial starting values to estimate y_{n+k} in the function f are called the predictor while (1.3) is called the corrector.

Implicit methods in general entail a substantially greater computational effort than do explicit methods even though implicit methods can be made more accurate than explicit ones and, moreover, enjoy more favorable stability properties.

1.9.2 *Collocation and Interpolation*

Definition 1.9.2: Collocation and Interpolation (Lambert, 1973)

Collocation is the evaluation of the differential system under consideration at some selected grid points while interpolation is the evaluation of the approximate solution at some selected grid points.

1.9.3 *Step-length*

Definition 1.9.3: Step-length (Lambert, 1973)

This is also known as mesh-size. We will adopt a principle of solving differential equation based on principle of discretization in which the approximate solutions are evaluated at each grid points. If we consider the sequence of points $\{x_n\}$ in the interval $I = [a, b]$ defined by $a = x_0 < x_1 < x_2 < \ldots < x_n = b, h_i = x_{i+1} - x_i;$ where $i = 0(1)n - 1$ then the parameter h_i is called the step-size.

1.9.4 ***Order and Error Constant***

Defining a linear difference operator L on equation (1.4) gives

$$L[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^n \beta_j y'(x + jh)\right] \tag{1.5}$$

where $y(x)$ is a function, continuously differentiable on $[a, b]$then we can formally define the order of accuracy of the operator and of the associated linear multistep method without involving the solution of the initial value problem.

Ideally, one would wish for a k-step linear multistep method of the highest possible order, but these goals are not always attainable due to Dahlquist barrier theorems (1956, 1959). For a given step number k, we are usually interested in choosing coefficients α_j, β_j which result in the method (1.3) having a reasonable high order; if not the highest order possible (Chawla and Sharma, 1981a; Cash, 1981 and Fatunla, 1988).

The difference operator and the associated linear multistep method are said to be of order p if

$$c_0 = c_1 = \ldots = c_{p+1} = 0; c_{p+2} \neq 0 \tag{1.6}$$

Definition 1.9.4: Zero Stability (Fatunla, 1988)

The linear multistep method is said to be zero stable if no root of the first characteristic polynomial $\rho(\xi)$ has modulus greater than one, and if every root with modulus one is simple.

An example of a zero stable linear multi-step method is available in Kayode (2009).

Theorem 1.9.1: Convergence (Lambert, 1973)

The linear multistep method is convergent if and only if it is consistent and zero-stable.

The proof of this theorem is available in Dahlquist (1956), Henrici (1962).

Definition 1.9.5: Convergence (Fatunla, 1988)

The linear multistep (1.3) is said to be convergent if for all initial value problems of the form (1.2) subject to the hypothesis of Theorem 1.9.1 we have that

$$\lim_{h\to 0} y_n = y(x_n),\ nh = x - a$$

holds for all $x \in [a, b]$ and for all solutions $\{y_n\}$ of the difference equation (1.3) satisfying starting conditions $y_\mu = \eta_\mu(h)$ for which

$$\lim_{h\to 0} \eta_\mu(h) = \eta,\ \mu = 0, 1, \ldots, k-1.$$

1.9.5 ***Consistency***

Equation (1.3) is said to be consistent if it has order $\rho \geq 1$ and if and only if

$$\rho(1) = \rho'(1) = \ldots = \rho^{(n-1)}(1) = 0 \ \textit{and}\ \rho^{(n)}(1) = n!\,\sigma(1)$$

where

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi_{n+j} \tag{1.7}$$

and

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi_{n+j} \tag{1.8}$$

are the first and second characteristic polynomials respectively (Lambert, 1973).

1.9.6 ***Stability of Linear Multistep Methods***

Definition 1.9.6 A(α) -Stability (Wildlund, 1967; Jeltsch, 1976; Lambert, 1973)

A numerical method is said to be A(α)-stable $\alpha\varepsilon(0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge

$$W_\alpha = \{h\lambda : -\alpha < \pi - \arg h\lambda < \alpha\}.$$

Definition 1.9.7 A-Stability (Dahlquist, 1956).

A numerical method is said to be A-stable if its absolute stability region contains the entire left half of the complex plane, meaning that it contains $Re\ \lambda h < 0$

Jeltsch (1976) applied this definition in developing a multistep multiderivetive method.

Definition 1.9.8 L-Stability (Lambert, 1973)

A one-step numerical method is sad to be L-stable if it is A-stable and in addition, when applied to the scalar test equation $y' = \lambda y, \lambda$ a complex number with $Re\,\lambda < 0$ it yields

$$y_{n+1} = R(h\lambda)y_n,$$
$$where\ |R(h\lambda)| \to 0\ as\ Re\,h\lambda \to -\infty.$$

Wanner *et al.* (1978) provides more materials for further studies on order and stability theorems.

1.9.7 ***Predictor-Corrector Methods***

We describe here a simple procedure called the predictor-corrector method. It should be noted that when using an implicit linear multistep method there is an additional difficulty because one cannot, in general, solve simply for the newest approximate y-value y_{n+k}. A general k -step implicit method involves, at the k^{th} time step

$$\alpha_k y_{n+k} + \ldots + \alpha_1 y_1 + \alpha_0 y_0 = h(\beta_k f_{n+k} + \ldots + \beta_1 f_1 + \beta_0 f_0)$$

and f depends on y in a complicated way, then it is not obvious how to calculate y_{n+k} out of $f_{n+k} = f(x_{n+k}, y_{n+k})$. One sure solution is the predictor-corrector method which involves

(i) The predictor step. We use an explicit method to obtain an approximation y^{p}_{n+k} to y_{n+k}.

(ii) The corrector step. We use an implicit method, but with the predicted value y^{p}_{n+k} on the right-hand side in the evaluation of f_{n+k}.

(iii) We can then go on to correct again and again. At each step we put the latest approximation to y_{n+k} in the right-hand side of the scheme to generate a new approximation on the left-hand side.

1.9.8 ***Block Method***

Block method was first proposed by Milne (1953). He advocated it's use as a means of obtaining starting values for predictor-corrector algorithm. Block method can be seen as a set of linear multistep method applied simultaneously to initial value problems and then put together to yield a more accurate approximation.

Equation (1.3) in its continuous form can be evaluated at different grid points to generate a method whose results give the independent solution at y_{n+k}.

Any method developed through this approach is called block method. According to Adesanya (2011) this method aside from reducing the computational burden experienced with the predictor-corrector method it also improves the performance and reduce the man hour wastage associated with the predictor-corrector method.

1.9.9 ***Block Predictor-Block Corrector Method***

It is obvious that the implementation of the predictor method in block form accounts for the improved performance of the block predictor-corrector method over that of predictor-corrector method. As a result it was deduced that if in addition we also implement the corrector in block, it will enhance the performance of the method. Consequent upon this we in this research proceed to implement both the predictor and corrector methods in block.

1.9.10 ***Power Series***

A series of the form $\sum_{k=0}^{\infty} a_k x^k$ where the coefficients $a_0, a_1, ..$ are constants is called power series. Power series approximation for a function interpolation has been widely applied in solving problems both theoretically and numerically because of their good approximation properties. In this research work, a polynomial in the form of power series of a single variable is used as the basis function throughout.

1.9.11 ***Matrices***

A matrix is defined as any array of numbers in the form $M \times N$ where M is the number of rows N is the number of columns and if $M = N$ a square matrix evolves.

Definition 1.9.9 Equality of Matrices:

Two matrices are said to be equal if the corresponding elements are equal.

Definition 1.9.10 Transpose of a Matrix:

If the rows and column of a matrix are interchanged then the new matrix formed is called the transpose of the original matrix. If the original matrix is A then A^T is the transpose of . That is, if, $A = [a_{ij}]$ and $B = [a_{ji}]$ then $B = A^T$.

Definition 1.9.11 Singular Matrix:

A matrix whose determinant is zero is called a singular matrix

Definition 1.9.12 Inverse of a Matrix:

The inverse of a matrix $|A|$ denoted by A^{-1} is given as

$$A^{-1} = \frac{adj(A)}{|A|}$$

where $adj(A)$ is the adjoint of the matrix A and $|A|$ is the determinant of the matrix A.

Definition 1.9.13: Adjoint of a Matrix

The matrix formed by taking the transpose of the cofactor matrix of a given original matrix. The adjoint of matrix A is often written $adjA$.

These principles were adopted in our work when solving generated systems of equations. Also adopted was the method of matrix inversion when solving the generated blocks.

CHAPTER TWO

LITERATURE REVIEW

2.0 **Introduction**

We had traced the developments in direct methods in order to offset the limitation of reduction method in chapter one. Various methods have been proposed by scholars for solving higher order ordinary differential equations directly. Notable among them were Chawla and Sharma (1981a, 1981b), Onumanyi and Oritz (1982), Twizell and Khaliq (1984), Jain *et al*. (1984), Fatunla (1988), Bun and Vasil'Yer (1992), Mchrkanoon (2011), Awoyemi (1994, 1996, 1999, 2003), Jator (2007), Adesanya *et al.* (2008), Kayode and Awoyemi (2010), Waeleh *et al.* (2012), Yakubu *et al.* (2013) and Odekunle *et al.* (2014) who had developed different forms of direct methods for solving general second order ODE's. Writing computer program is less burdensome since it no longer requires special skill to incorporate the subroutines which provides the starting values. As a result, this leads to reduced computer time and human effort.

We had taken advantage of the works of Omar and Suleiman (2003), Jator and Li (2009), Zarina *et al*. (2009) and Adesanya *et al.* (2013a) who did propose block methods to be used as a predictor.

2.1 **Existing Methods**

Numerical solutions to ordinary differential equations (ODEs) have great importance in scientific computation, as they are widely used to model real life problems. The common methods used to solve ODEs could be categorized as one-step (multistage) methods and multistep (one stage) methods, with Runge-Kutta methods representing the former group, and Adams-Bashforth-Molton method representing the latter group as suggested in Hairer and Wanner (1991).

According to Mustafa and Chollom (2010) the use of numerical techniques has become an integral part of modern engineering and scientific studies. The increasing importance of numerical methods in applied sciences has led to the developing of a new numerical method for the solution of applied problem. Several attempts were made by several researchers in developing reliable numerical method for solving ordinary differential equations. Hajjati *et al.* (2004) developed a multistep method for solving systems of ordinary differential equations. Hajjati *et al.* (2006) presented a new class of second derivative multistep methods with improved stability region. Onumanyi *et al.* (1993, 1999) resolved to continuous linear k-step methods which provide sufficient number of simultaneous discrete methods used as self starting single integrator.

Fatokun (2007) considered what he referred to as a continuous method for the derivation of self-starting multistep methods. According to him, the approach was based on collocation to obtain a set of k-multistep methods that required single step methods to obtain the $(k-1)$ starting values. He claimed that the methods were expected to be well suited for stiff differential equations because of the A-stability property. This claim is supported by Enright (1974) and Brown (1974).

Thus we have identified two basic types of existing method of solutions which includes, one step methods (Runge-Kutta and the Euler methods) and the multistep methods (linear and non linear multistep methods). The linear multistep methods include the following: explicit and implicit methods, the implicit methods have the following types: predictor-corrector, block predictor corrector. We will attempt to give a rundown of scholar's contributions in each case.

2.2 **One Step Methods**

In this family we have the Runge-Kutta and Euler's methods. Euler's methods are not very popular but the Runge-Kutta is one of the foremost methods and it still enjoys comfortable patronage by scholars.

Butcher and Rattenbury (2004) developed a method called Almost Runge-Kutta. This method formed a sub-class of general linear methods, with properties very close to those of Runge-Kutta methods. A characteristic feature of this method is that, although some of the data computed in a step have order limited to two, this inaccuracy does not affect the order of the subsequent steps.

Yakubu *et al.,* (2013) proposed stable two-step Runge-Kutta methods for the solution of systems, of first order initial value problems in ordinary differential equations whose oscillation is very high. They based their methods on multistep collocation at Gaussian points which were shown to be self starting and convergent with large region of absolute stability. They claimed that their method compared favorably well with the standard integrators, both in the quality of the numerical solutions and the computational effort. That is, the schemes were stable and of orders six and eight with five and seven stages respectively.

Mechee *et al.,* (2013) on the other hand developed a three-stage fifth-order Runge-Kutta method for the integration of a special third order ordinary differential equation. They compared their result with that of a method of reduction thus reducing the same problem to a first order equation whereby they concluded that their new method exhibited a higher level of accuracy than that of reduction method..

Ahmed and Yaacob (2005) developed an explicit Runge-kutta-like method which was shown to be efficient not only for stiff but also for ordinary differential equation.

2.3 **Multistep Methods**

In multistep method family there are linear and non linear cases. We express greater interest in linear multistep method to which we compare our method to see which has greater level of accuracy.

2.3.1 ***Predictor-Corrector Method***

This approach is amongst the recent methods in the theory of numerical solution of ODEs. It is an area that has received great attention from scholars in recent times.

Predictor-corrector method for solving higher order ordinary differential equations has been proposed by many scholars. These authors include Awoyemi (2005), Awoyemi and Kayode (2005), Awoyemi and Idowu (2005), Adesanya, *et al.* (2008). They proposed continuous implicit linear multistep method where separate reducing orders of accuracy predictors were developed and Taylor series expansion provided the starting value. The major setback of this method is the cost of developing predictors. Moreover, the predictors developed are of lower order to the corrector, hence it has a great effect on the accuracy of the results.

Awoyemi (2005) discussed a multiderivative collocation method for direct solution of the general initial value problems of ordinary differential equations of the form (1.2). Taylor series expansion was used to calculate the values at $y_{n+i}, i = 1(1)3$ and their derivatives which also appear in the main method. The result showed that a collocation method which produced a family of order six multiderivative schemes performed better than those obtained by Onumanyi and Ortiz (1982) and Wright *et al.* (1991).

Adesanya *et al.*, (2013c) developed a starting Hybrid Stomer-Cowell using the method of interpolation of the approximate solution and collocation of the differential system. The method was implemented in predictor-corrector mode in which Adams-Bashfort method was developed to provide a non-reducing order predictor. This was an improvement of Adesanya *et al.* (2012b).

2.3.2 ***Block Methods***

Block methods give solution at each grid point within the interval of integration without overlapping and the burden of developing separate predictors is removed. The authors who proposed block methods include Jator (2007), Jator and Li (2009), Adesanya (2011), Awoyemi *et al.* (2011) and Mchrkanoon (2011). These authors proposed discrete block method, which did not enable evaluation at all points within the interval of integration.

Waeleh *et al.*, (2012) developed a code based on point block method for the solution of higher order initial value problems of ordinary differential equations. Their block method was developed using interpolation approach which is similar to that of Adams Moulton type. This method is capable of solving higher order ordinary differential equations in a single code using variable step size and implementing in a predictor corrector mode. In addition, their method acted as simultaneous numerical integrator.

Zanariah *et al.*, (2012a) presented a two-point block one step method for solving general second order ODEs without reduction. They concluded that their method is faster in evaluation than the pre-existing methods.

Fudzial *et al.*, (2009) developed explicit and implicit 3-point block methods of constant step size using linear difference operator for solving special second order ordinary differential equations. The methods compute the solutions of the ODEs at three points simultaneously.

Zanariah *et al., (*2012b) developed a two-point four step direct implicit block method in which they applied the simple form of Adams-Moulton method for solving general third order ordinary differential equations using variable step size. Their results showed that their method performed better in terms of maximum error at all tested tolerance and lesser total number of steps as the tolerance is getting smaller compared to the existing methods

Adetola *et al.*, (2012) developed a four steps continuous method for the solution of$y'' = f(x, y, y')$. Power series approximate solution was adopted to derive a continuous implicit linear multistep method. Continuous block method was used to derive the independent solution which is evaluated at selected grid points to generate the discrete block method. Their method performed efficiently when compared with other existing methods.

Nur *et al.*, (2012) presented a four point direct block, one-step method for solving general second order non-stiff initial value problems of ODEs. Their results showed that the method developed reduced the total steps and the total functions to almost half compared to existing ones. What this means is that their developed method had fewer computational steps and functions than the existing method.

Omar and Suleiman (2003) derived a parallel 2 point explicit block method for solving ordinary differential equations directly. They positioned that most of the direct methods available are sequential in nature, meaning that the numerical solution to (1.2) is computed at one point at a time. Some of these methods are developed in Gear (1978), Hall and Suleiman (1981) and Suleiman (1989). They reported that parallel block one of the existing methods is as discussed in Chu and Hamilton (1987) and Tam (1989). In addition, they also reported that in a parallel block method, a set of new values that are obtained by each application of the formula is referred to as "block".

2.3.3 ***Block -Predictor Corrector Methods***

This is in the family of predictor-corrector method. The main difference is that the predictor in this case is implemented in block form.

Zarina *et al*., (2012) developed a code based on predictor corrector 2-point block of backward differentiation formula using variable step size. The method allows for the storage of the differential coefficients thus avoiding the coefficients calculating at each step but was robust enough to allow for step size variation. They concluded that the method reduces computational cost since the coefficients of the methods need not be recomputed at every step.

Adetola *et al*., (2012) considered a block predictor-corrector method for solving equations of the type (1.2).The method of collocation and interpolation was used to derive a linear multistep method with continuous coefficient. Block method was later adopted to generate the independent solution at selected grid points. The method performed more efficiently in terms of accuracy when compared with the self starting predictor-corrector method proposed by Shampine and Watts (1969) and Abbas (2006). They further noted that the method had greater accuracy when the step size is within the stability interval.

Adesanya *et al*., (2013c) developed a one step, five point Stomer Cowell method using the interpolation and collocation approach. They generated a continuous linear multistep method which served as the corrector to Adams-Bashforth method that in itself served as a constant order predictor. The method was able to combine the properties of the conventional predictor-corrector method, in which the predictors are in reducing order of accuracy, and block method.

Adesanya *et al*., (2012a) developed a method which combined the properties of hybrid method and the constant order predictor-corrector method. The predictors were implemented as a block method while the corrector gave the solution at an overlapping interval. The paper further submitted that the method was able to exhaust all interpolation points which resulted in the development of higher order methods without increasing the grid point as discussed by Adetola *et al*. (2012).

James *et al*., (2013) developed an order seven hybrid method in which only the predictor is implemented as a block method. Their method was found to exhibit a higher level of accuracy than that of Areo *et al*. (2011).

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CHAPTER THREE

METHODOLOGY

3.0 **Introduction**

There are two major parts in this research. The first part involved second order initial value problems of ordinary differential equation for step numbers $k = 3(1)5$. The second part involved third order initial value ordinary differential equation for step numbers of $k = 4,5$. It is important to note that for all cases we developed a predictor method as well as the corrector method and implemented them in block.

Systems of non-linear equation which are generated by the use of collocation and interpolation techniques were solved using the Gaussian elimination approach. The results generated were substituted into the approximate solution to give a continuous linear multistep method of the form (1.3). Since (1.3) is continuous and differentiable it is possible to evaluate along the derivatives at specified grid points.

3.1 **Development of the Continuous Linear Multistep Methods (LMM)**

Our ultimate desire is to develop methods that can solve general second and third order initial value problem directly. Thus in this research work, we examined the solution to:

(i) general second order initial value problem of the form

$$y'' = f(x, y, y'), y(x_0) = \eta_1, y'(x_0) = \eta_2 \tag{3.1}$$

and

(ii) general third order initial value problem of the form

$$y''' = f(x, y, y', y''), y(x_0) = \eta_1, y'(x_0) = \eta_2, y''(x_0) = \eta_3 \tag{3.2}$$

In developing our methods, we considered a polynomial as the approximate solution in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{3.3}$$

where r and s are the number of interpolation and collocation points respectively; x is a variable and a_j are constants $\forall j$.

The nth derivative of (3.3) gives

$$y^{(n)}(x) = \sum_{j=n}^{r+s-1} j(j-1)(j-2)\ldots(j-n)a_j x^{j-n-1} \tag{3.4}$$

Substituting (3.4) into (1.2) gives

$$f(x, y, y', y'', \ldots, y^{(n-1)}) = \sum_{j=3}^{r+s-1} j(j-1)(j-2)\ldots(j-n)a_j x^{j-n-1} \tag{3.5}$$

It must be noted that $n = 2$ and 3 in this work.

Interpolating (3.3) and collocating (3.5) at some selected grid points gives a system of non linear polynomials in the variable x equations in the form

$$Ax = b; \tag{3.6}$$

where the dimension of $A = (r + s) \times (r + s)$ and that of $x = b = 1 \times (r + s)$.

When $n = 2$, (3.6) gives

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{r+s-1} \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & \ldots & y_{n+r} & f_n & f_{n+1} & \ldots & f_{n+s} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^{r+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \ldots & x_{n+1}^{r+s-1} \\ . & . & . & . & . & . \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & \ldots & x_{n+r}^{r+s-1} \\ 0 & 0 & 2 & 6x_n & \ldots & (s+r-1)(s+r-2)x_n^{r+s-1} \\ 0 & 0 & 2 & 6x_{n+1} & \ldots & (s+r-1)(s+r-2)x_{n+1}^{r+s-1} \\ . & . & . & . & . & . \\ 0 & 0 & 2 & 6x_{n+s} & \ldots & (s+r-1)(s+r-2)x_{n+s}^{r+s-1} \end{bmatrix}$$

and when $n = 3$, (3.6) gives

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{r+s-1} \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & \ldots & y_{n+r} & f_n & f_{n+1} & \ldots & f_{n+s} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \ldots & x_n^{r+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \ldots & x_{n+1}^{r+s-1} \\ . & . & . & . & . & . & . \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r}^4 & \ldots & x_{n+r}^{r+s-1} \\ 0 & 0 & 0 & 6 & 24x_n & \ldots & (s+r-1)(s+r-2)(s+r-3)x_n^{r+s-1} \\ 0 & 0 & 0 & 6 & 24x_{n+1} & \ldots & (s+r-1)(s+r-2)(s+r-3)x_{n+1}^{r+s-1} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 6 & 24x_{n+s} & \ldots & (s+r-1)(s+r-2)(s+r-3)x_{n+s}^{r+s-1} \end{bmatrix}$$

Solving (3.6) for the unknown constants $a_j's$ using Guassian elimination method and substituting back into (3.3) gives a continuous linear multistep method in the form

$$y(t) = \sum_{j=0}^{r} \alpha_j(t) y_{n+j} + h^{\rho} \sum_{j=0}^{s} \beta_j(t) f_{n+j} \qquad (3.7)$$

where $\alpha_j(t)$ and $\beta_j(t)$ are polynomials, ρ is the order of the differential equation,

$y_{n+j} = y(x_n + jh)$, $f_{n+j} = (f(x_n + jh),\ y(x_n + jh),\ y'(x_n + jh), \ldots, y^{(n-1)}(x_n + jh))$,

$t = \frac{x - x_n}{h}$, *h is the step size.*

3.2 **Development of the Block Predictor Method**

In developing the block predictor, we solve (3.7) for the independent solutions at some selected grid points to give a continuous block formula of the form

$$y_{n+j} = \sum_{i=0}^{1} \frac{(jh)^i}{i!} y_n^{(i)} + h^{\rho} \sum_{j=0}^{r} \sigma_j(x) f_{n+j} \qquad (3.8)$$

Evaluating (3.8) at selected grid points give a discrete block formula in the form

$$A^{(0)} Y_m^{(i)} = \sum_{i=0}^{\rho-1} \left[\frac{(jh)^i}{i!} e_i y_n^{(i)} + h^{\rho} [d_i f(y_n) + b_i F(Y_m)] \right] \qquad (3.9)$$

where

$$Y_m^{(i)} = \begin{bmatrix} y_n^{(i)} & y_{n+1}^{(i)} & y_{n+2}^{(i)} & . & .. & y_{n+r}^{(i)} \end{bmatrix}^T$$

$$F(Y_m) = \begin{bmatrix} f_n & f_{n+1} & f_{n+2} & . & .. & f_{n+r} \end{bmatrix}^T$$

$$Y_n^{(i)} = \begin{bmatrix} y_{n-1}^{(i)} & y_{n-2}^{(i)} & y_{n-3}^{(i)} & . & .. & y_n^{(i)} \end{bmatrix}^T$$

$e_i = r \times r$ matrix, $A^{(0)} = r \times r$ identity matrix.

3.3 **Development of the Block Corrector Method**

In developing the block corrector, we implement (3.7) and its derivative in block form to give a discrete block formula of the form

$$A^{(0)} Y_m = A^{(1)} Y_{m-1} + A^{(2)} Y_{m-2} + h (B^{(0)} f_{m-1} + B^{(1)} f_m) \qquad (3.10)$$

where $A^{(0)} = r \times r$ identity matrix

$$Y_m = [y_{n+1} \quad y_{n+2} \quad y_{n+3} \quad . \quad .. \quad y_{n+s}]^T$$

$$f_{m-1} = [f_{n-1} \quad f_{n-2} \quad f_{n-3} \quad . \quad .. \quad f_n]^T$$

$$Y_{m-1} = [y_{n-1} \quad y_{n-2} \quad y_{n-3} \quad . \quad .. \quad y_n]^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_{n-2} & y'_{n-3} & \dots & y'_n & y'_{n+k} & y''_n \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_n & f_{n+1} & f_{n+2} & . & .. & f_{n+s} \end{bmatrix}^T$$

In this research work, we consider three cases for second order when the step number $(k) = 3(1)5$ and two cases for third order when the step length $(k) = 4,5$

3.4 **Development of Methods for Second Order ODEs**

3.4.1 ***Development of the Block Predictor Method for k = 3***

Interpolating (3.3) at $x_{n+r}, r = 0,1$ and collocating (3.5) at $x_{n+s}, s = 0(1)3,$ (3.6) reduces to

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+1} & f_{n+2} & f_{n+3} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ and substituting back into (3.3), (3.7) reduces to

$$y(t) = \sum_{j=0}^{1} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j(t) f_{n+j} \qquad (3.11)$$

where

$$\alpha_0 = 1 - t$$

$$\alpha_1 = t$$

$$\beta_0 = -\frac{1}{360}\left(3t^5 - 30t^4 + 110t^3 - 180t^2 + 97t\right)$$

$$\beta_1 = \frac{1}{120}\left(3t^5 - 25t^4 + 60t^3 - 38t\right)$$

$$\beta_2 = -\frac{1}{120}\left(3t^5 - 20t^4 + 30t^3 - 13t\right)$$

$$\beta_3 = \frac{1}{360}\left(3t^5 - 15t^4 + 20t^3 - 8t\right)$$

Solving for the independent solution in (3.11), (3.8) reduces to

$$y_{n+j} = \sum_{i=0}^{1} \frac{(jh)^i}{i!} y_n^{(i)} + h^2 \sum_{j=0}^{3} \sigma_j(t) f_{n+j} \qquad (3.12)$$

where

$$\alpha_0 = -\frac{1}{360}\left(3t^5 - 30t^4 + 110t^3 - 180t^2\right)$$

$$\alpha_1 = \frac{1}{120}\left(3t^5 - 25t^4 + 60t^3\right)$$

$$\alpha_2 = -\frac{1}{120}\left(3t^5 - 20t^4 + 30t^3\right)$$

$$\alpha_3 = \frac{1}{360}\left(3t^5 - 15t^4 + 20t^3\right)$$

Evaluating (3.12) at the selected grid points, the parameters in (3.9) becomes:

$A^{(0)} = 3 \times 3$ identity matrix

$$Y_m^{(0)} = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} \end{bmatrix}^T$$

$$Y_m^{'} = \begin{bmatrix} y_{n+1}^{'} & y_{n+2}^{'} & y_{n+3}^{'} \end{bmatrix}^T$$

When $i = 0$,

$$e_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & \frac{97}{360} \\ 0 & 0 & \frac{28}{45} \\ 0 & 0 & \frac{39}{40} \end{bmatrix}, \qquad b_0 = \begin{bmatrix} \frac{19}{60} & -\frac{13}{120} & \frac{1}{45} \\ \frac{22}{15} & -\frac{2}{15} & \frac{2}{45} \\ \frac{27}{10} & \frac{27}{40} & \frac{3}{20} \end{bmatrix}$$

When $i = 1$,

$$e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad d_1 = \begin{bmatrix} 0 & 0 & \frac{3}{8} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{3}{8} \end{bmatrix}, \qquad b_1 = \begin{bmatrix} \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{bmatrix}$$

3.4.2 ***Development of the Block Corrector Method for k = 3***

Interpolating (3.3) at $x_{n+r}, r = 0(1)2$ and collocating (3.5) at $x_{n+s}, s = 0(1)3$, (3.6) reduces to

$$x = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6]^T$$

$$b = [y_n \quad y_{n+1} \quad y_{n+2} \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3}]^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \end{bmatrix}$$

Solving for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t) = \sum_{j=0}^{2} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j(t) f_{n+j} \tag{3.13}$$

where

$$\alpha_0 = -\frac{1}{6}(2t^6 - 18t^5 + 55t^4 - 60t^3 + 27t - 6)$$

$$\alpha_1 = \frac{1}{3}(2t^6 - 18t^5 + 55t^4 - 60t^3 + 24t)$$

$$\alpha_2 = -\frac{1}{6}(2t^6 - 18t^5 + 55t^4 - 60t^3 + 21t)$$

$$\beta_0 = \frac{1}{360}(10t^6 - 93t^5 + 305t^4 - 410t^3 + 180t^2 + 8t)$$

$$\beta_1 = \frac{1}{360}(100t^6 - 891t^5 + 2675t^4 - 2820t^3 + 936t)$$

$$\beta_2 = \frac{1}{360}(10t^6 - 99t^5 + 335t^4 - 390t^3 + 144t)$$

$$\beta_3 = \frac{1}{360}(3t^5 - 15t^4 + 20t^2 - 8t)$$

Evaluating (3.13) at $t = 3$ will give the equation

$$y_{n+3} = \frac{9}{2}y_{n+2} - 103y_{n+1} + \frac{11}{2}y_n - \frac{h^2}{360}(127f_n + 1359f_{n+1} - 279f_{n+2} - 127f_{n+3}) \tag{3.14}$$

Evaluating the first derivatives of (3.13) at $t = 0$ and 1 gives the following equations

$$hy'_n = -\frac{7}{3}y_{n+2} + 8y_{n+1} - \frac{9}{2}y_n + \frac{h}{45}\left(-f_{n+3} + 18f_{n+2} + 117f_{n+1} + f_n\right) \quad (3.15)$$

$$hy'_{n+1} = -\frac{17}{6}y_{n+2} - 1\frac{14}{3}y_{n+1} + \frac{11}{6}y_n + \frac{h}{360}\left(f_{n+3} - 121f_{n+2} - 679f_{n+1} - 47f_n\right) \quad (3.16)$$

If we write (3.14) to (3.16) in block form, the parameters in (3.10) give the following results

$A^{(0)} = 3 \times 3$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_n \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A^{(2)} = \begin{bmatrix} 0 & \frac{17}{38} & \frac{21}{38} \\ 0 & \frac{14}{19} & \frac{24}{19} \\ 0 & \frac{33}{38} & \frac{81}{38} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{851}{13680} \\ 0 & 0 & \frac{127}{855} \\ 0 & 0 & \frac{267}{1520} \end{bmatrix}, \qquad B^{(1)} = \begin{bmatrix} -\frac{29}{240} & \frac{31}{4560} & -\frac{11}{13680} \\ \frac{7}{15} & \frac{37}{285} & -\frac{7}{855} \\ \frac{81}{80} & \frac{1701}{1520} & \frac{93}{1520} \end{bmatrix}$$

3.4.3 ***Development of the Block Predictor Method for k = 4***

Again as it was in the case of $k = 3$, interpolating (3.3) at $x_{n+r}, r = 0,1$ and collocating (3.5) at $x_{n+s}, s = 0(1)4$, the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and simplifying gives

$$y(t) = \sum_{j=0}^{1} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{4} \beta_j(t) f_{n+j} \qquad (3.17)$$

where

$\alpha_0 = 1 - t$

$\alpha_1 = t$

$\beta_0 = \frac{1}{1440}\left(2t^6 - 30t^5 - 175t^4 - 500t^3 + 720t^2 - 367t\right)$

$\beta_1 = -\frac{1}{360}\left(2t^6 - 27t^5 + 130t^4 - 240t^3 + 135t\right)$

$\beta_2 = \frac{1}{240}\left(2t^6 - 24t^5 + 95t^4 - 120t^3 + 47t\right)$

$\beta_3 = -\frac{1}{360}\left(2t^6 - 21t^5 + 70t^4 - 80t^3 + 29t\right)$

$\beta_4 = \frac{1}{1440}\left(2t^6 - 18t^5 + 55t^4 - 60t^3 + 21t\right)$

Solving for the independent solution in (3.6),gives

$$y_{n+j} = \sum_{i=0}^{1} \frac{(jh)^i}{i!} y_n^{(i)} + h^2 \sum_{j=0}^{4} \sigma_j(x) f_{n+j} \qquad (3.18)$$

where

$$\varphi_0 = \frac{1}{1440}\left(2t^6 - 30t^5 - 175t^4 - 500t^3 + 720t^2\right)$$

$$\varphi_1 = -\frac{1}{360}\left(2t^6 - 27t^5 + 130t^4 - 240t^3\right)$$

$$\varphi_2 = \frac{1}{240}\left(2t^6 - 24t^5 + 95t^4 - 120t^3\right)$$

$$\varphi_3 = -\frac{1}{360}\left(2t^6 - 21t^5 + 70t^4 - 80t^3\right)$$

$$\varphi_4 = \frac{1}{1440}\left(2t^6 - 18t^5 + 55t^4 - 60t^3\right)$$

Evaluating (3.18) at the selected grid points, the parameters in (3.9) becomes , $A^{(0)} = 4 \times 4$ identity matrix

$$Y_m^{(0)} = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_m^{(i)} = \begin{bmatrix} y_{n+1}^{(i)} & y_{n+2}^{(i)} & y_{n+3}^{(i)} & y_{n+4}^{(i)} \end{bmatrix}^T$$

When $i = 0$

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{367}{1440} \\ 0 & 0 & 0 & \frac{53}{90} \\ 0 & 0 & 0 & \frac{147}{160} \\ 0 & 0 & 0 & \frac{56}{45} \end{bmatrix}, \qquad b_0 = \begin{bmatrix} \frac{3}{8} & -\frac{47}{240} & \frac{29}{360} & -\frac{7}{480} \\ \frac{8}{5} & -\frac{1}{3} & \frac{8}{45} & -\frac{1}{30} \\ \frac{117}{40} & \frac{27}{80} & \frac{3}{8} & -\frac{9}{160} \\ \frac{64}{15} & \frac{16}{15} & \frac{64}{45} & 0 \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad d_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{720} \\ 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{323}{360} & -\frac{11}{80} & \frac{53}{360} & -\frac{19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & -\frac{3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}$$

3.4.4 ***Development of the Block Corrector Methods for k = 4***

Here two cases (One and Two) will be considered

3.4.4.1 *Development of the Block Corrector Method for Case One*

Interpolating (3.3) at $x_{n+r}, r = 0(1)2$ and collocating (3.5) at $x_{n+s}, s = 0(1)4$, (3.6) reduces to

$$x = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7]^T$$

$$b = [y_n \quad y_{n+1} \quad y_{n+2} \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j{'s}$ using Guassian elimination method and substituting into (3.3), makes equation (3.8) reduced to

$$y(t) = \sum_{j=0}^{2} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{4} \beta_j(t) f_{n+j} \qquad (3.19)$$

where

$$\alpha_0 = \frac{1}{42}(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 149t + 42)$$

$$\alpha_1 = -\frac{1}{21}\left(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 128t\right)$$

$$\alpha_2 = \frac{1}{42}\left(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 107t\right)$$

$$\beta_0 = -\frac{1}{10080}\left(38t^7 - 546t^6 + 3003t^5 - 7875t^4 + 9884t^3 - 5040t^2 + 536t\right)$$

$$\beta_1 = -\frac{1}{1260}\left(51t^7 - 707t^6 + 3654t^5 - 8470t^4 + 7728t^3 - 2256t\right)$$

$$\beta_2 = -\frac{1}{720}\left(2t^7 - 34t^6 + 219t^5 - 635t^4 + 696t^3 - 248t\right)$$

$$\beta_3 = -\frac{1}{1260}\left(t^7 - 7t^6 + 70t^4 - 112t^3 + 48t\right)$$

$$\beta_4 = \frac{1}{10080}\left(2t^7 - 14t^6 + 21t^5 + 35t^4 - 84t^3 + 40t\right)$$

Evaluating (3.19) at $t = 3,4$ gives the following equations

$$y_{n+3} = 2y_{n+2} - y_{n+1} + h^2\left(-\frac{1}{240}f_n + \frac{1}{10}f_{n+1} + \frac{97}{120}f_{n+2} + \frac{1}{10}f_{n+3} - \frac{1}{240}f_{n+4}\right) \quad (3.20)$$

$$y_{n+4} = -y_n + 2y_{n+2} + h^2\left(\frac{1}{15}f_n + \frac{16}{15}f_{n+1} + \frac{26}{15}f_{n+2} + \frac{16}{15}f_{n+3} + \frac{1}{15}f_{n+4}\right) \quad (3.21)$$

Evaluating the first derivatives of (3.19) at $t = 0$ and 1 give the following equations

$$hy'_n = -\frac{149}{42}y_n + \frac{128}{21}y_{n+1} - \frac{107}{42}y_{n+2} + h\left(\begin{array}{c} -\frac{67}{1260}f_n + \frac{188}{105}f_{n+1} \\ +\frac{31}{90}f_{n+2} - \frac{4}{105}f_{n+3} + \frac{1}{252}f_{n+4} \end{array}\right) \quad (3.22)$$

$$hy'_{n+1} = \frac{20}{21}y_n - \frac{61}{21}y_{n+1} + \frac{41}{21}y_{n+2} + h\left(\begin{array}{c} -\frac{613}{10080}f_n - \frac{1433}{1260}f_{n+1} \\ -\frac{41}{144}f_{n+2} + \frac{43}{1260}f_{n+3} - \frac{37}{10080}f_{n+4} \end{array}\right) \quad (3.23)$$

Writing (3.20) to (3.23) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 4 \times 4$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y_{n-2} & y'_n & y'_{n+1} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^{(1)} = \begin{bmatrix} 0 & 0 & \frac{82}{189} & \frac{107}{189} \\ 0 & 0 & \frac{122}{189} & \frac{256}{189} \\ 0 & 0 & \frac{6}{7} & \frac{15}{7} \\ 0 & 0 & \frac{244}{189} & \frac{512}{189} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{15649}{272160} \\ 0 & 0 & 0 & \frac{397}{3402} \\ 0 & 0 & 0 & \frac{577}{3360} \\ 0 & 0 & 0 & \frac{2552}{8505} \end{bmatrix} \qquad B^{(1)} = \begin{bmatrix} -\frac{4523}{34020} & \frac{533}{45360} & -\frac{19}{6804} & \frac{97}{272160} \\ \frac{3272}{8505} & \frac{463}{2835} & -\frac{184}{8505} & \frac{41}{17010} \\ \frac{421}{420} & \frac{629}{560} & \frac{5}{84} & \frac{1}{3360} \\ \frac{15616}{8505} & \frac{1168}{567} & \frac{8704}{8505} & \frac{608}{8505} \end{bmatrix}$$

3.4.4.2 *Development of the Block Corrector Method for Case Two*

Interpolating (3.3) at $x_{n+r}, r = 0(1)3$ and collocating (3.5) at $x_{n+s}, s = 0(1)$ the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & y_{n+3} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$A=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.8) reduces to

$$y(t)=\sum_{j=0}^{3}\alpha_j(t)y_{n+j}+h^2\sum_{j=0}^{4}\beta_j(t)f_{n+j} \qquad (3.24)$$

where the values of the $\alpha_i^{'}s$ and $\beta_j^{'}s$ are computed for $i=0(1)3$, and j$=0(1)4$.

Evaluating (3.24) at $t=4$ will give the equation

$$\begin{aligned} y_{n+4} &= -y_n-\frac{128}{31}y_{n+1}+\frac{318}{31}y_{n+2}-\frac{128}{31}y_{n+3}+ \\ &\frac{h^2}{465}\left(23f_n+688f_{n+1}+2358f_{n+2}+688f_{n+3}+23f_{n+4}\right) \end{aligned} \qquad (3.25)$$

Evaluating the first derivatives of (3.24) at $t=0(1)2$, gives the following equations

$$\begin{aligned} hy_n^{'} &= \frac{-149}{42}y_n+\frac{864}{217}y_{n+1}+\frac{729}{434}y_{n+2}-\frac{1376}{651}y_{n+3}- \\ &\frac{h}{4340}\left(269f_n-8688f_{n+1}-8910f_{n+2}-752f_{n+3}+21f_{n+4}\right) \end{aligned} \qquad (3.26)$$

$$\begin{aligned} hy_{n+1}^{'} &= \frac{20}{21}y_n-\frac{337}{434}y_{n+1}-\frac{500}{217}y_{n+2}+\frac{2771}{1302}y_{n+3}- \\ &\frac{h}{39060}\left(2029f_n+52736f_{n+1}+78318f_{n+2}+6980f_{n+3}-203f_{n+4}\right) \end{aligned} \qquad (3.27)$$

$$\begin{aligned} hy_{n+2}^{'} &= \frac{-37}{42}y_n-\frac{312}{217}y_{n+1}+\frac{1961}{434}y_{n+2}-\frac{1432}{651}y_{n+3}+ \\ &\frac{h}{39060}\left(1905f_n+43808f_{n+1}+87246f_{n+2}-7104f_{n+3}-203f_{n+4}\right) \end{aligned} \qquad (3.28)$$

Writing (3.25) to (3.28) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 4 \times 4$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} & y'_{n+2} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & \frac{11843}{30464} & \frac{953}{1904} & \frac{3373}{30464} \\ 0 & \frac{97}{238} & \frac{120}{119} & \frac{139}{238} \\ 0 & \frac{17763}{30464} & \frac{3321}{1904} & \frac{20493}{30464} \\ 0 & \frac{20}{119} & \frac{128}{119} & \frac{328}{119} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{24509}{548352} \\ 0 & 0 & 0 & \frac{117}{2380} \\ 0 & 0 & 0 & \frac{28617}{304640} \\ 0 & 0 & 0 & \frac{-20}{1071} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -\frac{77671}{342720} & -\frac{1593}{38080} & \frac{667}{342720} & -\frac{397}{2741760} \\ \frac{-586}{5355} & -\frac{71}{595} & \frac{2}{595} & -\frac{1}{4284} \\ \frac{16497}{38080} & \frac{6075}{7616} & \frac{3363}{38080} & -\frac{837}{304640} \\ -\frac{2656}{5355} & \frac{432}{595} & \frac{6112}{5355} & \frac{316}{5355} \end{bmatrix}$$

3.4.5 Development of the Block Predictor Method for k = 5

Interpolating (3.3) at $x_{n+r}, r = 0,1$ and collocating (3.5) at $x_{n+s}, s = 0(1)5$, the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(x)=\sum_{j=0}^{1}\alpha_j(x)y_{n+j}+h^2\sum_{j=0}^{5}\beta_j(x)f_{n+j} \qquad (3.29)$$

where the values of the α_i's and β_j's are computed for $i=0,1$ and j= $0(1)5$.

Solving for the independent solution in (3.29), makes (3.8) reduced to

$$y_{n+j}=\sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)}+h^2\sum_{j=0}^{5}\sigma_j(x)f_{n+j} \qquad (3.30)$$

where the values of the σ_j's are computed for j= $0(1)5$.

Evaluating (3.30) at the selected grid points, the parameters in (3.9) gives the following results:

$A^{(0)}=5\times 5$ identity matrix

$$Y_m^{(0)}=\begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_m^{(i)}=\begin{bmatrix} y'_{n+1} & y'_{n+2} & y'_{n+3} & y'_{n+4} & y'_{n+5} \end{bmatrix}^T$$

When $i=0$

$$e_0=\begin{bmatrix} 0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1 \end{bmatrix},\quad e_1=\begin{bmatrix} 0&0&0&0&1\\0&0&0&0&2\\0&0&0&0&3\\0&0&0&0&4\\0&0&0&0&5 \end{bmatrix}$$

$$
d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1231}{5040} \\ 0 & 0 & 0 & 0 & \frac{71}{126} \\ 0 & 0 & 0 & 0 & \frac{123}{140} \\ 0 & 0 & 0 & 0 & \frac{376}{315} \\ 0 & 0 & 0 & 0 & \frac{1525}{1008} \end{bmatrix}, \quad
b_0 = \begin{bmatrix} \frac{863}{2016} & -\frac{761}{2520} & \frac{941}{5040} & -\frac{341}{5040} & \frac{107}{10080} \\ \frac{544}{315} & -\frac{37}{63} & \frac{136}{315} & -\frac{101}{630} & \frac{8}{315} \\ \frac{3501}{1120} & -\frac{9}{140} & \frac{87}{112} & -\frac{9}{35} & \frac{9}{224} \\ \frac{1424}{315} & \frac{176}{315} & \frac{608}{315} & -\frac{16}{63} & \frac{16}{315} \\ \frac{11875}{2016} & \frac{625}{504} & \frac{3125}{1008} & \frac{625}{1008} & \frac{275}{2016} \end{bmatrix}
$$

When $i = 1$

$$
e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad
d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{95}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{51}{160} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{95}{288} \end{bmatrix}
$$

$$
b_1 = \begin{bmatrix} \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} & \frac{3}{160} \\ \frac{43}{30} & \frac{7}{45} & \frac{7}{45} & -\frac{1}{15} & \frac{1}{90} \\ \frac{219}{160} & \frac{57}{80} & \frac{57}{80} & -\frac{21}{160} & \frac{3}{160} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{125}{96} & \frac{125}{144} & \frac{125}{144} & \frac{125}{96} & \frac{95}{288} \end{bmatrix}
$$

3.4.6 *Development of the Block Corrector Methods for k = 5*

Here there are three cases (One, Two and Three) to be considered.

3.4.6.1 *Development of the Block Corrector Method for Case One*

Interpolating (3.3) at $x_{n+r}, r = 0(1)2$, and collocating (3.5) at $x_{n+s}, s = 0(1)5$, (3.6) reduced to

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8\\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8\\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8\\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6\\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6\\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6\\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6\\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6\\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6\end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t)=\sum_{j=0}^{2}\alpha_j(t)y_{n+j}+h^2\sum_{j=0}^{5}\beta_j(t)f_{n+j} \tag{3.31}$$

where the values of the $\alpha_i's$ and $\beta_j's$ are computed for $i=0(1)2$ and $j=0(1)5$

Evaluating (3.31) at $t=3(1)5$ gives the following equations

$$y_{n+3}=-\frac{31}{221}y_n-\frac{159}{221}y_{n+1}+\frac{411}{221}y_{n+2}+\frac{h^2}{53040}\begin{pmatrix}337f_n+11783f_{n+1}+42998f_{n+2}\\ +5738f_{n+3}-407f_{n+4}+31f_{n+5}\end{pmatrix} \tag{3.32}$$

$$y_{n+4}=-\frac{93}{221}y_n-\frac{256}{221}y_{n+1}+\frac{570}{221}y_{n+2}+\frac{h^2}{3315}\begin{pmatrix}77f_n+1864f_{n+1}+5714f_{n+2}\\ +3424f_{n+3}+269f_{n+4}-8f_{n+5}\end{pmatrix} \tag{3.33}$$

$$y_{n+5}=\frac{66}{221}y_n-\frac{795}{221}y_{n+1}+\frac{950}{442}y_{n+2}-\frac{h^2}{5304}\begin{pmatrix}163f_n-35f_{n+1}-14206f_{n+2}\\ -10162f_{n+3}-5653f_{n+4}-347f_{n+5}\end{pmatrix} \tag{3.34}$$

Evaluating the first derivatives of (3.31) at $t=0,1$ gives the following equations

$$hy_n'=-\frac{1437}{442}y_n+\frac{1216}{221}y_{n+1}-\frac{995}{442}y_{n+2}-\frac{h}{278460}\begin{pmatrix}20999f_n-426680f_{n+1}-94538f_{n+2}\\ +15424f_{n+3}-3169f_{n+4}+344f_{n+5}\end{pmatrix} \tag{3.35}$$

$$h y'_{n+1} = \frac{17}{26}y_n - \frac{30}{13}y_{n+1} + \frac{43}{26}y_{n+2} - \frac{h}{131040}\begin{pmatrix} 5035 f_n + 114965 f_{n+1} + 36658 f_{n+2} \\ -6754 f_{n+3} + 1459 f_{n+4} - 163 f_{n+5} \end{pmatrix} \quad (3.36)$$

Writing (3.32) to (3.36) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_{n-4} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_{n-2} & y'_{n-3} & y'_n & y'_{n+1} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_n \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(k)} = \begin{bmatrix} 0 & 0 & 0 & \frac{731}{1726} & \frac{995}{1726} \\ 0 & 0 & 0 & \frac{510}{863} & \frac{1216}{863} \\ 0 & 0 & 0 & \frac{1371}{1726} & \frac{3807}{1726} \\ 0 & 0 & 0 & \frac{892}{863} & \frac{2560}{863} \\ 0 & 0 & 0 & \frac{1755}{1726} & \frac{6875}{1726} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{313679}{5799360} \\ 0 & 0 & 0 & 0 & \frac{10733}{108738} \\ 0 & 0 & 0 & 0 & \frac{291909}{1933120} \\ 0 & 0 & 0 & 0 & \frac{19496}{90615} \\ 0 & 0 & 0 & 0 & \frac{692425}{3479616} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} -\frac{166091}{1159872} & \frac{152063}{8699040} & -\frac{18133}{2899680} & \frac{9271}{5799360} & -\frac{3373}{17398080} \\ \frac{17978}{54369} & \frac{52613}{271845} & -\frac{10844}{271845} & \frac{4873}{543690} & -\frac{278}{271845} \\ \frac{1817379}{1933120} & \frac{1119303}{966560} & \frac{37209}{966560} & \frac{3033}{386624} & -\frac{2277}{1933120} \\ \frac{143264}{90615} & \frac{598768}{271845} & \frac{84928}{90615} & \frac{1856}{18123} & -\frac{1312}{271845} \\ \frac{6761375}{3479616} & \frac{5997875}{1739808} & \frac{3074125}{1739808} & \frac{3822625}{3479616} & \frac{214775}{3479616} \end{bmatrix}$$

3.4.6.2 *Development of the Block Corrector Method for Case Two*

Interpolating (3.3) at $x_{n+r}, r = 0(1)3$, and collocating (3.5) at $x_{n+s}, s = 0(1)5$, (3.6) reduced to

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & y_{n+3} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t) = \sum_{j=0}^{3} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{5} \beta_j(t) f_{n+j} \tag{3.37}$$

where the values of the $\alpha_i's$ and $\beta_j's$ are computed for $i = 0(1)3$ and $j = 0(1)5$

Evaluating (3.37) at $t = 4$ and 5 gives the following equations

$$y_{n+4} = -y_n - \frac{128}{31} y_{n+1} + \frac{318}{31} y_{n+2} - \frac{128}{31} y_{n+3} + \frac{h^2}{465} \begin{pmatrix} 23 f_n + 688 f_{n+1} + 2358 f_{n+2} \\ + 688 f_{n+3} + 23 f_{n+4} \end{pmatrix} \tag{3.38}$$

$$y_{n+5} = \frac{128}{31}y_n + \frac{15423}{961}y_{n+1} - \frac{44672}{961}y_{n+2} + \frac{26242}{961}y_{n+3} - \frac{h^2}{14415}\begin{pmatrix} 2944f_n + 87351f_{n+1} + 280496f_{n+2} + \\ 14966f_{n+3} - 18384f_{n+4} - 713f_{n+5} \end{pmatrix} \quad (3.39)$$

Evaluating the first derivatives of (3.37) at $t = 0(1)2$ gives the following equations

$$hy'_n = \frac{-3793}{930}y_n + \frac{6048}{4805}y_{n+1} + \frac{83781}{9610}y_{n+2} - \frac{85024}{14415}y_{n+3} - \frac{h}{672700}\begin{pmatrix} 25519 f_n - 1912224 f_{n+1} - 3444954 f_{n+2} - \\ 391984 f_{n+3} + 22791 f_{n+4} - 1488 f_{n+5} \end{pmatrix} \quad (3.40)$$

$$hy'_{n+1} = \frac{736}{465}y_n + \frac{23611}{9610}y_{n+1} - \frac{51232}{4805}y_{n+2} + \frac{190927}{28830}y_{n+3} - \frac{h}{6054300}\begin{pmatrix} 487376 f_n + 14218759 f_{n+1} + 34197264 f_{n+2} + \\ 4025494 f_{n+3} - 240256 f_{n+4} + 15903 f_{n+5} \end{pmatrix} \quad (3.41)$$

$$hy'_{n+2} = -\frac{1393}{930}y_n - \frac{22112}{4805}y_{n+1} + \frac{122021}{9610}y_{n+2} - \frac{95104}{14415}y_{n+3} - \frac{h}{6054300}\begin{pmatrix} 464449 f_n + 12705306 f_{n+1} + 35108126 f_{n+2} + \\ 3981596 f_{n+3} - 235779 f_{n+4} + 15562_{n+5} \end{pmatrix} \quad (3.42)$$

Writing (3.38) to (3.42) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_{n-4} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_{n-2} & y'_n & y'_{n+1} & y'_{n+2} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_n \end{bmatrix}^T$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & \frac{792431}{2091360} & \frac{31306}{65355} & \frac{297137}{2091360} \\ 0 & 0 & \frac{25706}{65355} & \frac{63872}{65355} & \frac{41132}{65355} \\ 0 & 0 & \frac{343461}{697120} & \frac{33696}{21785} & \frac{669627}{697120} \\ 0 & 0 & \frac{28492}{65355} & \frac{108544}{65355} & \frac{124384}{65355} \\ 0 & 0 & \frac{523235}{418272} & \frac{58750}{13071} & -\frac{311875}{418272} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{13846067}{329389200} \\ 0 & 0 & 0 & 0 & \frac{1865753}{41173650} \\ 0 & 0 & 0 & 0 & \frac{848073}{12199600} \\ 0 & 0 & 0 & 0 & \frac{1105448}{20586825} \\ 0 & 0 & 0 & 0 & \frac{3455275}{13175568} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -\frac{329732329}{1317556800} & -\frac{38625017}{658778400} & \frac{2811953}{658778400} & -\frac{420667}{658778400} & \frac{72343}{1317556800} \\ -\frac{2956022}{20586825} & -\frac{2956487}{20586825} & \frac{138308}{20586825} & -\frac{38999}{41173650} & \frac{1634}{20586825} \\ \frac{10555029}{48798400} & \frac{15699717}{24399200} & \frac{2674947}{24399200} & -\frac{177633}{24399200} & \frac{24597}{48798400} \\ \frac{3023456}{205868825} & \frac{24355376}{20586825} & \frac{22196416}{20586825} & \frac{1491376}{20586825} & -\frac{30752}{20586825} \\ \frac{132019375}{52702272} & \frac{101369375}{26351136} & \frac{45105625}{26351136} & \frac{29258125}{26351136} & \frac{3184175}{52702272} \end{bmatrix}$$

3.4.6.3 *Development of the Block Corrector Method for Case Three*

Interpolating (3.3) at $x_{n+r}, r = 0(1)4$, and collocating (3.5) at $x_{n+s}, s = 0(1)5$, (3.6) becomes

$$x = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10}]^T$$

$$b = [y_n \quad y_{n+1} \quad y_{n+2} \quad y_{n+3} \quad y_{n+4} \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4} \quad f_{n+5}]^T$$

$$A=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 & 90x_{n+2}^8 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 & 90x_{n+3}^8 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 & 90x_{n+4}^8 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 & 90x_{n+5}^8 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(x)=\sum_{j=0}^{4}\alpha_j(t)\,y_{n+j}+h^2\sum_{j=0}^{5}\beta_j(t)\,f_{n+j} \tag{3.43}$$

where the values of the $\alpha_i's$ and $\beta_j's$ are computed for $i=0(1)4$ and $j=0(1)5$

Evaluating (3.43) at $t=5$ will give equation

$$y_{n+5}=y_n+\frac{97}{31}y_{n+1}-\frac{446}{31}y_{n+2}+\frac{446}{31}y_{n+3}-\frac{97}{31}y_{n+4}-\frac{h^2}{465}\begin{pmatrix}23f_n+665f_{n+1}+1670f_{n+2}-1670f_{n+3}\\-665f_{n+4}-23f_{n+5}\end{pmatrix} \tag{3.44}$$

Evaluating the first derivatives of (3.43) at $t=0(1)3$ gives the following equations

$$hy'_n=-\frac{2125}{948}y_n+\frac{108288}{12245}y_{n+1}-\frac{123984}{12245}y_{n+2}+\frac{61952}{36735}y_{n+3}+\frac{89973}{48980}y_{n+4}-\frac{h}{428575}\begin{pmatrix}55198f_n-53460f_{n+1}+1797420f_{n+2}+\\915080f_{n+3}+53460f_{n+4}-948f_{n+5}\end{pmatrix} \tag{3.45}$$

$$hy'_{n+1}=-\frac{225}{316}y_n-\frac{515647}{73470}y_{n+1}+\frac{315387}{24490}y_{n+2}-\frac{69867}{24490}y_{n+3}-\frac{337201}{146940}y_{n+4}+\frac{h^2}{1714300}\begin{pmatrix}56583f_n+1794545f_{n+1}+10266150f_{n+2}\\+4680810f_{n+3}+262615f_{n+4}-4503f_{n+5}\end{pmatrix} \tag{3.46}$$

$$hy'_{n+2} = \frac{347}{474}y_n + \frac{169184}{36735}y_{n+1} - \frac{249243}{24490}y_{n+2} + \frac{95872}{36735}y_{n+3} + \frac{81916}{36735}y_{n+4} - \frac{h}{15428700}\begin{pmatrix} 518143f_n + 18526150f_{n+1} + 84996130f_{n+2} + \\ 40757540f_{n+3} + 2302595f_{n+4} - 39658f_{n+5} \end{pmatrix} \tag{3.47}$$

$$hy'_{n+3} = -\frac{347}{474}y_n - \frac{110709}{24490}y_{n+1} + \frac{97011}{12245}y_{n+2} - \frac{26081}{73470}y_{n+3} - \frac{56691}{24490}y_{n+4} + \frac{h}{1714300}\begin{pmatrix} 57668f_n + 2050605f_{n+1} + 9035760f_{n+2} + \\ 4936870f_{n+3} + 263700f_{n+4} - 4503f_{n+5} \end{pmatrix} \tag{3.48}$$

Writing (3.44) to (3.48) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_{n-4} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} & y'_{n+2} & y'_{n+3} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_n \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & \frac{6841087}{19120320} & \frac{155083}{424896} & \frac{384281}{2124480} & \frac{1841969}{19120320} \\ 0 & \frac{12109}{33195} & \frac{5465}{6639} & \frac{22633}{33195} & \frac{1441}{11065} \\ 0 & \frac{261933}{708160} & \frac{125145}{141632} & \frac{840051}{708160} & \frac{396771}{708160} \\ 0 & \frac{147292}{298755} & \frac{13072}{6639} & \frac{59696}{33195} & -\frac{77776}{298755} \\ 0 & -\frac{148645}{424896} & -\frac{1763125}{424896} & \frac{933125}{424896} & \frac{1034375}{141632} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{49345421}{1338422400} \\ 0 & 0 & 0 & 0 & \frac{400784}{10456425} \\ 0 & 0 & 0 & 0 & \frac{1956279}{49571200} \\ 0 & 0 & 0 & 0 & \frac{707488}{10456425} \\ 0 & 0 & 0 & 0 & -\frac{6932075}{53536896} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -\frac{17\,385\,799}{53\,536\,896} & -\frac{2629\,933}{13\,384\,224} & -\frac{2157\,571}{66\,921\,120} & \frac{239\,231}{267\,684\,480} & -\frac{59\,681}{1338\,422\,400} \\ -\frac{1021\,711}{4182\,570} & -\frac{690\,083}{2091\,285} & -\frac{89\,164}{2091\,285} & \frac{2351}{2091\,285} & -\frac{1153}{20\,912\,850} \\ -\frac{2150\,361}{9914\,240} & -\frac{392\,499}{2478\,560} & -\frac{254\,553}{2478\,560} & \frac{16\,173}{9914\,240} & -\frac{3699}{49\,571\,200} \\ \frac{728\,072}{2091\,285} & \frac{3253\,232}{2091\,285} & \frac{492\,224}{418\,257} & \frac{28\,568}{418\,257} & -\frac{12\,808}{10\,456\,425} \\ -\frac{168\,194\,375}{53\,536\,896} & -\frac{88\,398\,125}{13\,384\,224} & -\frac{14\,134\,375}{13\,384\,224} & \frac{65\,661\,875}{53\,536\,896} & \frac{2830\,775}{53\,536\,896} \end{bmatrix}$$

All values needed for the execution of the methods with regards to second order has been calculated. We now proceed to do same for the case of third order.

3.5 **Development of Methods for Third Order ODEs**

3.5.1 ***Development of the Block Predictor Method for k = 4***

Interpolating (3.3) at $x_{n+r}, r = 0,1$ and collocating (3.5) at $x_{n+s}, s = 0(1)4$, (3.6) reduces to

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x^3 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{4} \beta_j(x) f_{n+j} \tag{3.49}$$

where

$$\alpha_0 = 1 - t$$

$$\alpha_1 = t(1 - t)$$

$$\alpha_2 = t^2$$

$$\beta_0 = \frac{1}{10\,080}\left(2t^7 - 35t^6 + 245t^5 - 875t^4 + 1680t^3 - 1017t^2\right)$$

$$\beta_1 = -\frac{1}{5040}\left(4t^7 - 63t^6 + 364t^5 - 840t^4 + 535t^2\right)$$

$$\beta_2 = \frac{1}{1680}\left(2t^7 - 28t^6 + 133t^5 - 210t^4 + 103t^2\right)$$

$$\beta_3 = -\frac{1}{5040}\left(4t^7 - 49t^6 + 196t^5 - 280t^4 + 129t^2\right)$$

$$\beta_4 = \frac{1}{10\,080}\left(2t^7 - 21t^6 + 77t^5 - 105t^4 + 47t^2\right)$$

Solving for the independent solution in (3.49), makes (3.8) reduced to

$$y_{n+j} = \sum_{i=0}^{2} \frac{(jh)^i}{i!} y_n^{(i)} + h^3 \sum_{j=0}^{4} \sigma_j(x) f_{n+j} \tag{3.50}$$

where

$$\sigma_0 = \frac{1}{10\,080}\left(2t^7 - 35t^6 + 245t^5 - 875t^4 + 1680t^3\right)$$

$$\sigma_1 = -\frac{1}{5040}\left(4t^7 - 63t^6 + 364t^5 - 840t^4\right)$$

$$\sigma_2 = \frac{1}{1680}\left(2t^7 - 28t^6 + 133t^5 - 210t^4\right)$$

$$\sigma_3 = -\frac{1}{5040}\left(4t^7 - 49t^6 + 196t^5 - 280t^4\right)$$

$$\sigma_4 = \frac{1}{10\,080}\left(2t^7 - 21t^6 + 77t^5 - 105t^4\right)$$

Evaluating (3.50) at the selected grid points, the parameters in (3.9) becomes:

$A^{(0)} = 4 \times 4$ identity matrix

$$Y_m^{(i)} = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_m^{(i)} = \begin{bmatrix} y'_{n+1} & y'_{n+2} & y'_{n+3} & y'_{n+4} \end{bmatrix}^T$$

When $i = 0$ we have that

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}; e_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{113}{1120} \\ 0 & 0 & 0 & \frac{331}{630} \\ 0 & 0 & 0 & \frac{1431}{120} \\ 0 & 0 & 0 & \frac{248}{105} \end{bmatrix}; b_0 = \begin{bmatrix} \frac{107}{1008} & -\frac{103}{1680} & \frac{43}{1680} & -\frac{47}{10080} \\ \frac{332}{315} & -\frac{8}{21} & \frac{52}{315} & -\frac{19}{630} \\ \frac{1863}{560} & -\frac{243}{560} & \frac{45}{112} & -\frac{81}{1120} \\ \frac{2176}{315} & \frac{32}{105} & \frac{128}{105} & -\frac{8}{65} \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}; d_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{367}{1440} \\ 0 & 0 & 0 & \frac{53}{90} \\ 0 & 0 & 0 & \frac{147}{160} \\ 0 & 0 & 0 & \frac{56}{45} \end{bmatrix};$$

$$b_1 = \begin{bmatrix} \frac{3}{8} & -\frac{47}{240} & \frac{29}{360} & -\frac{7}{480} \\ \frac{8}{5} & -\frac{1}{3} & \frac{8}{45} & -\frac{1}{30} \\ \frac{1117}{40} & \frac{27}{80} & \frac{3}{8} & -\frac{9}{160} \\ \frac{64}{15} & \frac{16}{15} & \frac{64}{45} & 0 \end{bmatrix}$$

When $i = 1$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; d_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{720} \\ 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}; b_2 = \begin{bmatrix} \frac{323}{360} & -\frac{11}{30} & \frac{53}{360} & -\frac{19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & -\frac{3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}$$

3.5.2 ***Development of the Block Corrector Method for k = 4***

Here two cases (One and Two) will be considered.

3.5.2.1 *Development of the Block Corrector Method for Case One*

Interpolating (3.3) at $x_{n+r}, r = 0(1)2$, and collocating (3.5) at $x_{n+s}, s = 0(1)4$, (3.6) reduced to

$$x = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7]^T$$

$$b = [y_n \quad y_{n+1} \quad y_{n+2} \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(x) = \sum_{j=0}^{2} \alpha_j(t) y_{n+j} + h^3 \sum_{j=0}^{4} \beta_j(t) f_{n+j} \tag{3.51}$$

where

$$\alpha_0 = \frac{1}{42}(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 149t + 42)$$

$$\alpha_1 = -\frac{1}{21}(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 128t)$$

$$\alpha_2 = \frac{1}{42}(2t^7 - 28t^6 + 147t^5 - 350t^4 + 336t^3 - 107t)$$

$$\beta_0 = -\frac{1}{10080}(38t^7 - 546t^6 + 3003t^5 - 7875t^4 + 9884t^3 - 5040t^2 + 536t)$$

$$\beta_1 = -\frac{1}{1260}(51t^7 - 707t^6 + 3654t^5 - 8470t^4 + 7728t^3 - 2256t)$$

$$\beta_2 = -\frac{1}{720}(2t^7 - 34t^6 + 219t^5 - 635t^4 + 696t^3 - 248t)$$

$$\alpha_3 = -\frac{1}{1260}\left(t^7 - 7t^6 + 70t^4 - 112t^3 + 48t\right)$$

$$\alpha_4 = \frac{1}{10080}\left(2t^7 - 14t^6 + 21t^5 + 35t^4 - 84t^3 + 40t\right)$$

Evaluating (3.51) at $t = 3,4$ gives the following equations

$$y_{n+3} = 2y_{n+2} - y_{n+1} + h^3\left(-\frac{1}{240}f_n + \frac{1}{10}f_{n+1} + \frac{97}{120}f_{n+2} + \frac{1}{10}f_{n+3} - \frac{1}{240}f_{n+4}\right) \qquad (3.52)$$

$$y_{n+4} = -y_n + 2y_{n+2} + h^3\left(\frac{1}{15}f_n + \frac{16}{15}f_{n+1} + \frac{26}{15}f_{n+2} + \frac{16}{15}f_{n+3} + \frac{1}{15}f_{n+4}\right) \qquad (3.53)$$

Evaluating the first derivatives of (3.51) at $t = 0$ and 1 give the following equations

$$hy_n' = \frac{-149}{42}y_n + \frac{128}{21}y_{n+1} - \frac{107}{42}y_{n+2} + h^2\left(-\frac{67}{1260}f_n + \frac{188}{105}f_{n+1} + \frac{31}{90}f_{n+2} - \frac{4}{105}f_{n+3} + \frac{1}{252}f_{n+4}\right) \qquad (3.54)$$

$$hy_{n+1}' = \frac{20}{21}y_n - \frac{61}{21}y_{n+1} + \frac{41}{21}y_{n+2} + h^2\left(-\frac{613}{10080}f_n - \frac{1433}{1260}f_{n+1} - \frac{41}{144}f_{n+2} + \frac{43}{1260}f_{n+3} - \frac{37}{10080}f_{n+4}\right) \qquad (3.55)$$

Writing (3.52) to (3.55) in block form, the parameters in (3.10) gives the following results

$A^{(0)} = 4 \times 4$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_{n-2} & y'_n & y'_{n+1} \end{bmatrix}^T$$

$$F(Y_m) = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$F(y_n) = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad A^{(1)} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & \frac{9}{2} \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{113}{1120} \\ 0 & 0 & 0 & \frac{331}{630} \\ 0 & 0 & 0 & \frac{1431}{1120} \\ 0 & 0 & 0 & \frac{248}{105} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} \frac{107}{1008} & -\frac{103}{1680} & \frac{43}{1680} & -\frac{47}{10080} \\ \frac{332}{315} & -\frac{8}{21} & \frac{52}{315} & -\frac{19}{630} \\ \frac{1863}{560} & -\frac{243}{560} & \frac{45}{112} & -\frac{81}{1120} \\ \frac{2176}{315} & \frac{32}{105} & \frac{128}{105} & -\frac{8}{63} \end{bmatrix}$$

3.5.2.2 *Development of the Block Corrector Method for Case Two*

Interpolating (3.3) at $x_{n+r}, r = 0(1)3$, and collocating (3.5) at $x_{n+s}, s = 0(1)4$, the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & y_{n+3} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t) = \sum_{j=0}^{3} \alpha_j(t) y_{n+j} + h^3 \sum_{j=0}^{4} \beta_j(t) f_{n+j} \qquad (3.56)$$

where the values of the $\alpha_i's$ and $\beta_j's$ are computed for $i = 0(1)3$ and $j = 0(1)4$

Evaluating (3.56) at $t = 4$ will give the equation

$$y_{n+4} = y_n - 2y_{n+1} + 0y_{n+2} + 2y_{n+3} + \frac{h^3}{120}\left(f_n + 56f_{n+1} + 126f_{n+2} + 56f_{n+3} + 2f_{n+4}\right) \qquad (3.57)$$

Evaluating the first derivatives of (3.56) at $t = 0,1$ and second derivative at $t = 0$ gives the following equations

$$hy'_n = \frac{11}{21}y_n - \frac{57}{14}y_{n+1} + \frac{39}{7}y_{n+2} - \frac{85}{42}y_{n+3} + \frac{h^2}{3360}\left(233f_n + 4344f_{n+1} + 3342f_{n+2} - 8f_{n+3} + 9f_{n+4}\right) \qquad (3.58)$$

$$hy'_{n+1} = -\frac{31}{42}y_n + \frac{5}{7}y_{n+1} - \frac{3}{14}y_{n+2} + \frac{5}{21}y_{n+3} - \frac{h^2}{10\,080}\left(89f_n + 2692f_{n+1} + 1314f_{n+2} - 20f_{n+3} + 5f_{n+4}\right) \qquad (3.59)$$

$$h^2y''_n = -\frac{89}{14}y_n + \frac{281}{14}y_{n+1} - \frac{295}{14}y_{n+2} + \frac{103}{14}y_{n+3} - \frac{h}{10\,080}\left(3571f_n + 44\,328f_{n+1} + 36\,330f_{n+2} - 88f_{n+3} + 99f_{n+4}\right) \qquad (3.60)$$

Writing (3.57) to (3.60) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 4 \times 4$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} & y'_{n+2} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A^{(1)} = \begin{bmatrix} 0 & \frac{289}{428} & \frac{75}{428} & \frac{139}{428} \\ 0 & -\frac{10}{107} & -\frac{10}{107} & \frac{224}{107} \\ 0 & -\frac{903}{428} & -\frac{261}{428} & \frac{2187}{428} \\ 0 & -\frac{596}{107} & -\frac{168}{107} & \frac{1024}{107} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{15\,637}{862\,848} \\ 0 & 0 & 0 & -\frac{61}{7490} \\ 0 & 0 & 0 & -\frac{11\,799}{479\,360} \\ 0 & 0 & 0 & -\frac{520}{6741} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -\frac{3373}{215\,712} & \frac{549}{239\,680} & -\frac{611}{1078\,560} & \frac{317}{4314\,240} \\ \frac{9064}{33\,705} & \frac{326}{11\,235} & -\frac{8}{2247} & \frac{5}{13\,482} \\ \frac{169\,047}{119\,840} & \frac{135\,837}{239\,680} & -\frac{1179}{119\,840} & \frac{1053}{479\,360} \\ \frac{111\,872}{33\,705} & \frac{1632}{749} & \frac{15\,104}{33\,705} & \frac{424}{33\,705} \end{bmatrix}$$

We have successfully calculated all the parameters needed for the two cases in $k =$ 3 and 4 for both block predictor and block corrector. We now determine the parameters for k = 5.

3.5.3 ***Development of the Block Predictor Method for k = 5***

Interpolating (3.3) at $x_{n+r}, r = 0(1)2$, and collocating (3.5) at $x_{n+s}, s = 0(1)5$, (3.6) reduces to

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 \end{bmatrix}$$

Solving the above system for the unknown constants $a_j's$ using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(x) = \sum_{j=0}^{2} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{5} \beta_j(x) f_{n+j} \qquad (3.61)$$

where the values of the $\alpha_i^{'}s$ and $\beta_j^{'}s$ are computed for $i = 0(1)2$ and $j = 0(1)5$.

Solving for the independent solution in (3.61), makes (3.8) reduced to

$$y_{n+j} = \sum_{i=0}^{2} \frac{(jh)^i}{i!} y_n^{(i)} + h^3 \sum_{j=0}^{5} \sigma_j(x) f_{n+j} \qquad (3.62)$$

where the values of the $\sigma_j^{'}s$ are computed for $j = 0(1)5$

Evaluating (3.62) at the selected grid points, the parameters in (3.9) gives the following results:

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m^{(0)} = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_m^{(1)} = \begin{bmatrix} y'_{n+1} & y'_{n+2} & y'_{n+3} & y'_{n+4} & y'_{n+5} \end{bmatrix}^T$$

When $i = 0$

$$e_0 = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&2\\ 0&0&0&0&3\\ 0&0&0&0&4\\ 0&0&0&0&5 \end{bmatrix}; e_2 = \begin{bmatrix} 0&0&0&0&\frac{1}{2}\\ 0&0&0&0&2\\ 0&0&0&0&\frac{9}{2}\\ 0&0&0&0&8\\ 0&0&0&0&\frac{25}{2} \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0&0&0&0&\frac{1231}{5040}\\ 0&0&0&0&\frac{71}{126}\\ 0&0&0&0&\frac{123}{140}\\ 0&0&0&0&\frac{376}{315}\\ 0&0&0&0&\frac{1525}{1008} \end{bmatrix}, \quad b_0 = \begin{bmatrix} \frac{863}{2016}&-\frac{761}{2520}&\frac{941}{5040}&-\frac{341}{5040}&\frac{107}{10080}\\ \frac{544}{315}&-\frac{37}{63}&\frac{136}{315}&-\frac{101}{630}&\frac{8}{315}\\ \frac{3501}{1120}&-\frac{9}{140}&\frac{87}{112}&-\frac{9}{35}&\frac{9}{224}\\ \frac{1424}{315}&\frac{176}{315}&\frac{608}{315}&-\frac{16}{63}&\frac{16}{315}\\ \frac{11875}{2016}&\frac{625}{504}&\frac{3125}{1008}&\frac{625}{1008}&\frac{275}{2016} \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1 \end{bmatrix}; \quad e_2 = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&2\\ 0&0&0&0&3\\ 0&0&0&0&4\\ 0&0&0&0&5 \end{bmatrix}; d_1 = \begin{bmatrix} 0&0&0&0&\frac{95}{288}\\ 0&0&0&0&\frac{14}{45}\\ 0&0&0&0&\frac{51}{160}\\ 0&0&0&0&\frac{14}{45}\\ 0&0&0&0&\frac{95}{288} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{1427}{1440}&-\frac{133}{240}&\frac{241}{720}&-\frac{173}{1440}&\frac{3}{160}\\ \frac{43}{30}&\frac{7}{45}&\frac{7}{45}&-\frac{1}{15}&\frac{1}{90}\\ \frac{219}{160}&\frac{57}{80}&\frac{57}{80}&-\frac{21}{160}&\frac{3}{160}\\ \frac{64}{45}&\frac{8}{15}&\frac{64}{45}&\frac{14}{45}&0\\ \frac{125}{96}&\frac{125}{144}&\frac{125}{144}&\frac{125}{96}&\frac{95}{288} \end{bmatrix}$$

When $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{95}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{51}{160} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{95}{288} \end{bmatrix}; b_2 = \begin{bmatrix} \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} & \frac{3}{160} \\ \frac{43}{30} & \frac{7}{45} & \frac{7}{45} & -\frac{1}{15} & \frac{1}{90} \\ \frac{219}{160} & \frac{57}{80} & \frac{57}{80} & -\frac{21}{160} & \frac{3}{160} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{125}{96} & \frac{125}{144} & \frac{125}{144} & \frac{125}{96} & \frac{95}{288} \end{bmatrix}$$

3.5.4 ***Development of the Block Corrector Method for k = 5***

3.5.4.1 *Development of the Block Corrector Method for Case One*

Interpolating (3.3) at $x_{n+r}, r = 0(1)3$, and collocating (3.5) at $x_{n+s}, s = 0(1)5$, the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & y_{n+3} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x^3 & 210x_n^4 & 336x_n^5 & 504x^6 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t) = \sum_{j=0}^{3} \alpha_j(t) y_{n+j} + h^3 \sum_{j=0}^{5} \beta_j(t) f_{n+j} \tag{3.63}$$

where the values of the $\alpha_i^{'}s$ and $\beta_j^{'}s$ are computed for $i = 0(1)3$ and $j = 0(1)5$

Evaluating (3.63) at $t = 4,5$ gives the equations

$$y_{n+5} = \frac{61}{32}y_n - \frac{87}{32}y_{n+1} - \frac{73}{32}y_{n+2} + \frac{131}{32}y_{n+3} + \frac{h^3}{15\,360}\begin{pmatrix} 247f_n + 13\,783f_{n+1} + 38\,638f_{n+2} + \\ 30\,518f_{n+3} + 7403f_{n+4} + 131f_{n+5} \end{pmatrix} \tag{3.64}$$

and

$$y_{n+4} = \frac{31}{32}y_n - \frac{61}{32}y_{n+1} - \frac{3}{32}y_{n+2} + \frac{65}{32}y_{n+3} + \frac{h^3}{15\,360}\begin{pmatrix} 125f_n + 6941f_{n+1} + 15\,866f_{n+2} + \\ 7186f_{n+3} + 121f_{n+4} + f_{n+5} \end{pmatrix} \tag{3.65}$$

Evaluating the first derivative of (3.63) at $t = 0{,}1$ and second derivative at $t = 0$ gives the following equations

$$hy_n' = \frac{3023}{1920}y_n - \frac{4623}{640}y_{n+1} + \frac{5583}{640}y_{n+2} - \frac{5903}{1920}y_{n+3} + \frac{h^2}{2150\,400}\begin{pmatrix} 163\,241f_n + 3848\,649f_{n+1} + 3372\,114f_{n+2} - \\ 89\,846f_{n+3} + 38\,709f_{n+4} - 4707f_{n+5} \end{pmatrix} \tag{3.66}$$

$$hy_{n+1}' = -\frac{1747}{1920}y_n + \frac{787}{640}y_{n+1} - \frac{467}{640}y_{n+2} + \frac{787}{1920}y_{n+3} - \frac{h^2}{6451\,200}\begin{pmatrix} 63\,887f_n + 2247\,023f_{n+1} + 1445\,918f_{n+2} - \\ 54\,362f_{n+3} + 19\,363f_{n+4} - 2309f_{n+5} \end{pmatrix} \tag{3.67}$$

$$h^2y_n'' = -\frac{1313}{128}y_n + \frac{4067}{128}y_{n+1} - \frac{4195}{128}y_{n+2} + \frac{1441}{128}y_{n+3} - \frac{h}{1290\,240}\begin{pmatrix} 488\,543f_n + 8054\,079f_{n+1} + 7397\,310f_{n+2} - \\ 199\,994f_{n+3} + 86\,067f_{n+4} - 10\,485f_{n+5} \end{pmatrix} \tag{3.68}$$

Writing (3.64) to (3.68) in block form, the parameters in (3.10) gives the following results:

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_{n-4} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} & y'_{n+2} & y'_{n+3} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_{n} \end{bmatrix}^T$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; A^{(1)} = \begin{bmatrix} 0 & 0 & \frac{20041}{29850} & \frac{2558}{14925} & \frac{9809}{29850} \\ 0 & 0 & -\frac{1574}{14925} & -\frac{1574}{14925} & \frac{31424}{14925} \\ 0 & 0 & -\frac{21909}{9950} & -\frac{3492}{4975} & \frac{51759}{9950} \\ 0 & 0 & -\frac{85708}{14925} & -\frac{26008}{14925} & \frac{145408}{14925} \\ 0 & 0 & -\frac{12655}{1194} & -\frac{1850}{597} & \frac{18625}{1194} \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{10340809}{601776000} \\ 0 & 0 & 0 & 0 & -\frac{104143}{9402750} \\ 0 & 0 & 0 & 0 & -\frac{1049949}{22288000} \\ 0 & 0 & 0 & 0 & -\frac{80104}{671625} \\ 0 & 0 & 0 & 0 & -\frac{136325}{687744} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -\frac{297137}{17193600} & \frac{1038421}{300888000} & -\frac{386363}{300888000} & \frac{200701}{601776000} & -\frac{24551}{601776000} \\ \frac{248029}{940275} & \frac{153458}{4701375} & -\frac{27274}{4701375} & \frac{1589}{1343250} & -\frac{599}{4701375} \\ \frac{6112179}{4457600} & \frac{6625719}{11144000} & -\frac{301257}{11144000} & \frac{187839}{22288000} & -\frac{21789}{22288000} \\ \frac{3051488}{940275} & \frac{10488736}{4701375} & \frac{1955392}{4701375} & \frac{114008}{4701375} & -\frac{8608}{4701375} \\ \frac{28672625}{4814208} & \frac{11712025}{2407104} & \frac{4556425}{2407104} & \frac{2469025}{4814208} & \frac{23725}{4814208} \end{bmatrix}$$

3.5.4.2 *Development of the Block Corrector Method for Case Two*

Interpolating (3.3) at $x_{n+r}, r = 0(1)4$ and collocating (3.5) at $x_{n+s}, s = 0(1)5$, the parameters in (3.6) becomes

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix}^T$$

$$b = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & f_n & f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & . & . & . & x_n^9 & x_n^{10} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & . & . & . & x_{n+1}^9 & x_{n+1}^{10} \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & . & . & . & x_{n+2}^9 & x_{n+2}^{10} \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & . & . & . & x_{n+3}^9 & x_{n+3}^{10} \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & . & . & . & x_{n+4}^9 & x_{n+4}^{10} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & . & . & . & 504x^6 & 720x^7 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & . & . & . & 504x_{n+1}^6 & 720x_{n+1}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & . & . & . & 504x_{n+2}^6 & 720x_{n+2}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & . & . & . & 504x_{n+3}^6 & 720x_{n+3}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & . & . & . & 504x_{n+4}^6 & 720x_{n+4}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & . & . & . & 504x_{n+5}^{67} & 720x_{n+5}^7 \end{bmatrix}$$

Solving the above system for the unknown constants a_j's using Guassian elimination method and substituting into (3.3), makes (3.7) reduced to

$$y(t) = \sum_{j=0}^{4} \alpha_j(t) y_{n+j} + h^3 \sum_{j=0}^{5} \beta_j(t) f_{n+j} \tag{3.69}$$

where the values of the α_i's and β_j's are computed for $i = 0(1)4$ and $j = 0(1)5$.

Evaluating (3.69) at $t = 5$ gives the equation

$$y_{n+5} = y_n - \frac{29}{31} y_{n+1} - \frac{68}{31} y_{n+2} + \frac{68}{31} y_{n+3} + \frac{29}{31} y_{n+4} + \frac{h^3}{2480}\left(21 f_n + 1177 f_{n+1} + 3842 f_{n+2} + 3842 f_{n+3} + 1177 f_{n+4} + 21 f_{n+5}\right) \tag{3.70}$$

Evaluating the first and second derivatives of (3.69) at $t = 0,1$ gives the following equations

$$hy_n' = \frac{695}{12} y_n - \frac{18304}{155} y_{n+1} + \frac{507}{155} y_{n+2} + \frac{53504}{465} y_{n+3} - \frac{36059}{620} y_{n+4} + \frac{h^2}{32550}\begin{pmatrix} 17877 f_n + 913724 f_{n+1} + 2006504 f_{n+2} + \\ 884304 f_{n+3} + 15499 f_{n+4} + 52 f_{n+5} \end{pmatrix} \tag{3.71}$$

$$hy_{n+1}' = -\frac{197}{20} y_n + \frac{8752}{465} y_{n+1} + \frac{21}{155} y_{n+2} - \frac{2842}{155} y_{n+3} + \frac{3433}{372} y_{n+4} - \frac{h^2}{1041600}\begin{pmatrix} 88541 f_n + 4706529 f_{n+1} + 10162514 f_{n+2} + \\ 4488274 f_{n+3} + 78849 f_{n+4} + 253 f_{n+5} \end{pmatrix} \tag{3.72}$$

$$h^2 y_n'' = -\frac{2653}{12} y_n + \frac{41536}{93} y_{n+1} - \frac{767}{62} y_{n+2} - \frac{40064}{93} y_{n+3} + \frac{80957}{372} y_{n+4} - \frac{h}{39060}\begin{pmatrix} 83967 f_n + 4085092 f_{n+1} + 9004456 f_{n+2} + \\ 3970800 f_{n+3} + 69569 f_{n+4} + 236 f_{n+5} \end{pmatrix} \quad (3.73)$$

$$h^2 y_{n+1}'' = \frac{419}{12} y_n - \frac{13213}{186} y_{n+1} + \frac{143}{31} y_{n+2} + \frac{11945}{186} y_{n+3} - \frac{12169}{372} y_{n+4} + \frac{h}{624960}\begin{pmatrix} 188555 f_n + 9941847 f_{n+1} + 21787838 f_{n+2} + \\ 9554494 f_{n+3} + 167175 f_{n+4} + 571 f_{n+5} \end{pmatrix} \quad (3.74)$$

Writing (3.70) to (3.74) in block form, the parameters in (3.10) gives the following results

$A^{(0)} = 5 \times 5$ identity matrix

$$Y_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5} \end{bmatrix}^T$$

$$Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_{n-4} & y_n \end{bmatrix}^T$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_n & y'_{n+1} & y'_{n+2} & y'_{n+3} \end{bmatrix}^T$$

$$f_m = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T$$

$$f_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_n \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$A^{(2)} = \begin{bmatrix} 0 & \frac{270197}{594274} & \frac{354796}{4457055} & \frac{324077}{594274} & -\frac{556846}{4457055} \\ 0 & -\frac{162398}{297137} & -\frac{1302658}{4457055} & \frac{756672}{297137} & -\frac{1133312}{4457055} \\ 0 & -\frac{3647499}{594274} & -\frac{3519324}{1485685} & \frac{5430321}{594274} & -\frac{3370896}{1485685} \\ 0 & -\frac{3896636}{297137} & -\frac{21681416}{4457055} & \frac{5085184}{297137} & -\frac{18939904}{4457055} \\ 0 & -\frac{14831755}{594274} & -\frac{8183300}{891411} & \frac{17803125}{594274} & -\frac{7378750}{891411} \end{bmatrix}$$

$$B^{(0)}=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{39\,259\,747}{7188\,338\,304} \\ 0 & 0 & 0 & 0 & -\frac{3923\,629}{112\,317\,786} \\ 0 & 0 & 0 & 0 & -\frac{346\,094\,019}{1331\,173\,760} \\ 0 & 0 & 0 & 0 & -\frac{29\,088\,904}{56\,158\,893} \\ 0 & 0 & 0 & 0 & -\frac{7007\,732\,125}{7188\,338\,304} \end{bmatrix};$$

$$B^{(1)}=\begin{bmatrix} \frac{2471\,419\,751}{179\,708\,457\,600} & -\frac{10\,100\,299}{29\,951\,409\,600} & \frac{6396\,703}{89\,854\,228\,800} & -\frac{2326\,169}{179\,708\,457\,600} & \frac{1159}{950\,838\,400} \\ \frac{459\,022\,259}{1403\,972\,325} & \frac{3889\,126}{155\,996\,925} & -\frac{4272\,286}{1403\,972\,325} & \frac{1341\,713}{2807\,944\,650} & -\frac{2801}{66\,855\,825} \\ \frac{12\,877\,664\,229}{6655\,868\,800} & \frac{249\,953\,931}{475\,419\,200} & -\frac{8055\,603}{3327\,934\,400} & \frac{14\,216\,229}{6655\,868\,800} & -\frac{1428\,111}{6655\,868\,800} \\ \frac{6038\,310\,688}{1403\,972\,325} & \frac{983\,781\,536}{467\,990\,775} & \frac{648\,657\,088}{1403\,972\,325} & \frac{17\,501\,768}{1403\,972\,325} & -\frac{20\,896}{51\,998\,975} \\ \frac{57\,592\,981\,375}{7188\,338\,304} & \frac{1842\,853\,375}{399\,352\,128} & \frac{7126\,160\,375}{3594\,169\,152} & \frac{3521\,621\,375}{7188\,338\,304} & \frac{18\,478\,625}{2396\,112\,768} \end{bmatrix}$$

CHAPTER FOUR

ANALYSIS AND RESULTS

4.0 Introduction

In this chapter section, we analyze the basic properties of the method viz; the order, consistency, zero stability and convergence of the method.

The order of a method is determined when the linear operator and associated linear multi-step method are expanded in Taylor series.

Also in this chapter we tested our methods and obtained results which we here present in a tabular form.

4.1 **Order of the Method**

Let the linear operator $L\{y(x):h\}$ associated with the block method be defined as

$$L\{y(x):h\} = A^{(0)}Y_m - A^{(1)}Y_{m-1} - A^{(2)}Y_{m-2} - h^n\left[B^{(0)}F_{m-1} + B^{(1)}F_m\right] \tag{4.1}$$

where $n = 2,3$

Expanding (4.1) in Taylor's series gives

$$L\{y(x):h\} = C_0y(x) + C_1hy'(x) + \ldots + C_ph^py^{p}(x) + C_{p+1}h^{p+1}y^{p+1}(x) + \ldots$$

Definition 4.0 Order (Lambert, 1973)

The linear operator and associated method are said to be of order p if $C_0 = C_1 = \ldots = C_{p+1} = 0$ and $C_{p+2} \neq 0, C_{p+2}$ is called the error constant and implies that the local truncation error is given by

$$t_{n+k} = C_{p+2}h^{p+2}y^{p+2} + 0\left(h^{p+3}\right)$$

where

$0\left(h^{p+3}\right)$ implies the existence of finite constant c and $h_0 > 0$ such that $t_{n+k} \leq ch^{p+k}, \forall h \leq h_0$

4.2 **Analysis of Order for Predictor Methods**

4.2.1 ***Analysis of Predictor Methods for Second Order***

4.2.1.1 *Order of the Block Predictor Method for k = 3*

When $i = 0$ expanding the method (3.12) in Taylor series gives

$$\begin{bmatrix} \sum_{j} \frac{(1h)^j}{j!} y_n^{(j)} - y_n - h y_n' - \frac{97}{360} h^2 y_n'' - \\ \sum_{j} \frac{h^{j-2}}{j!} y_n^{(j-2)} \left[\frac{114}{360}(1) - \frac{39}{360}(2) + \frac{8}{360}(3) \right] \\ \sum_{j} \frac{(2h)^j}{j!} y_n^{(j)} - y_n - 2h y_n' - \frac{28}{45} h^2 y_n'' - \\ \sum_{j} \frac{h^{j-2}}{j!} y_n^{(j-2)} \left[\frac{66}{45}(1) - \frac{6}{45}(2) + \frac{2}{45}(3) \right] \\ \sum_{j} \frac{(3h)^j}{j!} y_n^{(j)} - y_n - 3h y_n' - \frac{39}{40} h^2 y_n'' - \\ \sum_{j} \frac{h^{j-2}}{j!} y_n^{(j-2)} \left[\frac{108}{40}(1) + \frac{27}{40}(2) + \frac{6}{40}(3) \right] \end{bmatrix} = 0$$

$y_{n+1} =$

$$\begin{bmatrix} 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^4}{4!} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{97}{360} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{19}{60} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \end{bmatrix} + \frac{13}{120} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \end{bmatrix} - \frac{1}{45} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{30} \end{bmatrix}$$

$y_{n+2} =$

$$\begin{bmatrix} 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^4}{4!} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{28}{45} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{22}{15} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \end{bmatrix} - \frac{2}{45} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{5} \end{bmatrix}$$

$y_{n+3} =$

$$\begin{bmatrix} 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \\ \frac{(3)^5}{5!} \\ \frac{(3)^6}{6!} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{39}{40} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{27}{10} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \end{bmatrix} - \frac{27}{40} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \end{bmatrix} - \frac{3}{20} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{9}{160} \end{bmatrix}$$

Hence the order of the method is five and the error constant is

$$\begin{bmatrix} \frac{1}{30} & \frac{2}{5} & -\frac{9}{160} \end{bmatrix}^T$$

When $i = 1$, we have that

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(1)^j}{j!} y_n^{(j)} - h y_n' - \frac{9}{24} h^2 y_n'' - \\ \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{19}{24}(1)^j - \frac{5}{24}(2)^j + \frac{1}{24}(3)^j \right] \\ \sum_{j=0}^{\infty} \frac{(2h)^j}{j!} y_n^{(j)} - h y_n' - \frac{1}{3} h^2 y_n'' - \\ \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{4}{3}(1)^j + \frac{1}{3}(2)^j \right] \\ \sum_{j=0}^{\infty} \frac{(3h)^j}{j!} y_n^{(j)} - h y_n' - \frac{3}{8} h^2 y_n'' - \\ \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{9}{8}(1)^j + \frac{9}{8}(2)^j + \frac{3}{8}(3)^j \right] \end{bmatrix} = 0$$

$y'_{n+1} =$

$$\begin{bmatrix} 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{8} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{19}{24} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \end{bmatrix} + \frac{5}{24} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \end{bmatrix} - \frac{1}{24} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{19}{720} \end{bmatrix}$$

$y'_{n+2} =$

$$\begin{bmatrix} 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^5}{5!} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{90} \end{bmatrix}$$

$y'_{n+3} =$

$$\begin{bmatrix} 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \\ \frac{(3)^5}{5!} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{9}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \end{bmatrix} - \frac{9}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \end{bmatrix} - \frac{3}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{3}{80} \end{bmatrix}$$

Hence the order of the method is five and the error constant is

$$\begin{bmatrix} -\frac{19}{480} & -\frac{1}{90} & -\frac{3}{80} \end{bmatrix}^T$$

4.2.1.2 *Order of the Block Predictor Method for k = 4*

When $i = 0$ the order of the method is six and the error constant is

$$\begin{bmatrix} \frac{107}{10080} & \frac{8}{315} & \frac{9}{224} & \frac{16}{315} \end{bmatrix}^T$$

Similarly when $i = 1$ expanding in Taylor series gives the order of the method as six and the error constant is

$$\begin{bmatrix} \frac{3}{160} & \frac{1}{90} & \frac{3}{160} & -\frac{8}{945} \end{bmatrix}^T$$

4.2.1.3 *Order of the Block Predictor Method for k = 5*

Adopting the method used above we have that when $i = 0$ the order of the method is seven and the error constant is

$$\begin{bmatrix} -\frac{199}{24192} & -\frac{19}{945} & -\frac{141}{4480} & -\frac{8}{189} & -\frac{1375}{24192} \end{bmatrix}^T$$

Also when $i = 1$ the order of the method is seven and the error constant is

$$\begin{bmatrix} -\frac{863}{60480} & -\frac{37}{3780} & -\frac{29}{2240} & -\frac{8}{945} & -\frac{275}{12096} \end{bmatrix}^T$$

4.2.2 ***Analysis of Predictor Methods for Third Order***

4.2.2.1 *Order of the Block Predictor Method for K = 4*

Adopting the method used in section 4.2.1.1 we have that the order is eight and error constants are:

when $i = 0$ we have

$$\begin{bmatrix} \frac{1}{30} & \frac{107}{10080} & \frac{3}{160} & \frac{2}{5} \end{bmatrix}^T$$

when $i = 1$ we have

$$\begin{bmatrix} \frac{8}{315} & \frac{1}{90} & \frac{27}{20} & \frac{9}{224} \end{bmatrix}^T$$

when $i = 2$ we have

$$\begin{bmatrix} \frac{3}{160} & \frac{32}{15} & \frac{16}{315} & -\frac{8}{945} \end{bmatrix}^T$$

4.2.2.2 *Order of the Block Predictor Method for K = 5*

Adopting the method used in section 4.2.1.1 we have that when $i = 0,1,2$ the order is nine and error constants are:

when $i = 0$ we have

$$\begin{bmatrix} \frac{1}{30} & -\frac{199}{24192} & -\frac{863}{60480} & \frac{2}{5} & -\frac{19}{945} \end{bmatrix}^T$$

when $i = 1$ we have

$$\begin{bmatrix} -\frac{37}{3780} & \frac{27}{20} & -\frac{141}{4480} & -\frac{29}{2240} & \frac{32}{15} \end{bmatrix}^T$$

when $i = 2$ we have

$$\begin{bmatrix} -\frac{8}{189} & -\frac{8}{945} & \frac{625}{144} & -\frac{1375}{24192} & -\frac{275}{12096} \end{bmatrix}^T$$

4.3 **Analysis of Order for Corrector Methods**

4.3.1 ***Analysis of Corrector Methods for Second Order***

4.3.1.1 *Order of the Block Corrector Method for k = 3*

Expanding the block of equations (3.14) to (3.16) in Taylor series gives

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^{(j)} - y_n - \frac{17}{38}hy_n' - \frac{21}{38}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+1)} - \frac{851}{13680}h^2y_n'' - \\ \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[-\frac{1653}{13680}(1)^j + \frac{93}{13680}(2)^j - \frac{11}{13680}(3)^j\right] \\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^{(j)} - y_n - \frac{14}{19}hy_n' - \frac{24}{19}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+1)} - \frac{127}{855}h^2y_n'' - \\ \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{399}{855}(1)^j + \frac{111}{855}(2)^j - \frac{7}{855}(3)^j\right] \\ \sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^{(j)} - y_n - \frac{33}{38}hy_n' - \frac{81}{38}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+1)} - \frac{267}{1520}h^2y_n'' - \\ \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{1539}{1520}(1)^j + \frac{1701}{1520}(2)^j + \frac{93}{1520}(3)^j\right] \end{bmatrix} = 0$$

Collecting the coefficients according to the powers of h gives

$y_{n+1} =$

$$\begin{bmatrix} 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \\ \frac{(1)^6}{6!} \\ \frac{(1)^7}{7!} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{17}{38}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{21}{38}\begin{bmatrix} 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \\ \frac{(1)^6}{6!} \end{bmatrix} - \frac{851}{13680}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{29}{240}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \end{bmatrix} - \frac{31}{4560}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^5}{5!} \end{bmatrix} + \frac{11}{13680}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \\ \frac{(3)^5}{5!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{97}{383040} \end{bmatrix}$$

$y_{n+2} =$

$$\begin{bmatrix} 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^5}{5!} \\ \frac{(2)^6}{6!} \\ \frac{(2)^7}{7!} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{14}{19}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{24}{19}\begin{bmatrix} 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \\ \frac{(1)^6}{6!} \end{bmatrix} - \frac{127}{855}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \end{bmatrix} - \frac{37}{285}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2)^1}{1!} \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^5}{5!} \end{bmatrix} + \frac{7}{855}\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(3)^1}{1!} \\ \frac{(3)^2}{2!} \\ \frac{(3)^3}{3!} \\ \frac{(3)^4}{4!} \\ \frac{(3)^5}{5!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{41}{23940} \end{bmatrix}$$

$y_{n+3} =$

$$\begin{bmatrix}1\\ \frac{(3)^1}{1!}\\ \frac{(3)^2}{2!}\\ \frac{(3)^3}{3!}\\ \frac{(3)^4}{4!}\\ \frac{(3)^5}{5!}\\ \frac{(3)^6}{6!}\\ \frac{(3)^7}{7!}\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} - \frac{33}{38}\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} - \frac{81}{38}\begin{bmatrix}0\\1\\ \frac{(1)^1}{1!}\\ \frac{(1)^2}{2!}\\ \frac{(1)^3}{3!}\\ \frac{(1)^4}{4!}\\ \frac{(1)^5}{5!}\\ \frac{(1)^6}{6!}\end{bmatrix} - \frac{267}{1520}\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix} - \frac{81}{80}\begin{bmatrix}0\\0\\1\\ \frac{(1)^1}{1!}\\ \frac{(1)^2}{2!}\\ \frac{(1)^3}{3!}\\ \frac{(1)^4}{4!}\\ \frac{(1)^5}{5!}\end{bmatrix} - \frac{1701}{1520}\begin{bmatrix}0\\0\\1\\ \frac{(2)^1}{1!}\\ \frac{(2)^2}{2!}\\ \frac{(2)^3}{3!}\\ \frac{(2)^4}{4!}\\ \frac{(2)^5}{5!}\end{bmatrix} - \frac{93}{1520}\begin{bmatrix}0\\0\\1\\ \frac{(3)^1}{1!}\\ \frac{(3)^2}{2!}\\ \frac{(3)^3}{3!}\\ \frac{(3)^4}{4!}\\ \frac{(3)^5}{5!}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\ \frac{9}{42560}\end{bmatrix}$$

Hence the order of the method is six and the error constant is

$$\begin{bmatrix} \frac{97}{383040} & \frac{41}{23940} & \frac{9}{42560} \end{bmatrix}^T$$

4.3.1.2 *Order of the Block Corrector Methods for k = 4*

4.3.1.2.1 *Order of the Block Corrector Method for Case One:*

If we take a Taylor series expansion as in above, we have that

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(1h)^j}{j!}y_n^{(j)} - y_n - \frac{82}{189}hy_n' - \frac{107}{189}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+1)}(1)^j - \frac{15649}{272160}h^2y_n'' - \\
\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[-\frac{4523}{34020}(1)^j + \frac{533}{45360}(2)^j - \frac{19}{6804}(3)^j + \frac{97}{272160}(4)^j\right] \\
\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^{(j)} - y_n - \frac{122}{189}hy_n' - \frac{256}{189}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+1)}(1)^j - \frac{397}{3402}h^2y_n'' - \\
\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[-\frac{3272}{8505}(1)^j - \frac{463}{2835}(2)^j - \frac{184}{8505}(3)^j + \frac{41}{17010}(4)^j\right] \\
\sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^{(j)} - y_n - \frac{6}{7}hy_n' - \frac{15}{7}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+1)}(1)^j - \frac{577}{3360}h^2y_n'' - \\
\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{421}{420}(1)^j + \frac{629}{560}(2)^j + \frac{5}{84}(3)^j + \frac{1}{3860}(4)^j\right] \\
\sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^{(j)} - y_n - \frac{244}{189}hy_n' - \frac{512}{189}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+1)}(1)^j - \frac{2552}{8505}h^2y_n'' - \\
\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{15616}{8505}(1)^j + \frac{1168}{567}(2)^j + \frac{8704}{8505}(3)^j + \frac{608}{8505}(4)^j\right]
\end{bmatrix} = 0$$

Hence the order of the method is seven and the error constant is

$$\begin{bmatrix} \frac{-3373}{22861440} & \frac{-139}{178605} & \frac{-253}{282240} & \frac{-656}{178605} \end{bmatrix}^T$$

4.3.1.2.2 *Order of the Block Corrector Method for Case Two:*

Adopting the method used in section 4.3.1.2.1 we have the order as eight and the error constant as

$$\begin{bmatrix} \frac{72343}{3454617600} & \frac{817}{26989200} & \frac{8199}{42649600} & -\frac{961}{1686825} \end{bmatrix}^T$$

4.3.1.3 *Order of the Block Corrector Method for k = 5*

4.3.1.3.1 *Order of the Block Corrector Method for Case One:*

If we adopt the method used in section 4.3.1.2.1 we have the order of this method is eight with error constant

$$\begin{bmatrix} \frac{297137}{3131654400} & \frac{1469}{3495150} & \frac{3543}{5523200} & \frac{15548}{12233025} & -\frac{62375}{125266176} \end{bmatrix}^T$$

4.3.1.3.2 *Order of the Block Corrector Method for Case Two:*

Also adopting the method used above we have the order of this method is nine with error constant

$$\begin{bmatrix} -\frac{1841969}{158106816000} & -\frac{1441}{9149700} & -\frac{132257}{195193600} & \frac{9722}{308802375} & -\frac{41375}{46846464} \end{bmatrix}^T$$

4.3.1.3.3 *Order of the Block Corrector Method for Case Three:*

Adopting the method used in section 4.3.1.2.1 we have the order of this method as ten with error constant as

$$\begin{bmatrix} \frac{13573207}{5300152704000} & \frac{251351}{82814886000} & \frac{80077}{21811328000} & \frac{135967}{5175930375} & -\frac{8474525}{42401221632} \end{bmatrix}^T$$

4.3.2 *Analysis* of Corrector Methods for Third Order

4.3.2.1 *Order of the Block Corrector Methods for k = 4*

4.3.2.1.1 *Order of the Block Corrector Method for Case One:*

The order of this method is seven with error constant

$$\begin{bmatrix} \frac{139}{40320} & \frac{1}{45} & \frac{243}{4480} & \frac{32}{315} \end{bmatrix}^T$$

4.3.2.1.2 *Order of the Block Corrector Method for Case Two:*

The order of this method is eight with error constant

$$\begin{bmatrix} -\frac{24551}{776563200} & -\frac{599}{6066900} & -\frac{7263}{9587200} & -\frac{2152}{1516725} \end{bmatrix}^T$$

4.3.2.2 *Order of the Block Corrector Methods for k = 5*

4.3.2.2.1 *Order of the Block Corrector Method for Case One:*

The order of this method is nine with error constant

$$\begin{bmatrix} \frac{278423}{13539960000} & \frac{8854}{211561875} & \frac{2601}{6965000} & \frac{147968}{211561875} & \frac{29515}{21663936} \end{bmatrix}^T$$

4.3.2.2.2 *Order of the Block Corrector Method for Case Two:*

The order of this method is ten with error constant

$$\begin{bmatrix} -\frac{238325557}{711645492096000} & \frac{51751187}{5559730407000} & \frac{165363273}{2928582272000} & \frac{55666568}{694966300875} & \frac{101384225}{813309133824} \end{bmatrix}^T$$

4.4 **Consistency of the Method**

A block method is said to be consistent if it has order $p \geq 1$ (Lambert, 1973) $\cdot$ From this, it shows clearly that our methods are consistent.

4.5 **Zero Stability:-**

A block method is said to be zero stable if the root $z_j; j = 1(1)k$ of the first characteristics polynomials $\overline{\rho(Z)} = 0$ that is $\overline{\rho(Z)} = det\left[ZA^{(0)} - E\right]$ satisfies $|Z_s| \leq 1$ and every root with $|Z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $\rho(Z) = Z^{w-\mu}(Z-1)^{\mu}$, where μ is the order of the differential equation, w is the order of the matrices $A^{(0)}$ and E. The main consequence of zero stability is to control the propagation of the error as the integration proceeds. (Lambert, 1973).

Thus we hereby show that

for k = 3

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = z^2(z-1) = 0$$

$Z = 0{,}0{,}1$. Hence the method is zero stable

For k = 4

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = z^3(z-1) = 0$$

$Z = 0{,}0{,}0{,}1$. Hence the method is zero stable

For k = 5

$$\rho(z)=\det\left[z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right]=z^4(z-1)=0$$

$Z = 0,0,0,0,1$. Hence the method is zero stable.

Thus we have shown that our methods are zero stable.

4.6 Convergence

A block method is said to be convergent if and only if it is consistent and zero stable (Lambert, 1973).

From this, it shows that our methods are convergent.

4.7 **Region of Absolute Stability (RAS)**

According to Lambert (1973), it is not possible to quantify the region of stability in a meaningful way other than by presenting diagrams showing the boundary $\partial\mathcal{R}$ of the region, plotted in the complex plane. If such a diagram is not available, knowledge of the interval of stability, that is the intersection of $\partial\mathcal{R}$ on the real line, can still give some indication of a safe choice for h even when a system is to be solved.

Definition 4.7.1 Absolute Stability (Lambert 1973)

A linear multistep method is said to be absolutely stable in a region $\mathcal{R}$ of the complex plane if, for all $\bar{h} \in \mathcal{R}$, all roots of the stability polynomial $\pi(r, \bar{h})$ associated with the method, satisfy

$|r_s| < 1, \quad s = 1,2,\dots,k \;\; / \;\; |r_s| < |r_1|, \quad s = 2,3,\dots,k.$

Definition 4.7.2 Region of Absolute Stability

Region of absolute stability is a region in the z-plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solution of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

To determine the RAS of the block integrator a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method according to Lambert (1973) is called the boundary locus method (BLM). This is achieved by substituting the test equation $y' = \lambda y$ into the block formula (3.10) from where the characteristic or stability polynomial is formed.

We present below all the stability polynomials in respect of the block predictor and block corrector methods.

4.7.1 Stability Polynomials for Second Order ODEs

4.7.1.1 *Predictor for* $k = 3$

$$\text{✍}h^3\left(\tfrac{1}{4}w^3 \ \text{🖃}\tfrac{1}{4}w^2\right) \ \text{✍}h^2\left(\tfrac{11}{12}w^2 \ \text{✍}\tfrac{11}{12}w^3\right) \ \text{✍}h\left(\tfrac{3}{2}w^3 \ \text{🖃}\tfrac{3}{2}w^2\right) \ \text{🖃}w^3$$

4.7.1.2 *Corrector for* $k = 3$

$$h^3\left(\tfrac{1}{380}w^3 \ \text{✍}\tfrac{641}{1710}w^2 \ \text{🖃}\tfrac{373}{3420}w\right) \ \text{✍}h^2\left(\tfrac{1}{120}w^3 \ \text{🖃}\tfrac{3353}{1140}w^2 \ \text{✍}\tfrac{253}{456}w\right) \ \text{✍}h\left(\tfrac{4}{57}w^3 \ \text{🖃}\tfrac{31}{6}w^2 \ \text{🖃}\tfrac{5}{38}w\right) \ \text{🖃}w^3 \ \text{✍}w^2$$

4.7.1.3 *Predictor for* $k = 4$

$$h^8\left(\tfrac{1}{75}w^4 - \tfrac{28}{225}w^3\right) - \tfrac{31}{54}h^7w^3 + h^6\left(\tfrac{1}{360}w^4 - \tfrac{697}{360}w^3\right) - \tfrac{133}{30}h^5w^3 + h^4\left(\tfrac{31}{360}w^4 - \tfrac{2671}{360}w^3\right)$$

$$\text{✍}9h^3w^3 \ \text{✍}h^2\left(\tfrac{5}{12}w^4 \ \text{🖃}\tfrac{91}{12}w^3\right) \ \text{✍}4hw^3 \ \text{🖃}w^4 \ \text{✍}w^3$$

4.7.1.4 *Corrector for* $k = 4$

4.7.1.4.1 *Case I*

$$-h^8\left(\tfrac{4}{4725}w^4 + \tfrac{128}{4725}w^3\right) + h^7\left(\tfrac{11}{2025}w^2 - \tfrac{3767}{14175}w^3\right) - h^6\left(-\tfrac{11}{5670}w^4 + \tfrac{3323}{5670}w^3 + \tfrac{62}{405}w^2\right) - h^5\left(\tfrac{12343}{4860}w^3 + \tfrac{593}{34020}w^2\right)$$

$$-h^4\left(-\tfrac{1}{315}w^4 + \tfrac{85}{27}w^3 + \tfrac{22}{27}w^2\right) - h^3\left(\tfrac{19426}{2835}w^3 + \tfrac{914}{2835}w^2\right) - h^2\left(\tfrac{61}{378}w^4 + \tfrac{277}{54}w^3 + \tfrac{4}{9}w^2\right) - h\left(\tfrac{674}{189}w^3 + \tfrac{82}{189}w^2\right) + w^4 - w^3$$

4.7.1.4.2 *Case II*

$$-h^8\left(\tfrac{57}{5950}w^3 - \tfrac{3}{11900}w^4\right) - h^7\left(\tfrac{797}{8400}w^3 + \tfrac{307}{9520}w^2\right) - h^6\left(\tfrac{37}{95200}w^4 + \tfrac{25843}{95200}w^3 - \tfrac{22149}{47600}w^2 + \tfrac{611}{428400}w\right) -$$

$$h^5\left(\tfrac{18369}{19040}w^3 + \tfrac{29993}{57120}w^2 - \tfrac{43}{340}w\right) + h^4\left(-\tfrac{449}{19040}w^4 - \tfrac{36783}{19040}w^3 + \tfrac{1405}{357}w^2 + \tfrac{127}{4760}w\right) - h^3\left(\tfrac{63927}{19040}w^3 + \tfrac{28723}{57120}w^2 + \tfrac{6}{17}w\right)$$

$$\text{🖃}h^2\left(\tfrac{1891}{9520}w^4 \ \text{✍}\tfrac{15331}{9520}w^3 \ \text{🖃}\tfrac{293}{56}w^2 \ \text{✍}\tfrac{179}{952}w\right) \ \text{🖃}h\left(\tfrac{247}{272}w^2 \ \text{✍}\tfrac{1335}{272}w^3\right) \ \text{🖃}w^4 \ \text{✍}w^3$$

4.7.1.5 *Predictor for* $k = 5$

$$-h^{10}\left(\tfrac{1}{126}w^5 + \tfrac{8793019}{119070000}w^4\right) - \tfrac{1847}{7560}h^9w^4 - h^8\left(\tfrac{1481}{75600}w^5 + \tfrac{222263737}{571536000}w^4\right) - \tfrac{1019267}{907200}h^7w^4 - h^6\left(\tfrac{311}{15120}w^5 + \tfrac{169280551}{190512000}w^4\right)$$

$$\text{✍}\tfrac{131239}{60480}h^5w^4 \ \text{🖃}h^4\left(\tfrac{1}{8}w^5 \ \text{✍}\tfrac{102089}{3175200}w^4\right) \ \text{🖃}\tfrac{14443}{10080}h^3w^4 \ \text{✍}h^2\left(\tfrac{1}{2}w^5 \ \text{🖃}\tfrac{2573}{2016}w^4\right) \ \text{✍}5hw^4 \ \text{🖃}w^5 \ \text{✍}w^4$$

4.7.1.6 *Corrector for* $k = 5$

4.7.1.6.1 *Case I*

$$h^{10}\left(\tfrac{2}{6041}w^5 - \tfrac{328}{18123}w^4\right) - h^9\left(\tfrac{19286}{90615}w^4 + \tfrac{356}{90615}w^3\right) - h^8\left(\tfrac{349}{1812300}w^5 + \tfrac{281867}{604100}w^4 - \tfrac{35029}{233010}w^3\right) -$$

$$h^7\left(\tfrac{5775199}{2174760}w^4 + \tfrac{1747}{2174760}w^3\right) + h^6\left(\tfrac{4829}{7249200}w^5 - \tfrac{25491829}{7249200}w^4 + \tfrac{1935}{1726}w^3\right) + h^5\left(\tfrac{123643}{310680}w^3 - \tfrac{3367073}{310680}w^4\right)$$

$$+h^4\left(\frac{4781}{207120}w^5-\frac{2043581}{207120}w^4+\frac{1807}{863}w^3\right)+h^3\left(\frac{53603}{51780}w^3-\frac{772933}{51780}w^4\right)-h^2\left(\frac{13099}{51780}w^5+\frac{427901}{51780}w^4-\frac{950}{863}w^3\right)$$

$$+h\left(\frac{29}{1726}w^3-\frac{8659}{1726}w^4\right)+w^5-w^4$$

4.7.1.6.2 *Case II*

$$-h^{10}\left(\frac{2}{30499}w^5+\frac{184}{30499}w^4\right)+h^9\left(\frac{73544}{2287425}w^3-\frac{803171}{9149700}w^4\right)-h^8\left(-\frac{1039}{9149700}w^5+\frac{498029}{2614200}w^4+\frac{4737848}{6862275}w^3+\frac{14081}{54898200}w^2\right)$$

$$+h^7\left(-\frac{17124251}{15685200}w^4+\frac{35694391}{54898200}w^3+\frac{41911}{784260}w^2\right)-h^6\left(-\frac{7509}{2439920}w^5+\frac{1051501}{609980}w^4+\frac{63742547}{8234730}w^3-\frac{295639}{65877840}w^2\right)$$

$$-h^5\left(\frac{26892301}{5228400}w^4-\frac{1863511}{871400}w^3+\frac{13793}{65355}w^2\right)-h^4\left(\frac{78211}{1742800}w^5+\frac{8675789}{1742800}w^4+\frac{42720367}{1960650}w^3+\frac{365357}{7842600}w^2\right)-$$

$$h^3\left(\frac{1759669}{174280}w^4+\frac{64512}{21785}w^3-\frac{13121}{17428}w^2\right)+h^2\left(\frac{13191}{87140}w^5-\frac{3362889}{348560}w^4-\frac{733132}{65355}w^3+\frac{265327}{1045680}w^2\right)-h\left(\frac{30671}{21785}w^4+\frac{78254}{21785}w^3\right)$$

$$+w^5-w^4$$

4.7.1.6.3 *Case III*

$$h^{10}\left(\frac{1}{30982}w^5-\frac{107}{46473}w^4\right)-h^9\left(\frac{286109}{9294600}w^4+\frac{1107143}{27883800}w^3\right)-h^8\left(-\frac{1151}{18589200}w^5+\frac{5127103}{55767600}w^4-\frac{10276487}{18589200}w^3+\frac{12207}{885200}w^2\right)$$

$$-h^7\left(\frac{49487567}{111535200}w^4+\frac{427322209}{334605600}w^3+\frac{3377191}{167302800}w^2+\frac{56869}{167302800}w\right)-$$

$$h^6\left(\frac{73649}{12392800}w^5+\frac{368211727}{334605600}w^4-\frac{778623533}{111535200}w^3+\frac{21551483}{111535200}w^2+\frac{10507}{239004}w\right)$$

$$-h^5\left(\frac{711923}{354080}w^4+\frac{317695807}{28680480}w^3+\frac{3975607}{2868048}w^2-\frac{23591}{2868048}w\right)+h^4\left(\begin{array}{l}\frac{3793}{177040}w^5-\frac{5113337}{1593360}w^4+\frac{20151379}{796680}w^3\\+\frac{294971}{796680}w^2+\frac{7663}{79668}w\end{array}\right)$$

$$-h^3\left(\frac{465901}{66390}w^4+\frac{12129221}{597510}w^3+\frac{1877761}{298755}w^2+\frac{20479}{298755}w\right)+h^2\left(\begin{array}{l}\frac{225283}{354080}w^5+\frac{3192467}{354080}w^4+\frac{78980357}{3186720}w^3\\-\frac{16528907}{3186720}w^2\end{array}\right)$$

$$+h\left(\frac{23679}{4426}w^3-\frac{45809}{4426}w^4\right)+w^5-w^4$$

4.7.2 Stability Polynomials for Third Order ODEs

4.7.2.1 *Predictor for* $k=4$

$$h^{12}\left(\frac{1}{2625}w^4-\frac{41}{2625}w^3\right)-\frac{8}{75}h^{11}w^3-\frac{14}{45}h^{10}w^3+h^9\left(\frac{47}{4725}w^4-\frac{3329}{4725}w^3\right)-\frac{49}{30}h^8w^3-\frac{119}{36}h^7w^3+$$

$$h^6\left(\frac{23}{1680}w^4-\frac{28741}{5040}w^3\right)-\frac{128}{15}h^5w^3-\frac{32}{3}h^4w^3-\frac{32}{3}h^3w^3-8h^2w^3-4hw^3+w^4-w^3$$

4.7.2.1 *Corrector for* $k=4$

4.7.2.1.1*Case I*

$$-h^{12}\left(\frac{1}{187250}w^4+\frac{15277}{561750}w^3\right)-\frac{347}{40125}h^{11}w^3-\frac{8399}{240750}h^{10}w^3-h^9\left(\frac{5}{53928}w^4+\frac{1414411}{1348200}w^3\right)-$$

$$\frac{5767}{32100}h^8w^3-\frac{29809}{38520}h^7w^3-h^6\left(\frac{2029}{5392800}w^4+\frac{50608531}{5392800}w^3\right)-\frac{59}{535}h^5w^3-\frac{3569}{3210}h^4w^3-$$

$$h^3\left(\frac{69}{4280}w^4\ \square\ \frac{198193}{12840}w^3\right)\ \square\ \frac{168}{107}h^2w^3\ \square\ \frac{596}{107}hw^3\ \square\ w^4\ \text{✍}\ w^3$$

4.7.2.2 *Predictor for* $k=5$

$$-h^{15}\left(\frac{1}{7056}w^5+\frac{1697}{169344}w^4\right)-\frac{35611}{483840}h^{14}w^4-\frac{3265457}{14515200}h^{13}w^4-h^{12}\left(\frac{144761}{31752000}w^5+\frac{128451287}{254016000}w^4\right)-$$

$$\frac{241375}{193536}h^{11}w^4-\frac{19625}{6912}h^{10}w^4-h^9\left(\frac{125}{5376}w^5+\frac{22075}{4032}w^4\right)-\frac{78065}{8064}h^8w^4-\frac{7811}{504}h^7w^4-h^6\left(\frac{1}{1680}w^5+\frac{27343}{1260}w^4\right)$$

$$\text{✍}\frac{625}{24}h^5w^4\ \text{✍}\frac{625}{24}h^4w^4\ \text{✍}\frac{125}{6}h^3w^4\ \text{✍}\frac{25}{2}h^2w^4\ \text{✍}5hw^4\ \square w^5\ \text{✍}w^4$$

4.7.2.3 *Corrector for* $\boldsymbol{k=5}$

4.7.2.3.1 *Case I*

$$h^{15}\left(\frac{1}{877590}w^5-\frac{4199}{10531080}w^4\right)+h^{14}\left(\frac{108803}{150444000}w^3-\frac{3771049}{263277000}w^4\right)-h^{13}\left(\frac{108898253}{6318648000}w^4+\frac{32597}{313425}w^3\right)-$$

$$h^{12}\left(-\frac{54133}{1974577500}w^5+\frac{625075439}{15796620000}w^4+\frac{632287}{25074000}w^3\right)-h^{11}\left(\frac{1068989}{1741250}w^4+\frac{8162311}{107460000}w^3\right)-h^{10}\left(\frac{912563923}{1805328000}w^4-\frac{917933}{1002960}w^3\right)$$

$$+h^9\left(\frac{459887}{2256660000}w^5-\frac{4490249149}{4513320000}w^4+\frac{171221}{1432800}w^3\right)+h^8\left(\frac{261633059}{601776000}w^3-\frac{4642872559}{601776000}w^4\right)-h^7\left(\frac{360934633}{120355200}w^4+\frac{2137}{398}w^3\right)-$$

$$h^6\left(\frac{5107}{10746000}w^5+\frac{167249161}{21492000}w^4+\frac{3821}{5970}w^3\right)-h^5\left(\frac{19250791}{716400}w^4+\frac{547171}{179100}w^3\right)+h^4\left(\frac{100133}{28656}w^4+\frac{1796}{597}w^3\right)-$$

$$h^3\left(\frac{209}{11940}w^5+\frac{38854}{2985}w^4-\frac{2012}{2985}w^3\right)+h^2\left(\frac{20242}{14925}w^3-\frac{413609}{29850}w^4\right)+\frac{12655}{1194}hw^4+w^5-w^4$$

4.7.2.3.2 *Case II*

$$-h^{15}\left(\frac{4}{43679139}w^5+\frac{6772}{43679139}w^4\right)+h^{14}\left(\frac{9798739}{3275935425}w^4-\frac{185887}{133711650}w^3\right)-h^{13}\left(\begin{array}{l}\frac{23709337}{2183956950}w^4+\frac{590826343}{2620748340}w^3\\-\frac{1801685}{1048299336}w^2\end{array}\right)+$$

$$h^{12}\left(\frac{24172}{7019861625}w^5-\frac{1288087954}{49139031375}w^4+\frac{1042019057}{6551870850}w^3+\frac{370604}{2079959}w^2\right)+h^{11}\left(\begin{array}{l}\frac{60850291799}{589668376500}w^4+\frac{132204086527}{1179336753000}w^3\\-\frac{550153}{5425980}w^2\end{array}\right)-$$

$$h^{10}\left(\frac{3954004171}{9434694024}w^4+\frac{6177658693}{1871963100}w^3+\frac{1032521501}{2495950800}w^2\right)-h^9\left(\begin{array}{l}-\frac{167}{80226990}w^5+\frac{260897197}{280794465}w^4+\frac{28768842271}{2246355720}w^3\\+\frac{9812205}{4159918}w^2\end{array}\right)+$$

$$h^8\left(\frac{2311159}{4521650}w^4-\frac{180045793583}{14975704800}w^3+\frac{5112247}{3565644}w^2\right)-h^7\left(\frac{14798547409}{8985422880}w^4+\frac{1323190647}{166396720}w^3-\frac{5466518723}{1497570480}w^2\right)+$$

$$h^6\left(\frac{34137767}{22463557200}w^5-\frac{188526639017}{22463557200}w^4+\frac{581490115}{24959508}w^3+\frac{975000}{297137}w^2\right)-h^5\left(\begin{array}{l}\frac{293095457}{99838032}w^4-\frac{4654061641}{299514096}w^3\\+\frac{595720}{297137}w^2\end{array}\right)-$$

$$h^4\left(\square\frac{1623071357}{99838032}w^4\ \square\frac{10402148}{891411}w^3\ \square\frac{91784}{38757}w^2\right)\square h^3\left(\frac{670821}{11885480}w^5\ \square\frac{167295429}{11885480}w^4\ \square\frac{9355754}{297137}w^3\right)\square$$

$$h^2\left(\frac{1253306}{193785}w^4\ \square\frac{2338013}{387570}w^3\right)\square\frac{14831755}{594274}hw^4\ \square w^5\ \square w^4$$

With the aid of MATLAB 5.5 we plotted the RAS for the above stability polynomials and presented them as Figures 1 to 14.

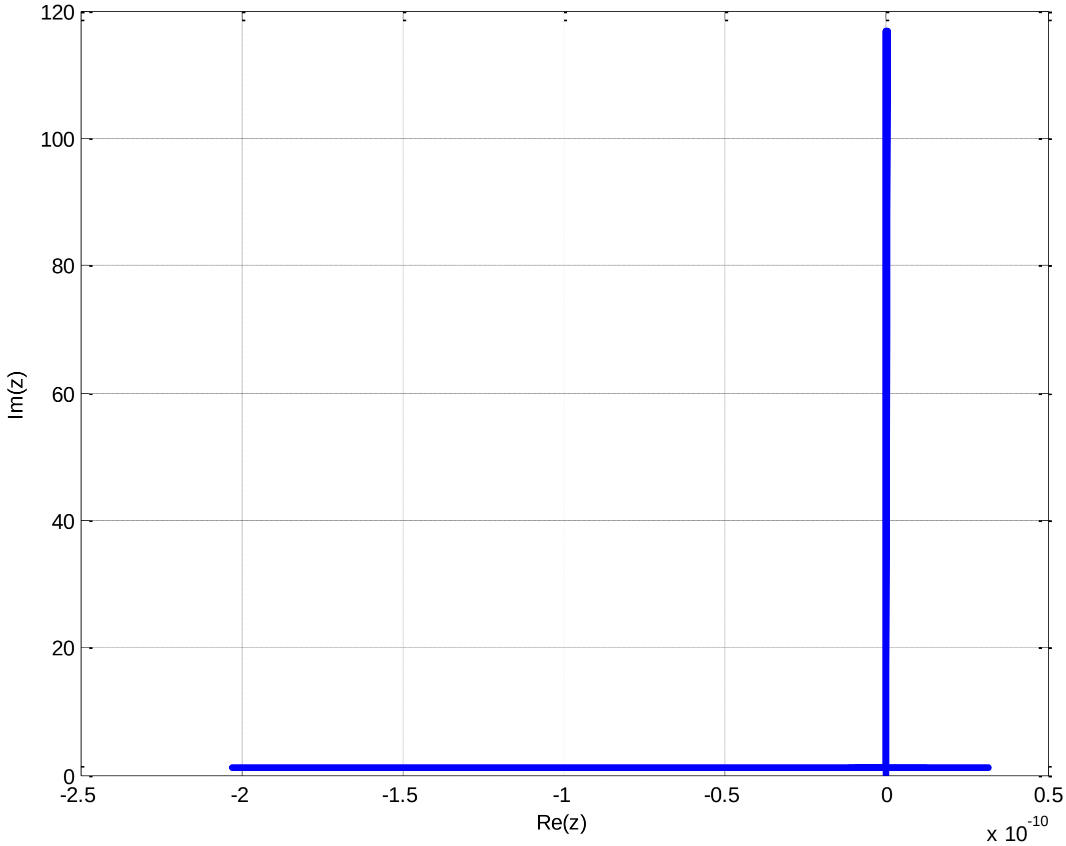

Figure 1: RAS of Block Predictor Method for Second Order when K = 3

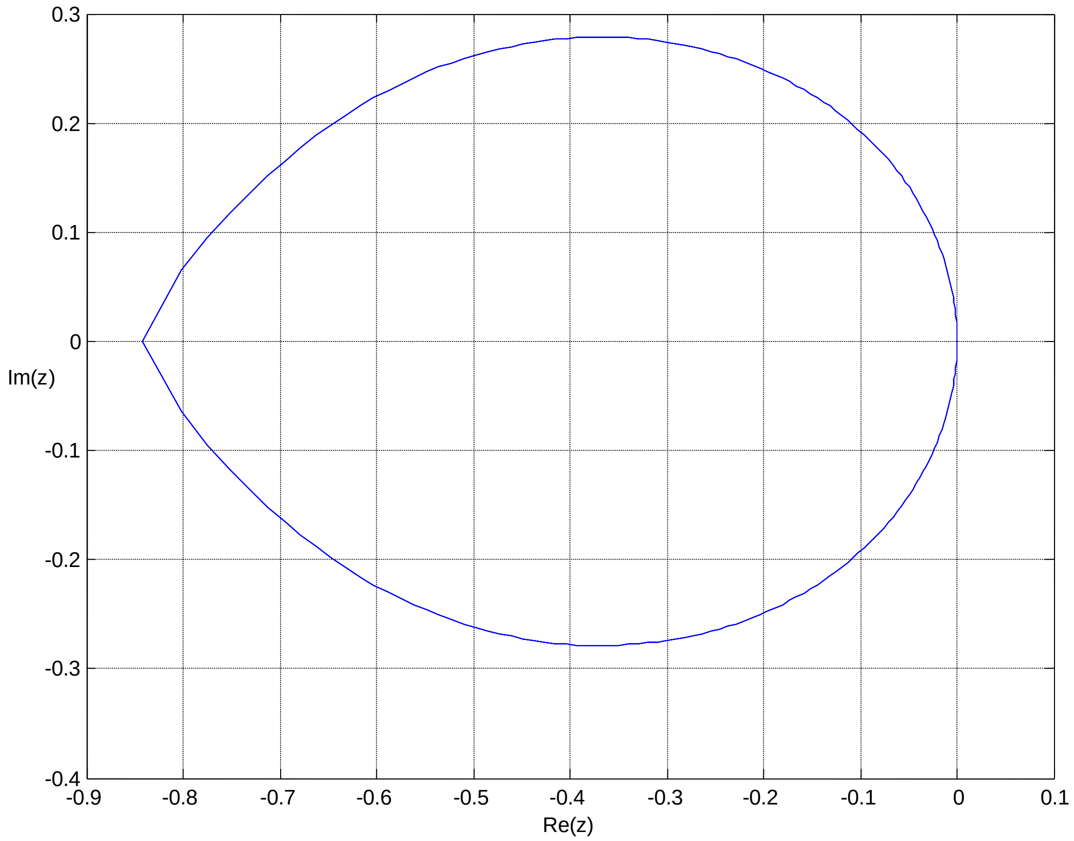

Figure 2: RAS of Block Corrector Method for Second Order when K = 3

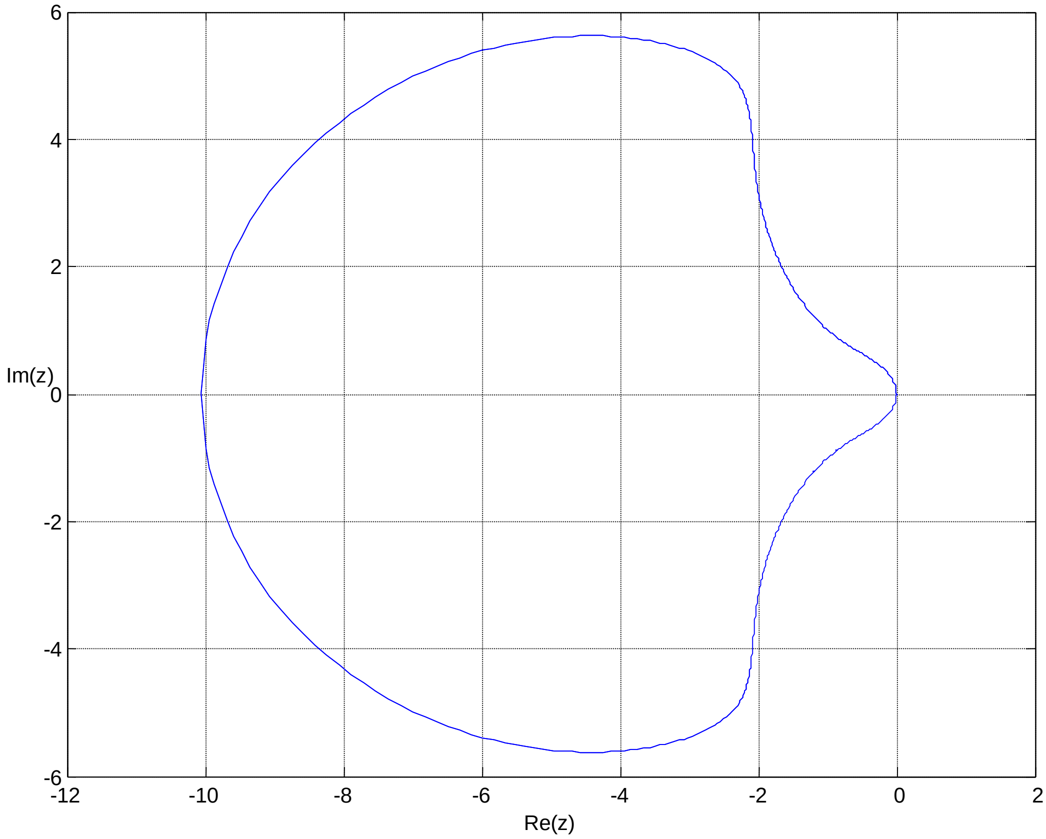

Figure 3: RAS of Block Predictor Block Corrector Method for Second Order when K =3

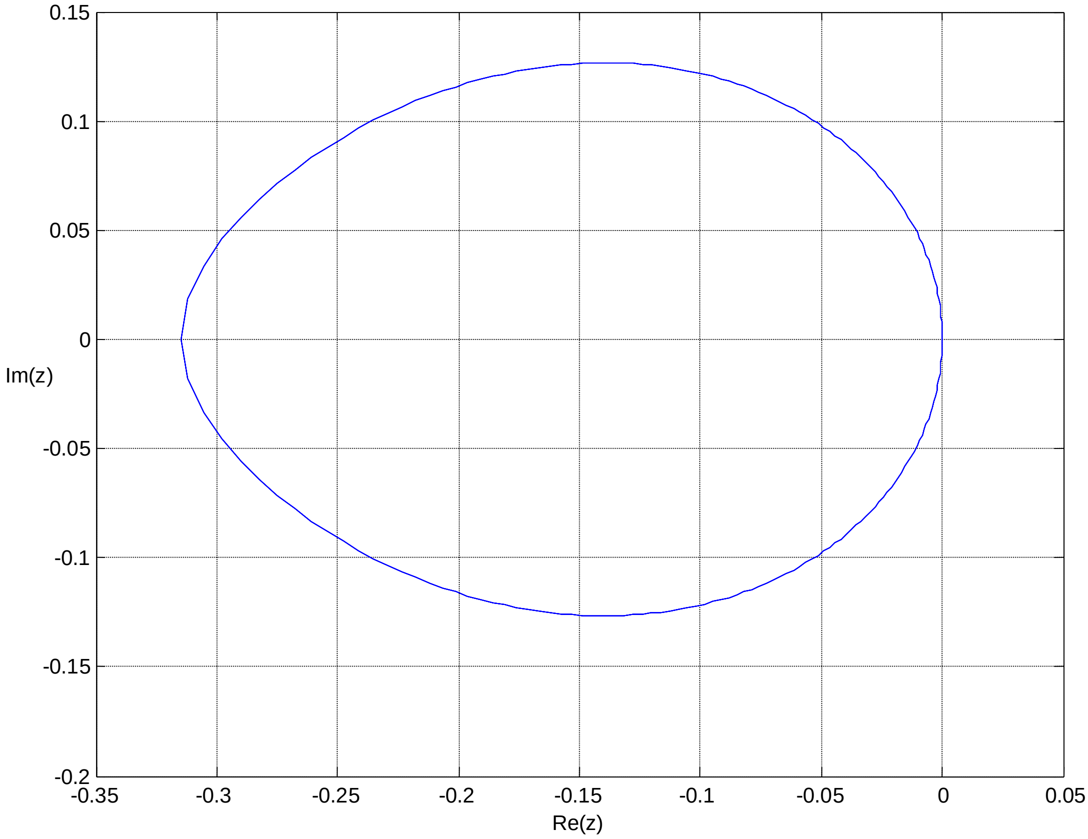

Figure 4: RAS of Block Predictor Method for Second Order when K = 4

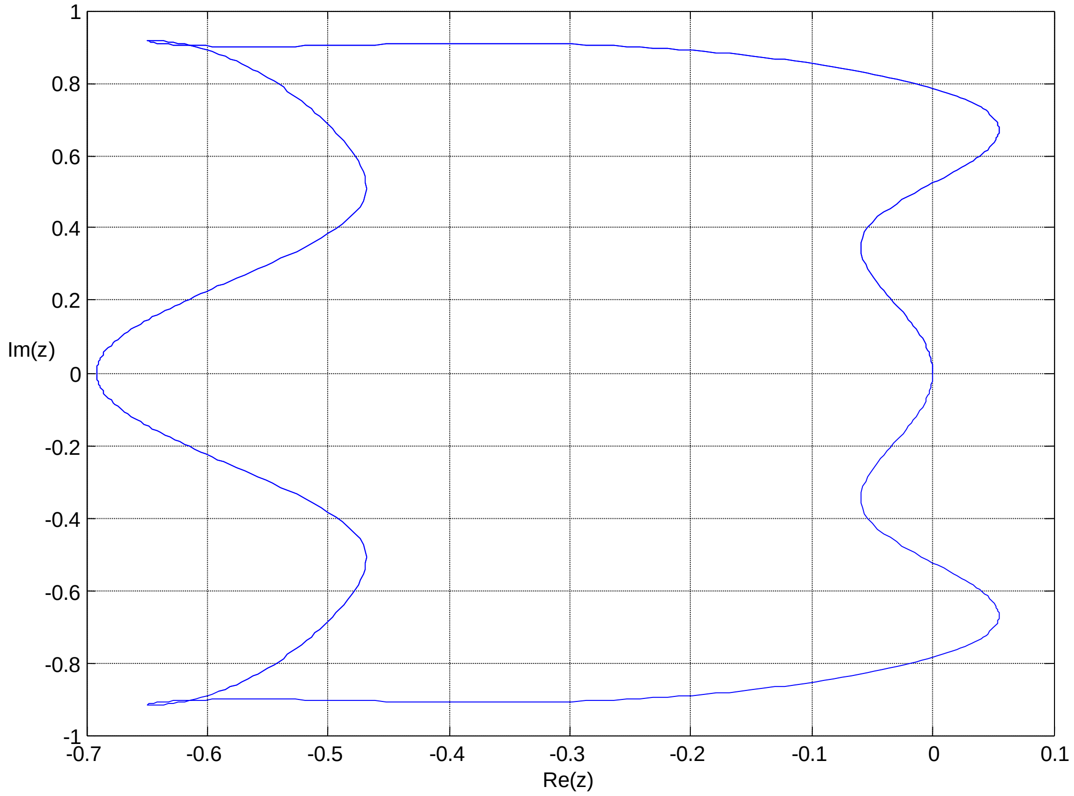

Figure 5: RAS of Block Corrector Case I for Second Order when K = 4

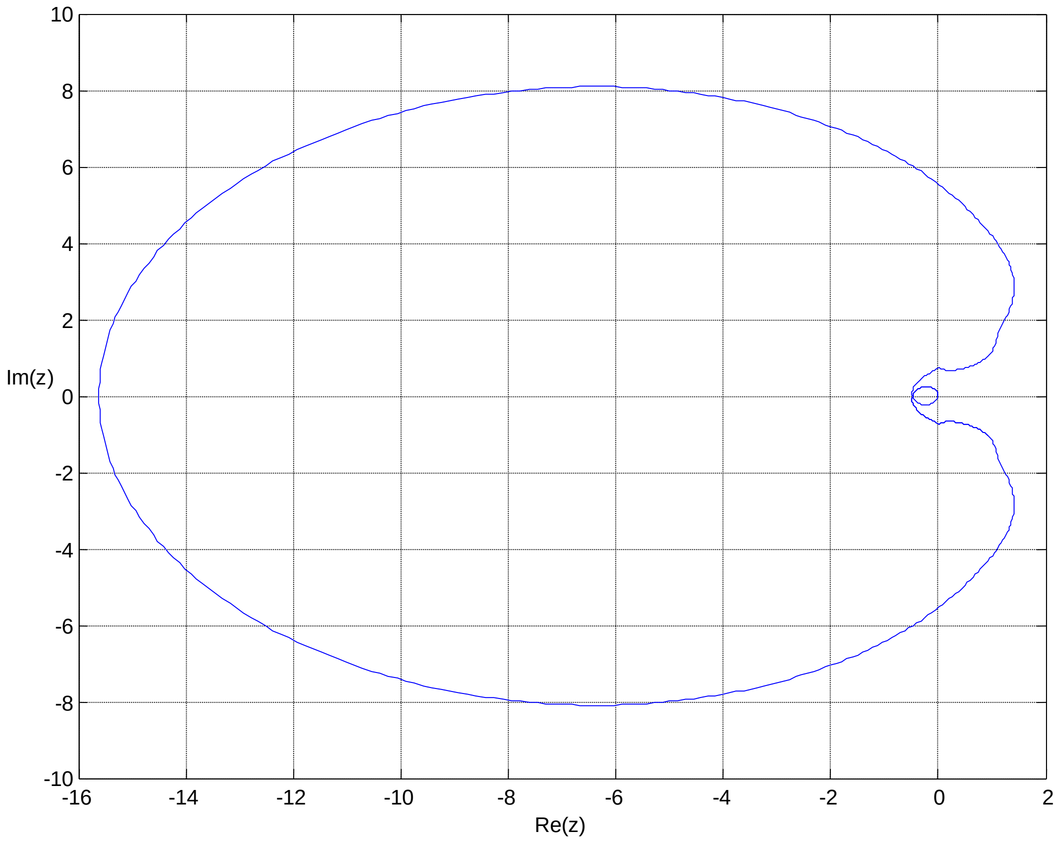

Figure 6: RAS of Block Corrector Method Case II for Second Order when K = 4

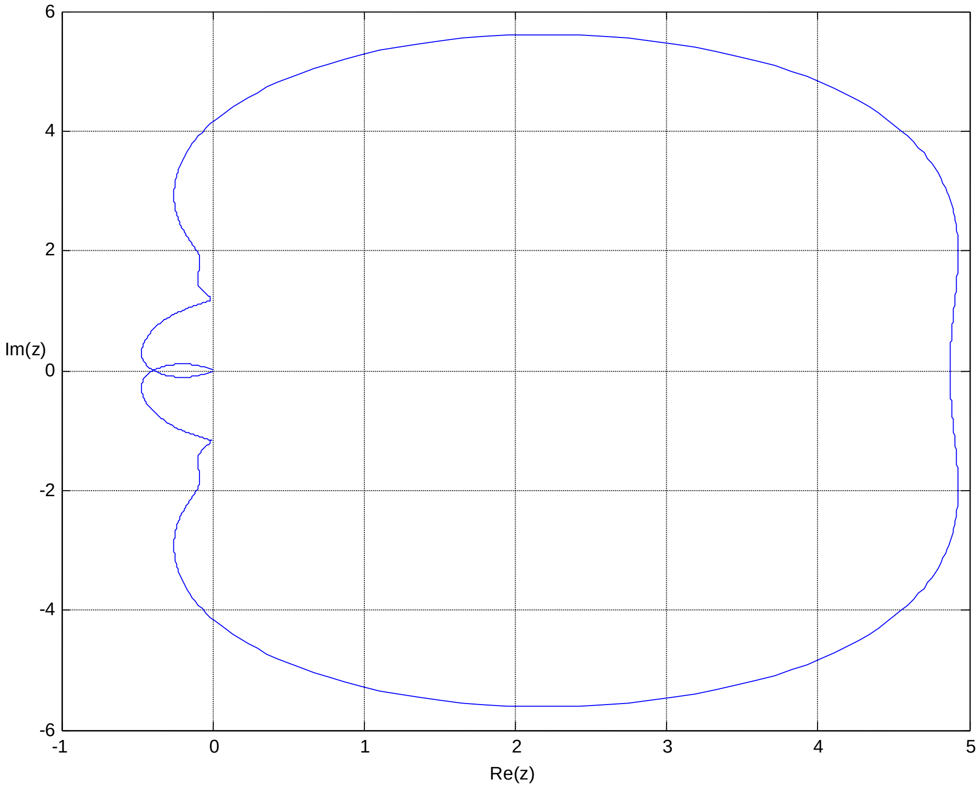

Figure 7: RAS of Block Predictor Method for Second Order when K = 5

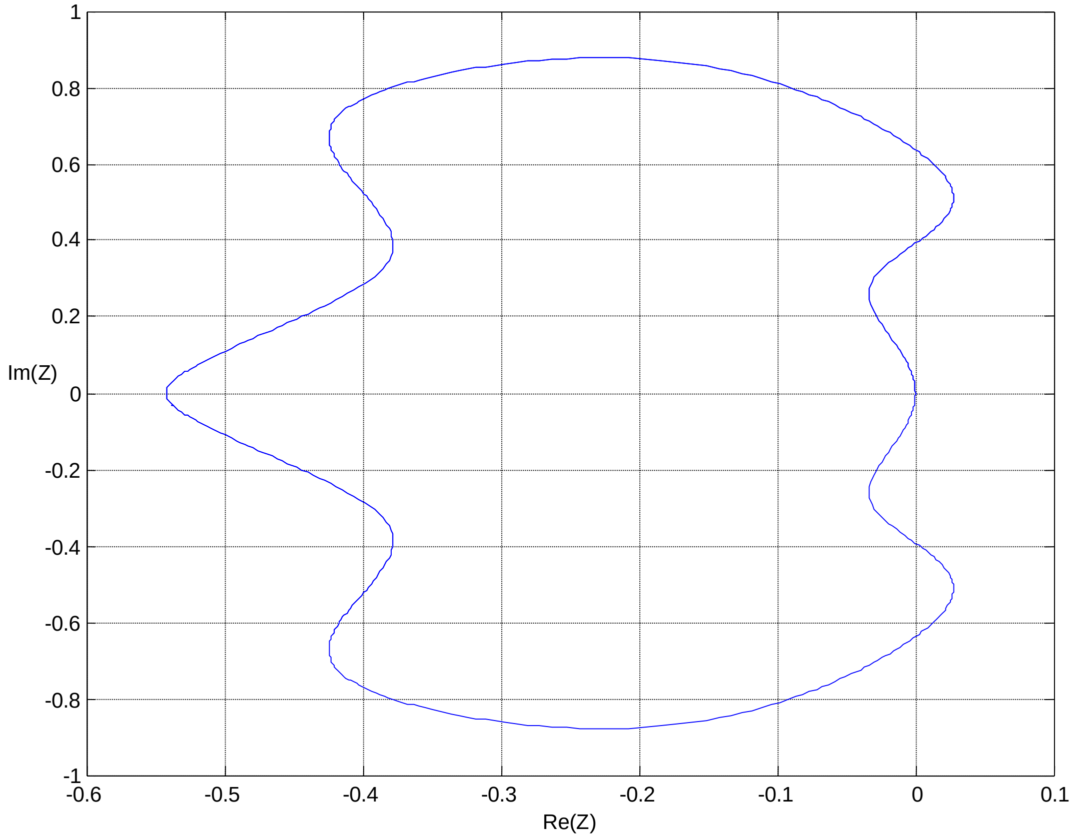

Figure 8: RAS of Block Corrector Method Case I for Second Order when K = 5

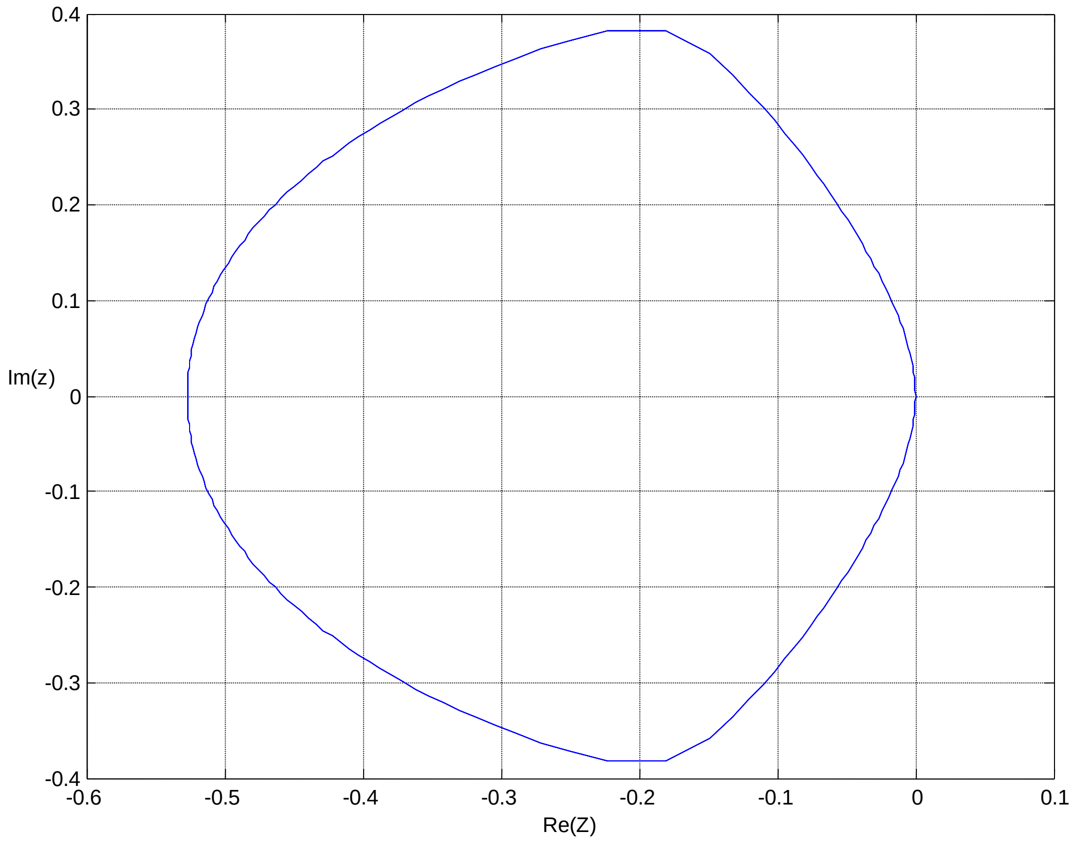

Figure 9: RAS of Block Corrector Method Case II for Second Order when K=5

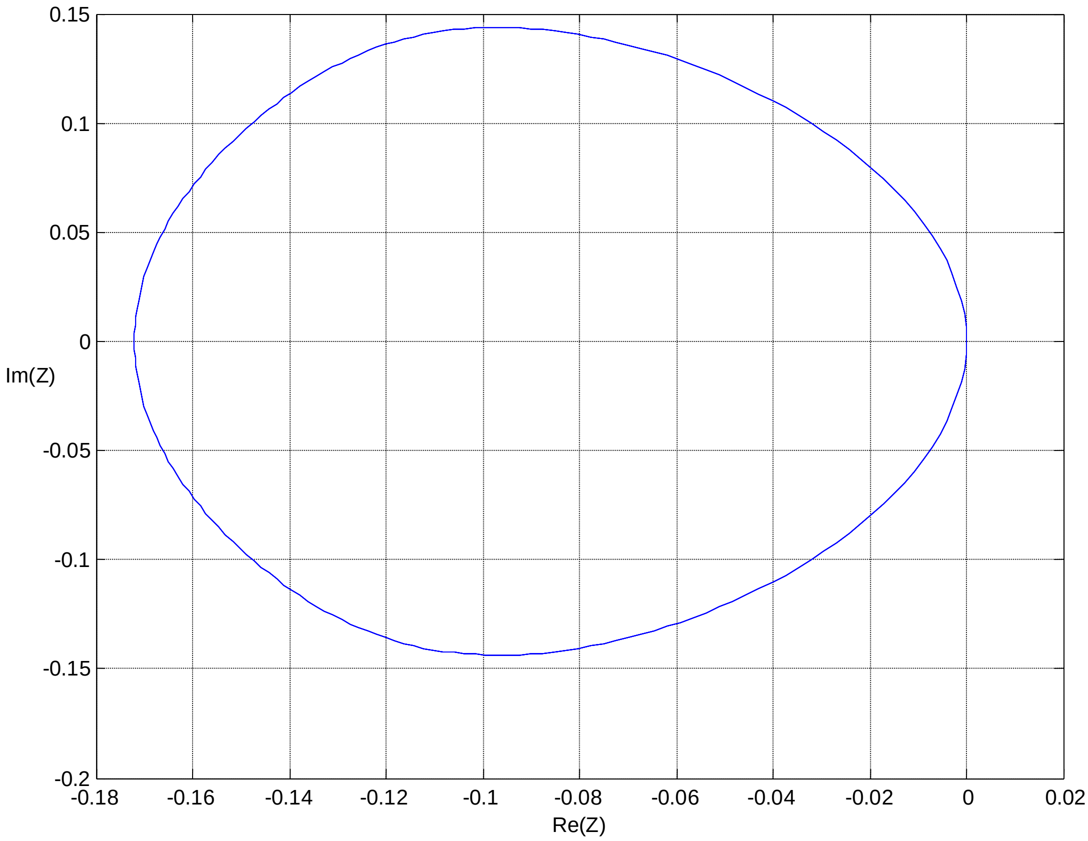

Figure 10: RAS of Block Corrector Method Case III for Second Order when K = 5

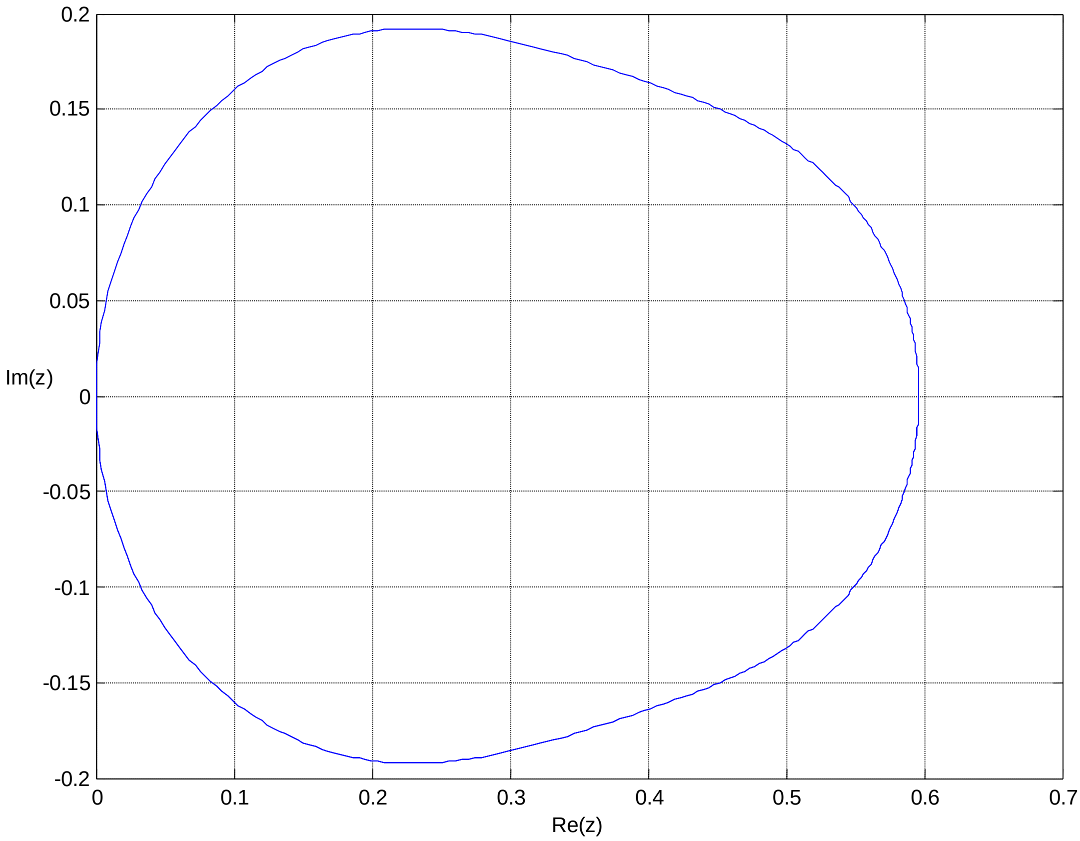

Figure 11: RAS of Block Corrector Method Case I for Third Order when K = 4

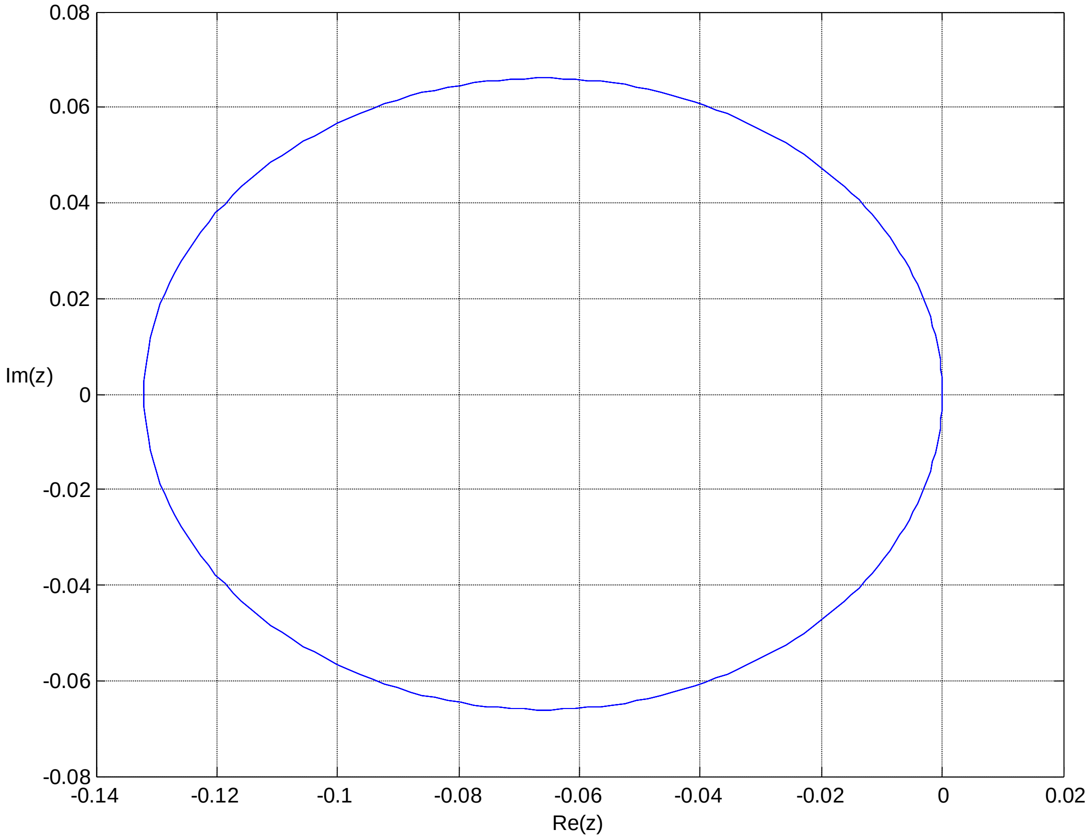

Figure 12: RAS of Block Predictor Method for Third Order when K = 5

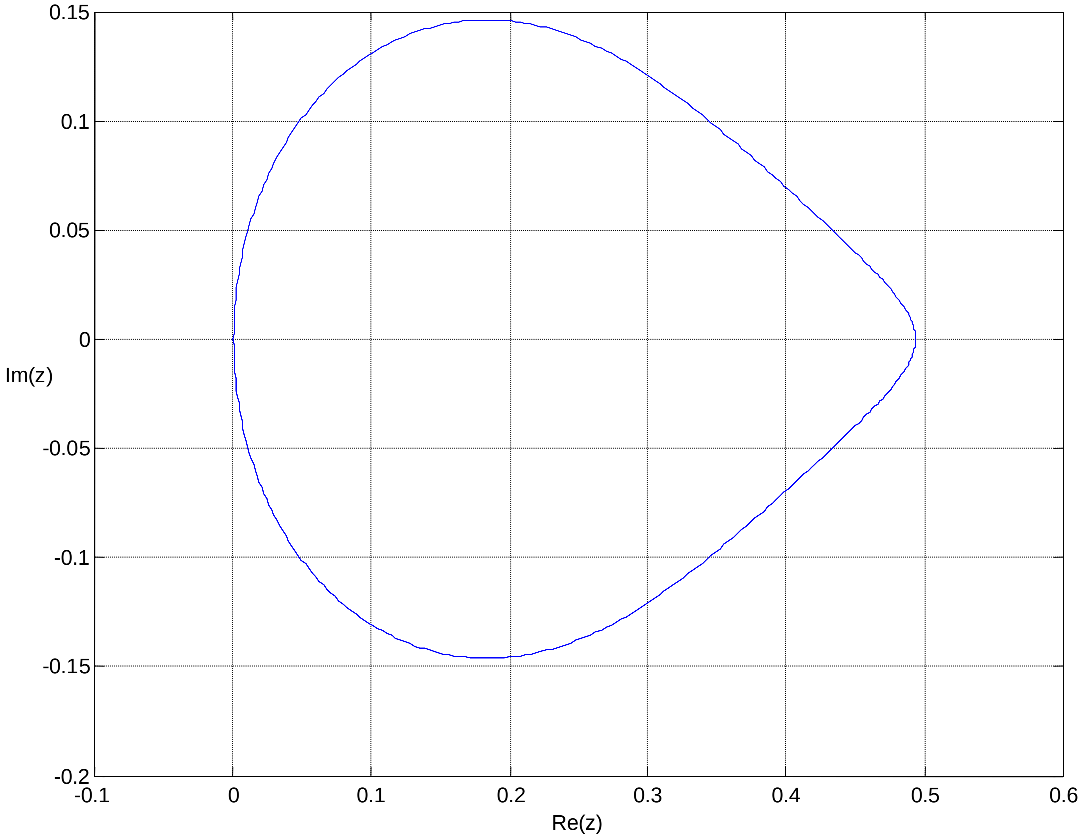

Figure 13: RAS of Block Corrector Method Case I for Third Order when K = 5

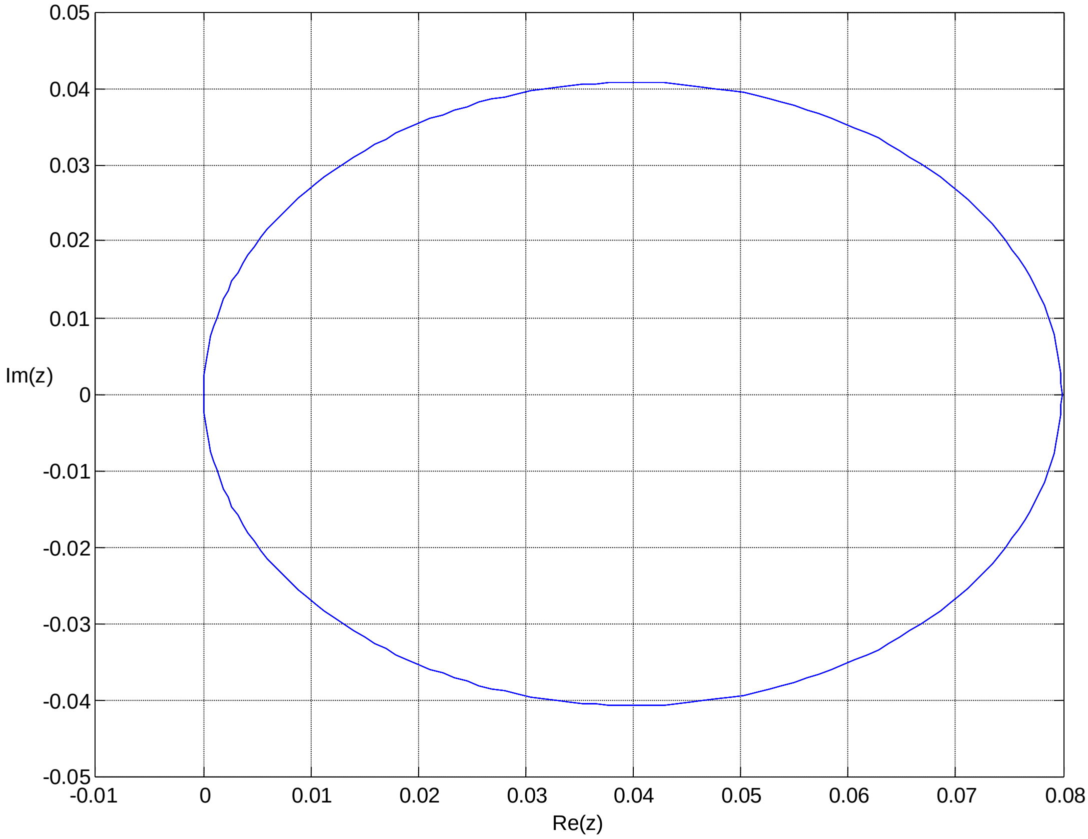

Figure 14: RAS of Block Corrector Method Case II for Third Order when K = 5

4.8 **Implementation and Numerical Experiments**

We implement the proposed methods to verify their efficacies over existing methods. To be considered are: for second order, two cases for k = 4 and three cases for k = 5; and for third order; two cases for k = 4 and two cases for k = 5. In all, seven examples were considered at h = 0.01 and some at h = 0.05. All computations were made with the usage of MATLAB 5.5. An error (Err) is defined in this research work as the absolute value of the difference between the computed (y computed) and exact values (y exact). The following keys are used in displaying our results on the tables for clarity.

Err I: Error in Block Predictor Block Corrector Method.

Err II: Error in Block -Predictor-Corrector Method.

Err III: Error in Block Method.

CASE One: Two interpolation points.

CASE Two: Three interpolation points.

CASE Three: Four interpolation points.

AWR: Results of Awoyemi *et al.*(2006).

ADR: Results of Adetola *et al.*, (2012).

OYR: Repsults of Olabode and Yusuf, (2009).

The concept of block predictor block corrector is new and our primary objective is the workability of the new methods in the solution to problems of the type (1.2). As such, the choice of our examples was only for the purpose of verification.

4.9 Test Problems

We have ten independent results to be tested for their respective effectiveness. Numerical problems in the areas of linear and non-linear second and third order ordinary differential equations were considered. Also considered was one real life problem on spring stiffness.

4.9.1 ***Test Problem I at $h = 0.05$***

Consider the non-linear ODE

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

Exact Solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$

4.9.2 ***Test Problem 2 at $h = 0.05$***

Consider the non-linear initial value problem

$$y'' = \frac{(y')^2}{2y} - 2y \; ; \quad y\left(\frac{\pi}{6}\right) = \frac{1}{4}, \; y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Exact Solution: $(x) = sin^2 x$.

4.9.3 *Test Problem 3*

Consider the initial value ODE

$$y'' = y + xe^{3x} \; ; \quad y(0) = -\tfrac{3}{32}, y'(0) = -\tfrac{5}{32}, h = 0.01.$$

Exact Solution: $(x) = \frac{4x-3}{32e^{-3x}}$.

4.9.4 *Test Problem 4*

Consider the initial value problem

$$y'' = -4y \; ; \quad y(0) = 1, y'(0) = 2, h = 0.01.$$

Exact Solution: $y(x) = \cos 2x + sin2x.$

4.9.5 *Test Problem 5*

Consider the linear third order initial value problem

$$y''' + y' = 0; y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.01.$$

Exact Solution: $y(x) = 2(1 - cosx) + sinx.$

4.9.6 *Test Problem 6*

Consider the special third order initial value problem

$$y''' = 3\,\mathrm{Si}nx; \quad y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.01.$$

Exact Solution: $y(x) = 3cosx + \frac{x^2}{2} - 2.$

4.9.7 *Test Problem 7*

A spring has one end attached to a vertical wall and the other end attached to a 1 kilogram mass. The mass lies on a horizontal surface and $x(t)$ denotes the displacement from equilibrium at time t. According to Hooke's Law, the force on the mass exerted by the spring is proportional to the displacement from equilibrium. The proportionality constant, (spring constant), which measures the stiffness of the spring is assumed to be 4.Find the position of the mass as a function of time if the initial displacement is 0 and the initial velocity is 1.

The differential equation is $\frac{d^2}{dt^2}x(t) = -4x(t); x(0) = 0; x'(0) = 1$

Exact Solution: $x(t) = \frac{1}{2}sin2t$

We have considered a total of seven test problems consisting of both linear and non-linear differential equations. All seven test problems were solved using the methods derived in chapter three. The results are presented in Tables1 to 24. These results confirm the accuracy of our methods and their superiority over the existing methods.

Table 1*:* Absolute errors of the new methods for different interpolation points at h = 0.05 for problem 1.

	Err for k = 4		Err for k = 5		
x-value	Case One	Case Two	Case One	Case Two	Case Three
0.1	1.148666(-10)	1.117957(-10)	6.959988(-12)	6.9773755(-12)	6.980638(-12)
0.2	2.304048(-10)	2.361691(-10)	1.471068(-11)	1.468337(-11)	1.470468(-11)
0.3	4.070080(-10)	4.174734(-10)	2.956990(-11)	2.969025(-11)	2.974887(-11)
0.4	5.871796(-10)	5.994316(-10)	6.953504(-11)	7.023626(-11)	7.063172(-11)
0.5	9.923316(-10)	9.767278(-10)	1.086504(-11)	1.177911(-10)	1.174840(-10)
0.6	1.414456(-09)	1.357228(-09)	2.316121(-10)	2.327660(-10)	2.347267(-10)
0.7	2.664834(-09)	2.329491(-09)	3.954923(-10)	3.931862(-10)	3.991147(-10)
0.8	4.005613(-09)	3.317707(-09)	5.231535(-10)	5.297960(-10)	5.332230(-10)
0.9	8.804608(-09)	6.384651(-09)	1.076466(-09)	1.115101(-09)	1.138269(-09)
1.0	1.414628(-08)	9.537446(-09)	1.379378(-09)	1.882982(-09)	1.864998(-09)

Table 2: Absolute error of the new method with existing methods when k = 4, at h = 0.01for problem 1.

X-value	Err I	Err II	Err III
0.1	9.103829e-15	4.862777e-14	9.992007e-15
0.2	1.110223e-14	2.160494e-13	8.149037e-14
0.3	1..576517e-14	5.255796e-13	4.700684e-13
0.4	1.798561e-14	1.025402e-12	1.637801e-12
0.5	2.775558e-14	1.803224e-12	4.664935e-12
0.6	4.352074e-14	3.007816e-12	1.116263e-11
0.7	5.595524e-14	4.899192e-12	2.501044e-11
0.8	3.397282e-13	7.946088e-12	5.215339e-11
0.9	5.551115e-14	1.302736e-11	1.076854e-10
1.0	1.461054e-14	2.188583e-11	2.170679e-10

Table 3: Absolute errors of the new methods for different interpolation points for problem 2.

	Err for k = 4		Err for k = 5		
x-value	Case One	Case Two	Case One	Case Two	Case Three
1.023	5.483646(-08)	1.001313(-09)	1.450115(-08)	1.450198(-08)	1.448318(-08)
1.123	7.475030(-08)	1.039073(-09)	2.041163(-08)	2.040973(-08)	2.040707(-08)
1.223	9.301246(-08)	4.935578(-09)	2.645463(-08)	2.644890(-08)	2.645422(-08)
1.323	1.104840(-07)	9.038723(-09)	3.120210(-08)	3.120180(-08)	3.120062(-08)
1.423	1.267508(-07)	1.454907(-08)	3.532345(-08)	3.532144(-08)	3.531459(-08)
1.523	1.408839(-07)	1.992274(-08)	3.884826(-08)	3.884982(-08)	3.876058(-08)
1.623	1.537364(-07)	2.583176(-08)	4.128344(-08)	4.128357(-08)	4.128494(-08)
1.723	1.633737(-07)	3.115442(-08)	4.305053(-08)	4.305124(-08)	4.304851(-08)
1.823	1.693086(-07)	3.594488(-08)	4.550337(-08)	4.550378(-08)	4.550700(-08)
2.023	1.686840(-07)	4.303219(-08)	4.830085(-08)	4.829873(-08)	4.854291(-08)

Table 4: Absolute error of the new method with existing methods when k = 5 for problem 2 at h = 0.01.

x-value	Err I	Err II	Err III
1.003	1.517786e-12	2.138789e-11	4.779190e-10
1.103	1.949219e-12	3.059430e-11	5.974645e-10
1.203	2.530198e-12	4.070444e-11	6.895838e-10
1.303	3.297806e-12	5.124245e-11	7.468689e-10
1.403	4.226508e-12	6.169587e-11	7.709915e-10
1.503	5.276668e-12	7.154077e-11	7.651488e-10
1.603	6.378194e-12	8.026635e-11	7.381171e-10
1.703	7.273737e-12	8.739887e-11	6.996770e-10
1.803	7.936540e-12	9.252354e-11	6.594617e-10
1.903	8.164913e-12	9.530710e-11	6.275389e-10
2.003	7.862899e-12	9.551315e-11	6.080769e-10

Table 5: Absolute Errors of the New Methods for Different Interpolation Points for Problem 3

	Err for k = 4		Err for k = 5		
x-value	Case One	Case Two	Case One	Case Two	Case Three
0.1	6.486478(-13)	6.483702(-12)	4.023171(-14)	4.025946(-14)	4.024558(-14)
0.2	1.532774(-12)	1.530470(-12)	1.813549(-13)	1.813272(-13)	1.813272(-13)
0.3	2.785883(-12)	2.778555(-12)	4.665435(-13)	4.665712(-13)	4.665757(-13)
0.4	4.523689(-12)	4.509226(-12)	9.578172(-13)	9.578172(-13)	7.577894(-13)
0.5	7.014500(-12)	6.983969(-12)	7.742495(-12)	7.742495(-12)	1.742412(-12)
0.6	1.048768(-11)	1.043124(-11)	2.944242(-12)	2.944242(-12)	2.944187(-12)
0.7	1.546147(-11)	1.536465(-11)	4.737225(-12)	4.737225(-12)	4.737134(-12)
0.8	2.238659(-11)	2.222950(-11)	7.365913(-12)	7.365913(-12)	7.365761(-12)
0.9	3.227908(-11)	3.202710(-11)	1.117295(-11)	1.117295(-11)	1.117278(-11)
1.0	4.601608(-11)	4.561729(-11)	1.663791(-11)	1.663791(-11)	1.663769(-11)

Table 6: Absolute Errors of the New Methods for Different Interpolation Points for Problem 4

	Err for k = 4		Err for k = 5		
x-value	Case One	Case Two	Case One	Case Two	Case Three
0.1	1.441269(-13)	1.472156(-13)	6.439294(-15)	6.661338(-15)	6.439294(-15)
0.2	2.682299(-13)	2.589040(-13)	2.575717(-14)	2.553513(-14)	2.553513(-14)
0.3	3.570477(-13)	3.250733(-13)	5.973000(-14)	5.950795(-14)	5.973000(-14)
0.4	4.125589(-13)	3.428369(-13)	1.072475(-13)	1.070255(-13)	1.072475(-13)
0.5	4.105605(-13)	3.077538(-13)	1.671996(-13)	1.669775(-13)	1.669775(-13)
0.6	3.550493(-13)	2.242651(-13)	2.362555(-13)	2.360334(-13)	2.358114(-13)
0.7	2.307043(-13)	9.547918(-13)	3.104184(-13)	3.104184(-13)	3.104184(-13)
0.8	7.049916(-13)	6.750156(-13)	3.864686(-13)	3.863576(-13)	3.863576(-13)
0.9	1.261213(-13)	2.562395(-13)	4.577486(-13)	4.577450(-13)	4.577450(-13)
1.0	3.397282(-13)	4.554135(-13)	5.193623(-13)	5.194734(-13)	5.194734(-13)

Table 7: Results of the Block Method when k = 4, for Problem 5.

x-value	Exact Solution	Computed solution	Error
0.1000	0.1098250860907767	0.1098250828184196	3.272357e-009
0.2000	0.2385361751125780	0.2385361580136287	1.709895e-008
0.3000	0.3848472284101276	0.3848471788665861	4.954354e-008
0.4000	0.5472963543028803	0.5472962426921204	1.116108e-007
0.5000	0.7242604148234575	0.7242602062205029	2.086030e-007
0.6000	0.9139712435756787	0.9139708936772020	3.498985e-007
0.7000	1.1145333126687142	1.1145327752334167	5.374353-007
0.8000	1.3239426722051924	1.3239418938809573	7.783242e-007
0.9000	1.5401069730861550	1.5401059022797503	1.070806e-006
1.0000	1.7608663730716176	1.7608649538699379	1.419202e-006

Table 8: Results of the Block Predictor-Corrector Method when K = 4, for Problem 5

x-value	Exact Solution	Computed solution	Error
0.1000	0.1098250860907767	0.1098250860671365	2.364028e-011
0.2000	0.2385361751125780	0.2385361749073337	2.052443e-010
0.3000	0.3848472284101276	0.3848472277158512	6.942765e-010
0.4000	0.5472963543028803	0.5472963526757231	1.627158e-009
0.5000	0.7242604148234575	0.7242604117003604	3.123098e-009
0.6000	0.9139712435756787	0.9139712382940540	5.281625e-009
0.7000	1.1145333126687142	1.1145333044882695	8.180446e-009
0.8000	1.3239426722051924	1.3239426603315196	1.187367e-008
0.9000	1.5401069730861550	1.5401069566956951	1.639046e-008
1.0000	1.7608663730716176	1.7608663513375686	2.173405e-008

Table 9: Results of the Block Predictor-Block Corrector Method when

K = 4, for Problem 5

x-value	Exact Solution	Computed solution	Error
0.1000	0.1098250860907767	0.1098250860901333	6.434575e-013
0.2000	0.2385361751125780	0.2385361751116449	9.330592e-013
0.3000	0.3848472284101276	0.3848472284100583	6.933343e-014
0.4000	0.5472963543028803	0.5472963543050361	2.155831e-012
0.5000	0.7242604148234575	0.7242604148298174	6.359913e-012
0.6000	0.9139712435756787	0.9139712435878882	1.220946e-011
0.7000	1.1145333126687142	1.1145333126889474	2.023315e-011
0.8000	1.3239426722051924	1.3239426722362888	3.109646e-011
0.9000	1.5401069730861550	1.5401069731315438	4.538880e-011
1.0000	1.7608663730716176	1.7608663731356975	6.407985e-011

Table 10: Results of the Block Method when K = 4, for Problem 6

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958865779	5.250089e-011
0.2000	0.9601997335237251	0.9601997337365135	2.127883e-010
0.3000	0.9110094673768181	0.9110094678533345	4.765164e-010
0.4000	0.8431829820086554	0.8431829828534525	8.447971e-010
0.5000	0.7577476856711183	0.7577476869830143	1.311896e-099
0.6000	0.6560068447290353	0.6560068466062442	1.877209e-009
0.7000	0.5395265618534655	0.5395265643873072	2.533842e-009
0.8000	0.4101201280414961	0.4101201313209933	3.279497e-009
0.9000	0.2698299048119930	0.2698299089183891	4.106396e-009
1.0000	0.1209069176044189	0.1209069226150592	5.010640e-009

Table 11: Results of the Block Predictor- Corrector Method when K = 4, for Problem 6

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958340764	6.661338e-016
0.2000	0.9601997335237251	0.9601997335237178	7.327472e-015
0.3000	0.9110094673768181	0.9110094673767926	2.553513e-014
0.4000	0.8431829820086554	0.8431829820085955	5.939693e-014
0.5000	0.7577476856711183	0.7577476856710023	1.155742e-013
0.6000	0.6560068447290353	0.6560068447288362	1.981748e-013
0.7000	0.5395265618534655	0.5395265618531518	3.127498e-013
0.8000	0.4101201280414961	0.4101201280410321	4.635763e-013
0.9000	0.2698299048119930	0.2698299048113384	6.541434e-013
1.0000	0.1209069176044189	0.1209069176035296	8.884005e—013

Table 12: Results of the Block Predictor- Block Corrector Method when k = 4, for Problem 6

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958340770	0.000000e+000
0.2000	0.9601997335237251	0.9601997335237242	8.881784e-016
0.3000	0.9110094673768181	0.9110094673768165	1.554312e-015
0.4000	0.8431829820086554	0.8431829820086523	3.108624e-015
0.5000	0.7577476856711183	0.7577476856711137	4.551914e-015
0.6000	0.6560068447290353	0.6560068447290284	6.883383e-015
0.7000	0.5395265618534655	0.5395265618534564	9.103829e-015
0.8000	0.4101201280414961	0.4101201280414846	1.154632e-014
0.9000	0.2698299048119930	0.2698299048119788	1.421085e-014
1.0000	0.1209069176044189	0.1209069176044014	1.743050e-014

Table 13: Results of the Block Method when k = 5, for Problem 6.

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958340774	3.330669e-016
0.2000	0.9601997335237251	0.9601997335237249	2.220446e-016
0.3000	0.9110094673768181	0.9110094673768181	0.000000e+000
0.4000	0.8431829820086554	0.8431829820086553	1.110223e-016
0.5000	0.7577476856711183	0.7577476856711183	0.000000e+000
0.6000	0.6560068447290353	0.6560068447290350	2.220446e-016
0.7000	0.5395265618534655	0.5395265618534654	1.110223e-016
0.8000	0.4101201280414961	0.4101201280414963	1.665335e-016
0.9000	0.2698299048119930	0.2698299048119932	1.665335e-016
1.0000	0.1209069176044189	0.1209069176044187	1.387779e-016

Table 14: Results of the Block Predictor-Corrector Method when k = 5, for Problem 6

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958340770	0.000000e+000
0.2000	0.9601997335237251	0.9601997335237245	6.661338e-016
0.3000	0.9110094673768181	0.9110094673768180	1.110223e-016
0.4000	0.8431829820086554	0.8431829820086551	1.110223e-016
0.5000	0.7577476856711183	0.7577476856711179	1.110223e-016
0.6000	0.6560068447290353	0.6560068447290346	2.220446e-016
0.7000	0.5395265618534655	0.5395265618534650	4.440892e-016
0.8000	0.4101201280414961	0.4101201280414956	5.551115e-017
0.9000	0.2698299048119930	0.2698299048119922	3.330669e-016
1.0000	0.1209069176044189	0.1209069176044173	6.938894e-016

Table 15: Results of the Block Predictor-Block Corrector Method when

k = 5, for Problem 6

x-value	Exact Solution	Computed Solution	Error
0.1000	0.9900124958340770	0.9900124958340774	3.330669e-016
0.2000	0.9601997335237251	0.9601997335237250	1.110223e-016
0.3000	0.9110094673768181	0.9110094673768181	0.000000e+000
0.4000	0.8431829820086554	0.8431829820086552	2.220446e-016
0.5000	0.7577476856711183	0.7577476856711182	1.110223e-016
0.6000	0.6560068447290353	0.6560068447290350	2.220446e-016
0.7000	0.5395265618534655	0.5395265618534653	2.220446e-016
0.8000	0.4101201280414961	0.4101201280414963	2.220446e-016
0.9000	0.2698299048119930	0.2698299048119932	2.220446e-016
1.0000	0.1209069176044189	0.1209069176044187	1.387779e-016

Table 16: Absolute Errors of the New Method with Existing Methods when $k = 4$ for Problem 5

x-value	Err I	Err II	Err III
0.1000	6.434575e-013	2.364028e-011	3.272357e-009
0.2000	9.330592e-013	2.052443e-010	1.709895e-008
0.3000	6.933343e-014	6.942765e-010	4.954354e-008
0.4000	2.155831e-012	1.627158e-009	1.116108e-007
0.5000	6.359913e-012	3.123098e-009	2.086030e-007
0.6000	1.220946e-011	5.281625e-009	3.498985e-007
0.7000	2.023315e-011	8.180446e-009	5.374353-007
0.8000	3.109646e-011	1.187367e-008	7.783242e-007
0.9000	4.538880e-011	1.639046e-008	1.070806e-006
1.0000	6.407985e-011	2.173405e-008	1.419202e-006

Table 17: Absolute Errors of the New Method with Existing Methods when $k = 4$. for Problem 6

x-value	Err I	Err II	Err III
0.1000	0.000000e+000	6.661338e-016	5.250089e-011
0.2000	8.881784e-016	7.327472e-015	2.127883e-010
0.3000	1.554312e-015	2.553513e-014	4.765164e-010
0.4000	3.108624e-015	5.939693e-014	8.447971e-010
0.5000	4.551914e-015	1.155742e-013	1.311896e-009
0.6000	6.883383e-015	1.981748e-013	1.877209e-009
0.7000	9.103829e-015	3.127498e-013	2.533842e-009
0.8000	1.154632e-014	4.635763e-013	3.279497e-009
0.9000	1.421085e-014	6.541434e-013	4.106396e-009
1.0000	1.743050e-014	8.884005e--013	5.010640e-009

Table 18: Absolute Errors of the New Method with Awoyemi *et al.*, (2006) and Adetola *et al.*, (2012) for Problem 5

x-value	Exact Solution	Err1	ADR	AWR
0.1000	0.1098250860907767	6.434575e-013	1.54055e-009	1.189947e-011
0.2000	0.2385361751125780	9.330592e-013	9.8455e-009	3.042207e-009
0.3000	0.3848472284101276	6.933343e-014	2.3652e-008	7.779556e-008
0.4000	0.5472963543028803	2.155831e-012	4.3273e-008	7.749556e-007
0.5000	0.7242604148234575	6.359913e-012	3.9018e-008	3.398961e-006
0.6000	0.9139712435756787	1.220946e-011	6.9700e-008	9.501398e-006
0.7000	1.1145333126687142	2.023315e-011	5.2032e-008	1.756558e-006
0.8000	1.3239426722051924	3.109646e-011	1.3527e-007	2.745889e-005
0.9000	1.5401069730861550	4.538880e-011	4.7483e-007	3.888082e-005
1.0000	1.7608663730716176	6.407985e-011	1.0693e-007	5.137153e-005

Table 19: Absolute Errors of the New Method with Olabode and Yusuf, (2009) and Adetola *et al* ., (2013) for Problem 6

x-value	Exact Solution	Err1	ADR	OYR
0.1000	0.9900124958340770	0.000000e+000	0.00000e+000	9.992007e-016
0.2000	0.9601997335237251	6.661338e-016	9.99200e-016	7.660538e-015
0.3000	0.9110094673768181	1.110223e-016	1.55431e-015	2.287059e-014
0.4000	0.8431829820086554	1.110223e-016	3.10862e-015	5.906386e-014
0.5000	0.7577476856711183	1.110223e-016	4.66293e-015	1.153521e-013
0.6000	0.6560068447290353	2.220446e-016	6.88338e-015	1.982858e-013
0.7000	0.5395265618534655	4.440892e-016	9.10382e-015	3.127498e-013
0.8000	0.4101201280414961	5.551115e-017	1.14908e-014	4.635736e-013
0.9000	0.2698299048119930	3.330669e-016	1.42108e-014	6.542544e-013
1.0000	0.1209069176044189	6.938894e-016	1.74582e-014	8.885253e-013

Table 20: Results for the Evaluation of a Real Life Problem, Problem 7 when k = 4.

t-value	Exact solution	Computed solution	Error
0.1000	0.0993346653975306	0.0993346654700283	7.249765e-011
0.2000	0.1947091711543253	0.1947091716158979	4.615726e-010
0.3000	0.2823212366975177	0.2823212378106521	1.113134e-009
0.4000	0.3586780454497613	0.3586780473843496	1.934588e-009
0.5000	0.4207354924039482	0.4207354952358620	2.831914e-009
0.6000	0.4660195429836131	0.4660195466751811	3.691568e-009
0.7000	0.4927248649942301	0.4927248693992306	4.405000e-009
0.8000	0.4997868015207526	0.4997868063878927	4.867140e-009
0.9000	0.4869238154390975	0.4869238204231041	4.984007e-009
1.0000	0.4546487134128407	0.4546487181003812	4.687540e-009

Table 21 Absolute Errors of when k = 4, 5 of Block Predictor-Corrector Method for Problem 6

x-value	Exact Solution	Error K =4	Error K = 5
0.1000	0.9900124958340770	6.661338e-016	0.000000e+000
0.2000	0.9601997335237251	7.327472e-015	6.661338e-016
0.3000	0.9110094673768181	2.553513e-014	1.110223e-016
0.4000	0.8431829820086554	5.939693e-014	1.110223e-016
0.5000	0.7577476856711183	1.155742e-013	1.110223e-016
0.6000	0.6560068447290353	1.981748e-013	2.220446e-016
0.7000	0.5395265618534655	3.127498e-013	4.440892e-016
0.8000	0.4101201280414961	4.635763e-013	5.551115e-017
0.9000	0.2698299048119930	6.541434e-013	3.330669e-016
1.0000	0.1209069176044189	8.884005e—013	6.938894e-016

Table 22: Absolute Errors of when k = 4, 5 of Block Predictor- Block Corrector Method for Problem 6

x-value	Exact Solution	Error K =4	Error K = 5
0.1000	0.9900124958340770	0.000000e+000	3.330669e-016
0.2000	0.9601997335237251	8.881784e-016	1.110223e-016
0.3000	0.9110094673768181	1.554312e-015	0.000000e+000
0.4000	0.8431829820086554	3.108624e-015	2.220446e-016
0.5000	0.7577476856711183	4.551914e-015	1.110223e-016
0.6000	0.6560068447290353	6.883383e-015	2.220446e-016
0.7000	0.5395265618534655	9.103829e-015	2.220446e-016
0.8000	0.4101201280414961	1.154632e-014	2.220446e-016
0.9000	0.2698299048119930	1.421085e-014	2.220446e-016
1.0000	0.1209069176044189	1.743050e-014	1.387779e-016

Table 23: Absolute Errors of when k = 4, 5 of Block Method for Problem 6

x-value	Exact Solution	Error k = 4	Error k = 5
0.1000	0.9900124958340770	5.250089e-011	3.330669e-016
0.2000	0.9601997335237251	2.127883e-010	2.220446e-016
0.3000	0.9110094673768181	4.765164e-010	0.000000e+000
0.4000	0.8431829820086554	8.447971e-010	1.110223e-016
0.5000	0.7577476856711183	1.311896e-099	0.000000e+000
0.6000	0.6560068447290353	1.877209e-009	2.220446e-016
0.7000	0.5395265618534655	2.533842e-009	1.110223e-016
0.8000	0.4101201280414961	3.279497e-009	1.665335e-016
0.9000	0.2698299048119930	4.106396e-009	1.665335e-016
1.0000	0.1209069176044189	5.010640e-009	1.387779e-016

Table 24: Absolute Errors of the new Method with those of Existing Methods when $k = 5$ for Problem 6

x-value	Err I	Err II	Err III
0.1000	3.330669e-016	0.000000e+000	3.330669e-016
0.2000	1.110223e-016	6.661338e-016	2.220446e-016
0.3000	0.000000e+000	1.110223e-016	0.000000e+000
0.4000	2.220446e-016	1.110223e-016	1.110223e-016
0.5000	1.110223e-016	1.110223e-016	0.000000e+000
0.6000	2.220446e-016	2.220446e-016	2.220446e-016
0.7000	2.220446e-016	4.440892e-016	1.110223e-016
0.8000	2.220446e-016	5.551115e-017	1.665335e-016
0.9000	2.220446e-016	3.330669e-016	1.665335e-016
1.0000	1.387779e-016	6.938894e-016	1.387779e-016

CHAPTER FIVE
DISCUSSION

5.0 Introduction

The derivations of Schemes (method) for the solution of differential equations are very necessary but the level of accuracy is the only parameter that ascertains the viability of any such scheme. Thus we tested our methods on known differential equations so as to ascertain their viabilities and reported the results as seen above.

The results were presented in a total of twenty four tables. Each table presents only one general kind of relationship and form a valid generalization of findings inherent in the experiments. We considered only second and third order initial value problems in this research work. Problems 1 to 4 and 7 were second order initial value problems while the rest were third order initial value problems.

5.1 Discussion of Results

It was discovered that varying the step length improves the accuracy of the method. Solving problem 1 we have that at x = 0.4 say, for case one, k = 4 gave 5.871796(-10) while k = 5 gives 6.953504(-11). A similar performance occurs for case two where for k = 4, we had 5.994316(-10) as against 7.023626(-11) for k = 5.

The concept of varying step length to improve the performance of a method was shown not to be order bias. The reason is that the performance is same in all cases both for second and third order methods. For instance solving problem 1 at h = 0.05 and problems 2 and 3, for k = 4 and k = 5 respectively confirmed this. This performance was replicated when problem 6 was solved.

The opinion raised in Odekunle *et al.*(2014) that block-predictor block-corrector (Err I) method is more accurate than those of the existing two considered methods (Err II and Err III), was supported when we solved problem 1 at $h = 0.01$ for k = 4. At $x = 0.5$ say, the new method gave 2.775558(-14) and that of block method was 4.664935(-12). When problem 2 was solved at $h = 0.01$ for $k = 5$ the same observation occurred. Similarly, solving problem 5 at $x = 0.2$ for $= 4$, Err I gave 9.330592(-13), Err II as 2.052443(-10) and Err III gave 2 .709895(-08). Also for problem 6 we had that for $k = 4$ at $x = 0.6$, Err I gave 6.883383(-15), Err II gave 1.981748(-13) and Err III gave 1.877209(-09.)

Solving problem 1 at $h = 0.05$ enabled us compare results of our second order methods for different interpolation points. It was found out that increasing the number of interpolation points does not significantly improve the result. This was obvious when for $k = 4$ at $x = 1.123$ we had that case one gave 7.475030(-08) and case two gave 1.039073(-09) while for $k = 5$ case one gave 2.041163(-08), case two gave 2.040973(-08) and case three gave 2.040707(-08). As can be seen, the three results for $k = 5$ are the same up to two decimal places.

The above performance was uniform throughout since for problem 3 at $x = 0.5$ for $k = 4$, case one gave 7.014500(-12), case two gave 6.983969(-12) while for $k = 5$, case one gave 7.742495(-12), case two gave 7.742495(-12) and case three gave 7.742412(-12). Here the results are same up to four decimal places. Similarly, when problem 4 was solved for $k = 4\ and$ 5 we had that for $k = 4$ at $x = 0.7$, case one gave 2.307043(-13), case two gave 9.547918(-13) while for $k = 5$ we had that case one, case two and case three all gave the same result 3.104184(-13). Thus there is no need in increasing the number of interpolation points when developing a method for the numerical solution of problems. We compared our result with those of Awoyemi *et al.* (2006) and Adetola *et al.* (2012) for problem 5 and found the new method (Err I) exhibiting a better performance. Equally when our method was compared with those of Adetola *et al.* (2012) which we code (ADR) and Olabode and Yusuf (2009), which we code (OYR), for problem 6, the performance of (Err 1) was 2.220446e-016, that of ADR was 6.88338e-015 and that of OYR was 1.982858e-013.The implication of this is that Err 1 method is more accurate than those of ADR and OYR.

We presented results for third order ODEs for problems 5 and 6 on block, block-predictor corrector and block- predictor block- corrector methods respectively. In all it was found that Err 1 still performed better than the existing methods.

A real life case was considered as in problem 7. The performance of our new method on real life problem was very satisfactory. This implies that our methods are useful in the provision of numerical solutions to problems of real life. Hence any real life problem that is modeled into a differential equation (second or third order) can be solved numerically.

We also compared our method with existing third order methods when $k = 5$ for problem 6. All methods developed for third order exhibited the same observations made for second order methods. That is, the block-predictor block-corrector method (Err I) is more accurate than those of block and block-predictor corrector methods for both second and third order cases.

Generally, all methods developed were zero stable and consistent, hence, convergent and they allowed for the evaluation at non overlapping intervals. It was found that the number of interpolation points used within a specific step length in developing a method is not significantly advantageous rather it contributes in the increase in the number of variables involved in the evaluation of such a method. Sampled numerical examples of second and third order initial value problems were chosen to test the performance of our methods. All our methods performed comparatively in terms of level of accuracies than existing ones

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

We presented our work in six chapters. In chapter one concept relevant to the research work were defined and in chapter two we reviewed the existing methods by several scholars. The methodology of the research was discussed in chapter three and in chapter four we presented all the results for cases we considered. Then in chapter five we discussed the properties of the method and the results for all cases and finally in this last chapter we summarize, conclude and made recommendations.

Basically we developed methods for the direct solution of problems in second and third order ordinary differential equations of the forms (3.1) and (3.2).The technique of collocation and interpolation of the differential system and power series approximate solution respectively at some selected grid points was considered and used to generate continuous linear multi-step methods. Two, three and four interpolation points were considered in order to generate continuous predictor and corrector methods which are implemented in block respectively. All methods developed were consistent, zero stable, convergent and of very high orders.

6.2 Conclusion

In this research work we have proposed ten block- predictor block-corrector methods. Block methods which have the properties of evaluation at all points within the interval of integration are adopted to give independent solutions at non overlapping intervals as predictors to the correctors.

Thus, the new methods have justified the need for their development for the solution of second and third order initial value problems of ordinary differential equations.

6.3 **Recommendations**

We therefore recommend that:

(i) The block- predictor block- corrector method be used in the quest for solutions to second and third order initial value problems of ordinary differential equations.

(ii) Further work should be done to investigate the effect of increasing the step number for higher order ordinary differential equations.

6.4 Contribution to knowledge

Sequel to the high performance of block- predictor corrector method we developed some new methods using the concept we named block- predictor block-corrector methods.

These new methods when compared with those of ordinary block and that of block predictor-corrector methods performed with greater accuracy. Our new methods, unlike the existing ones, enables us to achieve very high accuracy level performances as it made it possible for evaluation to be done at a non overlapping intervals.

Our results re-affirmed the claim of Adesanya *et al.* (2012a) that though block-predictor corrector method takes longer time to implement, it gives a more accurate approximation than the block method. The claim that block predictor-corrector method could not give highly accurate results (Awoyemi, 2005) due to the overlapping of the result which prompted our new method is verified.

Also affirmed was the fact that increasing the step number improves the accuracy of a method. This was evident when we increased the number of step number of the concept in Adesanya *et al.* (2012a) by one to get $k = 5$ for the second order case only. Interestingly the concept of self starting as discussed in Kayode (2009) and that of constant order are also reaffirmed in our new methods.

In the review we cited cases of scholars who worked on two points and three points methods. In this research we developed methods of four points two steps, five points three steps, five points four steps, six points four steps, six points five steps.

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APPENDICES (PROGRAMME TO SOLVE O.D.ES)

APPENDIX I

PROG. 1: Programme to solve O.D.Es with step length of four for the solution of $y'' = f(x, y, y')$.

```
clear
% initial conditions
x0 = (pi)/6; y0 = 1/4; z0 = sqrt(3)/2; h = 0.01;
% y' is represented by z
disp( 'X-value Exact solution Computed solution Error Time/s')
tic;
for k = 1 : 1 : 4;
for j = 1 : 1 : 10;
for i = 1 : 4;
f0 = g(x0, y0, z0);
x(i) = x0 + i*h;
z(i)=z0+i*h*f0+(((i*h)^2)/2)*dx(x0,y0,z0)+(((i*h)^3)/6)*ddx(x0,y0,z0)+
    (((i*h)^4)/24)*dddx(x0,y0,z0);
y(i)=y0+(i*h)*z0+(((i*h)^2)/2)*f0+((i*h)^3)/6*dx(x0,y0,z0)+
    ((i*h)^4)/24*ddx(x0,y0,z0)+(((i*h)^5)/120)*dddx(x0,y0,z0);
f(i)=g(x(i),y(i),z(i));
end
%predictor when i=1
yp1=z0+h*((251/720)*f0+(323/360)*f(1)-(11/30)*f(2)+(53/360)*f(3)-
    (19/720)*f(4));
yp2=z0+h*((29/90)*f0+(62/45)*f(1)+(4/15)*f(2)+(2/45)*f(3)-(1/90)*f(4));
yp3=z0+h*((27/80)*f0+(51/40)*f(1)+(9/10)*f(2)+(21/40)*f(3)-(3/80)*f(4));
yp4=z0+h*((14/45)*f0+(64/45)*f(1)+(8/15)*f(2)+(64/45)*f(3)+(14/45)*f(4));
%predictor when i=0
```

```
yr1=y0+h*z0+(h^2)*((367/1440)*f0+(3/8)*f(1)-(47/240)*f(2)+
      (29/360)*f(3)-(7/480)*f(4));
yr2=y0+2*h*z0+(h^2)*((53/90)*f0+(8/5)*f(1)-(1/3)*f(2)+(8/45)*f(3)-
       (1/30)*f(4));
yr3=y0+3*h*z0+((h^2))*((147/160)*f0+(117/40)*f(1)+(27/80)*f(2)+
       (3/8)*f(3)-(9/160)*f(4));
yr4=y0+4*h*z0+((h^2))*((56/45)*f0+(64/15)*f(1)+(16/15)*f(2)+
       (64/45)*f(3));
fr1=g(x(1),yr1,yp1);
fr2=g(x(2),yr2,yp2);
fr3=g(x(3),yr3,yp3);
fr4=g(x(4),yr4,yp4);
zr1=(1/h)*((20/21)*y0-(61/21)*yr1+(41/21)*yr2+(h^2)*((-613/10080)*f0-
       (1433/1260)*fr1-(41/144)*fr2+(43/1260)*fr3-(37/10080)*fr4));
zr2=(1/h)*((-37/42)*y0+(16/21)*yr1+(5/42)*yr2+(h^2)*((73/1260)*f0+
        (284/315)*fr1+(41/90)*fr2-(4/105)*fr3+(1/252)*fr4));
zr3=(1/h)*((20/21)*y0-(61/21)*yr1+(41/21)*yr2+(h^2)*((-145/2016)*f0-
        (319/420)*fr1+(707/720)*fr2+(173/420)*fr3-(149/10080)*fr4));
zr4=(1/h)*((-149/42)*y0+(128/21)*yr1-(107/42)*yr2+(h^2)*((65/252)*f0+
       (1012/315)*fr1+(79/90)*fr2+(436/315)*fr3+(397/1260)*fr4));
frr1=g(x(1),yr1,zr1);
frr2=g(x(2),yr2,zr2);
frr3=g(x(3),yr3,zr3);
frr4=g(x(4),yr4,zr4);
yrr1=y0+h*(82/189)*z0+h*(107/189)*zr1+(h^2)*((15649/272160)*f0-
       (4523/34020)*frr1+(533/45360)*frr2-(19/6804)*frr3+(97/272160)*frr4);
m1=toc;
err1=abs(gr(x(1))-yrr1);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(1),gr(x(1)),yrr1,err1,m1)
yrr2=y0+h*(122/189)*z0+h*(256/189)*zr1+(h^2)*((397/3402)*f0+
          (3272/8505)*frr1+(463/2835)*frr2-(184/8505)*frr3+(41/17010)*frr4);
m2=toc;
err2=abs(gr(x(2))-yrr2);
```

```
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(2),gr(x(2)),yrr2,err2,m2)
yrr3=y0+h*(6/7)*z0+h*(15/7)*zr1+(h^2)*((577/3360)*f0+(421/420)*frr1+
      (629/560)*frr2+(5/84)*frr3+(1/3360)*frr4);
m3=toc;
err3=abs(gr(x(3))-yrr3);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(3),gr(x(3)),yrr3,err3,m3)
yrr4=y0+h*(244/189)*z0+h*(512/189)*zr1+(h^2)*((2552/8505)*f0+
     (15616/8505)*fr1+(1168/567)*fr2+(8704/8505)*fr3+(608/8505)*fr4);
m4=toc;
err4=abs(gr(x(4))-yrr4);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(4),gr(x(4)),yrr4,err4,m4)
x0=x(4);y0=yrr4;z0=zr4;
end
end
```

APPENDIX 2

PROG. 2: Programme to solve O.D.Es with step length of five for the solution of $y'' = f(x, y, y')$.

```
clear
% initial conditions
x0 = pi/6; y0 = 1/4; z0 = sqrt(3)/2; h = 0.01;
% y' is represented by z
disp( 'X-value Exact solution Computed solution Error Time/s')
tic;
for k = 1 : 1 : 6;
for j = 1 : 1 : 5;
for i = 1 : 5;
f0 = g(x0, y0, z0);
x(i) = x0 + i*h;
z(i)=z0+i*h*f0+(((i*h)^2)/2)*dx(x0,y0,z0)+(((i*h)^3)/6)*ddx(x0,y0,z0)+
          (((i*h)^4)/24)*dddx(x0,y0,z0);
y(i)=y0+(i*h)*z0+(((i*h)^2)/2)*f0+((i*h)^3)/6*dx(x0,y0,z0)+
       ((i*h)^4)/24*ddx(x0,y0,z0)+(((i*h)^5)/120)*dddx(x0,y0,z0);
f(i)=g(x(i),y(i),z(i));
end
%predictor when i=1
yp1=z0+h*((95/288)*f0+(1427/1440)*f(1)-(133/240)*f(2)+
          (241/720)*f(3)-(173/1440)*f(4)+(3/160)*f(5));
yp2=z0+h*((14/45)*f0+(43/30)*f(1)+(7/45)*f(2)+(7/45)*f(3)-
         (1/15)*f(4)+(1/90)*f(5));
yp3=z0+h*((51/160)*f0+(219/160)*f(1)+(57/80)*f(2)+
            (57/80)*f(3)-(21/160)*f(4)+(3/160)*f(5));
yp4=z0+h*((14/45)*f0+(64/45)*f(1)+(8/15)*f(2)+
```

```
            (64/45)*f(3)+(14/45)*f(4));
yp5=z0+h*((95/288)*f0+(125/96)*f(1)+(125/144)*f(2)+
              (125/144)*f(3)+(125/96)*f(4)+(95/288)*f(5));
tr1=g(x(1),y(1),yp1);
tr2=g(x(2),y(2),yp2);
tr3=g(x(3),y(3),yp3);
tr4=g(x(4),y(4),yp4);
tr5=g(x(5),y(5),yp5);
%predictor when i=0
yr1=y0+h*z0+(h^2)*((1231/5040)*f0+(863/2016)*tr1-
 (761/2520)*tr2+(941/5040)*tr3-(341/5040)*tr4+(107/10080)*tr5);
m1=toc;
err1=abs(gr(x(1))-yr1);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(1),gr(x(1)),yr1,err1,m1)
yr2=y0+2*h*z0+(h^2)*((71/126)*f0+(544/315)*tr1-(37/63)*tr2+
           (136/315)*tr3-(101/630)*tr4+(8/315)*tr5);
m2=toc;
err2=abs(gr(x(2))-yr2);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(2),gr(x(2)),yr2,err2,m2)
yr3=y0+3*h*z0+((h^2))*((123/140)*f0+(3501/1120)*tr1-
           (9/140)*tr2+(87/112)*tr3-(9/35)*tr4+(9/224)*tr5);
m3=toc;
err3=abs(gr(x(3))-yr3);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(3),gr(x(3)),yr3,err3,m3)
yr4=y0+4*h*z0+((h^2))*((376/315)*f0+(1424/315)*tr1+
        (176/315)*tr2+(608/315)*tr3-(16/63)*tr4+(16/315)*tr5);
m4=toc;
err4=abs(gr(x(4))-yr4);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(4),gr(x(4)),yr4,err4,m4)
yr5=y0+5*h*z0+((h^2))*((1525/1008)*f0+(11875/2016)*tr1+
      (625/504)*tr2+(3125/1008)*tr3+(625/1008)*tr4+(275/2016)*tr5);
m5=toc;
err5=abs(gr(x(5))-yr5);
```

```
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(5),gr(x(5)),yr5,err5,m5)
%fr1=g(x(1),yr1,yp1);
%fr2=g(x(2),yr2,yp2);
%fr3=g(x(3),yr3,yp3);
%fr4=g(x(4),yr4,yp4);
%fr5=g(x(5),yr5,yp5);
%zr1=(1/h)*(((17/26)*y0-(30/13)*yr1+(43/26)*yr2)+(h^2)*
     ((-1007/26208)*f0-(22993/26208)*fr1-(18329/65520)*fr2+
      (3377/65520)*fr3-(1459/131040)*fr4+(163/131040)*fr5));
%zr2=(1/h)*(((-253/442)*y0+(32/221)*yr1+(189/442)*yr2)+
    (h^2)*((9689/278460)*f0+(88117/139230)*fr1+(62711/139230)*fr2-
      (781/13923)*fr3+(3253/278460)*fr4-(179/139230)*fr5));
%zr3=(1/h)*(((129/442)*y0-(350/221)*yr1+(571/442)*yr2)+(h^2)*
      ((-49867/2227680)*f0-(410597/2227680)*fr1+(1105991/1113840)*fr2+
        (501713/1113840)*fr3-(13943/445536)*fr4+(6131/2227680)*fr5));
%zr4=(1/h)*(((-413/442)*y0+(192/221)*yr1+(29/442)*yr2)+(h^2)*
     ((17249/278460)*f0+(13046/13923)*fr1+(116149/139230)*fr2+
       (85744/69615)*fr3+(105929/278460)*fr4-(758/69615)*fr5));
%zr5=(1/h)*(((101/34)*y0-(118/17)*yr1+(135/34)*yr2)+(h^2)*
      ((-36359/171360)*f0-(88549/34272)*fr1+(94579/85680)*fr2+
      (38533/85680)*fr3+(251729/171360)*fr4+(51871/171360)*fr5));
%frr1=g(x(1),yr1,zr1);
%frr2=g(x(2),yr2,zr2);
%frr3=g(x(3),yr3,zr3);
%frr4=g(x(4),yr4,zr4);
%frr5=g(x(5),yr5,zr5);
% yrr1=y0+h*(731/1726)*z0+h*(995/1726)*zr1+(h^2)*((313679/5799360)*f0
     -(166091/1159872)*frr1+(152063/8699040)*frr2-(18133/2899680)*frr3
      +(9271/5799360)*frr4-(3373/17398080)*frr5);
% m1=toc;
% err1=abs(gr(x(1))-yrr1);
% fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(1),gr(x(1)),yrr1,err1,m1)
% yrr2=y0+h*(510/863)*z0+h*(1216/863)*zr1+(h^2)*((10733/108738)*f0+
```

```
            (17978/54369)*frr1+(52613/271845)*frr2-(10844/271845)*frr3+
             (4873/543690)*frr4-(278/271845)*frr5);
% m2=toc;
% err2=abs(gr(x(2))-yrr2);
% fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(2),gr(x(2)),yrr2,err2,m2)
% yrr3=y0+h*(1371/1726)*z0+h*(3807/1726)*zr1+(h^2)*((291909/1933120)*f0+
            (1817379/1933120)*fr1+(1119303/966560)*fr2+(37209/966560)*fr3+
            (3033/386624)*fr4-(2277/1933120)*fr5);
% m3=toc;
% err3=abs(gr(x(3))-yrr3);
%fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(3),gr(x(3)),yrr3,err3,m3)
% yrr4=y0+h*(892/863)*z0+h*(2560/863)*zr1+(h^2)*((19496/90615)*f0+
      (143264/90615)*fr1+(598768/271845)*fr2+(84928/90615)*fr3+(1856/18123)*fr4-
      (1312/271845)*fr5);
% m4=toc;
% err4=abs(gr(x(4))-yrr4);
% fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(4),gr(x(4)),yrr4,err4,m4)
yrr5=y0+h*(1755/1726)*z0+h*(6875/1726)*zr1+(h^2)*((692425/3479616)*f0+
       (6761375/3479616)*frr1+(5997875/1739808)*frr2+(3074125/1739808)*frr3+
        (3822625/3479616)*frr4+(214775/3479616)*frr5);
% m5=toc;
%err5=abs(gr(x(5))-yrr5);
% fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(5),gr(x(5)),yrr5,err5,m5)
x0=x(5);y0=yr5;z0=yp5;
end
end
```

APPENDIX 3

Programme for the solution of the Block

PROG. 3: Programme to solve O.D.Es with step length of five for the solution of $y''' = f(x, y, y', y'')$.

```
% initial conditions
x0=0;y0=0;z0=1;t0=2;h=01;
% y' is represented by z, y=t
disp( 'X-value Exact solution Computed solution Error Time/s')
tic;
for j=1:1:5;
for k=1:1:4;
for i=1:5;
f0=g(x0,y0,z0,t0);
x(i)=x0+i*h;
y(i)=y0+(i*h)*z0+(((i*h)^2)/2)*t0+(((i*h)^3)/6)*f0+(((i*h)^4)/24)*dx(x0,y0,z0,t0)+
    (((i*h)^5)/120)*ddx(x0,y0,z0,t0)+(((i*h)^6)/720)*dddx(x0,y0,z0,t0)+
    (((i*h)^7)/5040)*ddddx(x0,y0,z0,t0);%+(((i*h)^8)/40320)*dddddx(x0,y0,z0,t0);
z(i)=z0+(i*h)*t0+(((i*h)^2)/2)*f0+(((i*h)^3)/6)*dx(x0,y0,z0,t0)+
    (((i*h)^4)/24)*ddx(x0,y0,z0,t0)+(((i*h)^5)/120)*dddx(x0,y0,z0,t0)+
    (((i*h)^6)/720)*ddddx(x0,y0,z0,t0);%+(((i*h)^7)/5040)*dddddx(x0,y0,z0,t0);
t(i)=t0+(i*h)*f0+(((i*h)^2)/2)*dx(x0,y0,z0,t0)+(((i*h)^3)/6)*ddx(x0,y0,z0,t0)+
    (((i*h)^4)/24)*dddx(x0,y0,z0,t0)+(((i*h)^5)/120)*ddddx(x0,y0,z0,t0);+
    (((i*h)^6)/720)*dddddx(x0,y0,z0,t0);
f(i)=g(x(i),y(i),z(i),t(i));
end
ypp1=t0+h*((95/288)*f0+(1427/1440)*f(1)-(133/240)*f(2)+(241/720)*f(3)-
    (173/1440)*f(4)+(3/160)*f(5));
ypp2=t0+h*((14/45)*f0+(43/30)*f(1)+(7/45)*f(2)+(7/45)*f(3)-(1/15)*f(4)+
```

```
(1/90)*f(5));
ypp3=t0+h*((51/160)*f0+(219/160)*f(1)+(57/80)*f(2)+(57/80)*f(3)-
(21/160)*f(4)+(3/160)*f(5));
ypp4=t0+h*((14/45)*f0+(64/45)*f(1)+(8/15)*f(2)+(64/45)*f(3)+(14/45)*f(4));
ypp5=t0+h*((95/288)*f0+(125/96)*f(1)+(125/144)*f(2)+(125/144)*f(3)+
(125/96)*f(4)+(95/288)*f(5));
yp1=z0+h*t0+(h^2)*((1231/5040)*f0+(863/2016)*f(1)-(761/2520)*f(2)+
(941/5040)*f(3)-(341/5040)*f(4)+(107/10080)*f(5));
yp2=z0+2*h*t0+(h^2)*((71/126)*f0+(544/315)*f(1)-(37/63)*f(2)+
(136/315)*f(3)-(101/630)*f(4)+(8/315)*f(5));
yp3=z0+3*h*t0+(h^2)*((123/140)*f0+(3501/1120)*f(1)-(9/140)*f(2)+
(87/112)*f(3)-(9/35)*f(4)+(9/224)*f(5));
yp4=z0+4*h*t0+(h^2)*((376/315)*f0+(1424/315)*f(1)+(176/315)*f(2)+
(608/315)*f(3)-(16/63)*f(4)+(16/315)*f(5));
yp5=z0+5*h*t0+(h^2)*((1525/1008)*f0+(11875/2016)*f(1)+(625/504)*f(2)+
(3125/1008)*f(3)+(625/1008)*f(4)+(275/2016)*f(5));
fr1=g(x(1),y(1),yp1,ypp1);
fr2=g(x(2),y(2),yp2,ypp2);
fr3=g(x(3),y(3),yp3,ypp3);
fr4=g(x(4),y(4),yp4,ypp4);
fr5=g(x(5),y(5),yp5,ypp5);
yr1=y0+h*z0+(1/2)*(h^2)*t0+(h^3)*((3929/40320)*f0+(995/8064)*fr1-
(1931/20160)*fr2+(173/2880)*fr3-(883/40320)*fr4+(139/40320)*fr5);
m1=toc;
err1=abs(gr(x(1))-yr1);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(1),gr(x(1)),yr1,err1,m1)
yr2=y0+2*h*z0+(2)*(h^2)*t0+(h^3)*((317/630)*f0+(367/315)*fr1-
(38/63)*fr2+(122/315)*fr3-(89/630)*fr4+(1/45)*fr5);
m2=toc;
err2=abs(gr(x(2))-yr2);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(2),gr(x(2)),yr2,err2,m2)
yr3=y0+3*h*z0+(9/2)*(h^2)*t0+(h^3)*((783/640)*f0+(16119/4480)*fr1-
(2187/2240)*fr2+(423/448)*fr3-(1539/4480)*fr4+(243/4480)*fr5);
```

```
m3=toc;
err3=abs(gr(x(3))-yr3);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(3),gr(x(3)),yr3,err3,m3)
yr4=y0+4*h*z0+(8)*(h^2)*t0+(h^3)*((712/315)*f0+(2336/315)*fr1-
    (32/45)*fr2+(704/315)*fr3-(40/63)*fr4+(32/315)*fr5);
m4=toc;
err4=abs(gr(x(4))-yr4);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(4),gr(x(4)),yr4,err4,m4)
yr5=y0+5*h*z0+(25/2)*(h^2)*t0+(h^3)*((29125/8064)*f0+(101875/8064)*fr1+
    (625/4032)*fr2+(19375/4032)*fr3-(625/1152)*fr4+(1375/8064)*fr5);
m5=toc;
err5=abs(gr(x(5))-yr5);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(5),gr(x(5)),yr5,err5,m5)
x0=x(5); y0=yr5; z0=yp5; t0=ypp5;
end
end
```

APPENDIX 4

Programme for the solution of the Block Predictor-corrector

PROG. 4: Programme to solve O.D.Es with step length of five for the solution of

$y''' = f(x, y, y', y'')$

% initial conditions

$x0 = 0; y0 = 0; z0 = 1; t0 = 2; h = 01$;

% y' is represented by z, y=t

disp('X-value Exact solution Computed solution Error Time/s')

tic;

$for j = 1 : 1 : 10$;

$for k = 1 : 1 : 10;$

$for i = 1 : 5;$

$f0 = g(x0, y0, z0, t0);$

$x(i) = x0 + i * h;$

```
y(i)=y0+(i*h)*z0+(((i*h)^2)/2)*t0+(((i*h)^3)/6)*f0+(((i*h)^4)/24)*dx(x0,y0,z0,t0)+
        (((i*h)^5)/120)*ddx(x0,y0,z0,t0)+(((i*h)^6)/720)*dddx(x0,y0,z0,t0)+
         (((i*h)^7)/5040)*ddddx(x0,y0,z0,t0);%+(((i*h)^8)/40320)*dddddx(x0,y0,z0,t0);
z(i)=z0+(i*h)*t0+(((i*h)^2)/2)*f0+(((i*h)^3)/6)*dx(x0,y0,z0,t0)+
             (((i*h)^4)/24)*ddx(x0,y0,z0,t0)+(((i*h)^5)/120)*dddx(x0,y0,z0,t0)+
              (((i*h)^6)/720)*ddddx(x0,y0,z0,t0);%+(((i*h)^7)/5040)*dddddx(x0,y0,z0,t0);
t(i)=t0+(i*h)*f0+(((i*h)^2)/2)*dx(x0,y0,z0,t0)+(((i*h)^3)/6)*ddx(x0,y0,z0,t0)+
            (((i*h)^4)/24)*dddx(x0,y0,z0,t0)+(((i*h)^5)/120)*ddddx(x0,y0,z0,t0);%+
             (((i*h)^6)/720)*dddddx(x0,y0,z0,t0);
f(i)=g(x(i),y(i),z(i),t(i));
end
ypp1=t0+h*((95/288)*f0+(1427/1440)*f(1)-(133/240)*f(2)+(241/720)*f(3)-
           (173/1440)*f(4)+(3/160)*f(5));
ypp2=t0+h*((14/45)*f0+(43/30)*f(1)+(7/45)*f(2)+(7/45)*f(3)-(1/15)*f(4)+
```

```
        (1/90)*f(5));
ypp3=t0+h*((51/160)*f0+(219/160)*f(1)+(57/80)*f(2)+(57/80)*f(3)-
        (21/160)*f(4)+(3/160)*f(5));
ypp4=t0+h*((14/45)*f0+(64/45)*f(1)+(8/15)*f(2)+(64/45)*f(3)+(14/45)*f(4));
ypp5=t0+h*((95/288)*f0+(125/96)*f(1)+(125/144)*f(2)+(125/144)*f(3)+
       (125/96)*f(4)+(95/288)*f(5));
yp1=z0+h*t0+(h^2)*((1231/5040)*f0+(863/2016)*f(1)-(761/2520)*f(2)+
        (941/5040)*f(3)-(341/5040)*f(4)+(107/10080)*f(5));
yp2=z0+2*h*t0+(h^2)*((71/126)*f0+(544/315)*f(1)-(37/63)*f(2)+(136/315)*f(3)-
       (101/630)*f(4)+(8/315)*f(5));
yp3=z0+3*h*t0+(h^2)*((123/140)*f0+(3501/1120)*f(1)-(9/140)*f(2)+
         (87/112)*f(3)-(9/35)*f(4)+(9/224)*f(5));
yp4=z0+4*h*t0+(h^2)*((376/315)*f0+(1424/315)*f(1)+(176/315)*f(2)+
        (608/315)*f(3)-(16/63)*f(4)+(16/315)*f(5));
yp5=z0+5*h*t0+(h^2)*((1525/1008)*f0+(11875/2016)*f(1)+(625/504)*f(2)+
         (3125/1008)*f(3)+(625/1008)*f(4)+(275/2016)*f(5));
fr1=g(x(1),y(1),yp1,ypp1);
fr2=g(x(2),y(2),yp2,ypp2);
fr3=g(x(3),y(3),yp3,ypp3);
fr4=g(x(4),y(4),yp4,ypp4);
fr5=g(x(5),y(5),yp5,ypp5);
yr1=y0+h*z0+(1/2)*(h^2)*t0+(h^3)*((3929/40320)*f0+(995/8064)*fr1-
       (1931/20160)*fr2+(173/2880)*fr3-(883/40320)*fr4+(139/40320)*fr5);
yr2=y0+2*h*z0+(2)*(h^2)*t0+(h^3)*((317/630)*f0+(367/315)*fr1-(38/63)*fr2+
       (122/315)*fr3-(89/630)*fr4+(1/45)*fr5);
yr3=y0+3*h*z0+(9/2)*(h^2)*t0+(h^3)*((783/640)*f0+(16119/4480)*fr1-
       (2187/2240)*fr2+(423/448)*fr3-(1539/4480)*fr4+(243/4480)*fr5);
yr4=y0+4*h*z0+(8)*(h^2)*t0+(h^3)*((712/315)*f0+(2336/315)*fr1-(32/45)*fr2+
         (704/315)*fr3-(40/63)*fr4+(32/315)*fr5);
yr5=y0+5*h*z0+(25/2)*(h^2)*t0+(h^3)*((29125/8064)*f0+(101875/8064)*fr1+
         (625/4032)*fr2+(19375/4032)*fr3-(625/1152)*fr4+(1375/8064)*fr5);
yrr5=y0-(29/31)*yr1-(68/31)*yr2+(68/31)*yr3+(29/31)*yr4+(h^3)*((21/2480)*f0+
```

```
(1177/2480)*fr1+(1921/1240)*fr2+(1921/1240)*fr3+(1177/2480)*fr4+(21/2480)*fr5);
m5=toc;
err5=abs(gr(x(5))-yrr5);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(5),gr(x(5)),yrr5,err5,m5)
x0=x(1); y0=yr1; z0=yp1; t0=ypp1;
end
end
```

APPENDIX 5

Programme for the solution of the Block Predictor-Block corrector

PROG. 5: Programme to solve O.D.Es with step length of five for the solution of $y''' = f(x, y, y', y'')$

```
clear
% initial conditions
x0 = 0; y0 = 0; z0 = 1; t0 = 2; h = 01;
% y' is represented by z, y=t
disp( 'X-value Exact solution Computed solution Error Time/s')
tic;
for j = 1 : 1 : 5;
for k = 1 : 1 : 4;
for i = 1 : 5;
f0 = g(x0, y0, z0, t0);
x(i) = x0 + i*h;
   y(i)=y0+(i*h)*z0+(((i*h)^2)/2)*t0+(((i*h)^3)/6)*f0+(((i*h)^4)/24)*dx(x0,y0,z0,t0)+
        (((i*h)^5)/120)*ddx(x0,y0,z0,t0)+(((i*h)^6)/720)*dddx(x0,y0,z0,t0)+
        (((i*h)^7)/5040)*ddddx(x0,y0,z0,t0)+(((i*h)^8)/40320)*dddddx(x0,y0,z0,t0);
z(i)=z0+(i*h)*t0+(((i*h)^2)/2)*f0+(((i*h)^3)/6)*dx(x0,y0,z0,t0)+
          (((i*h)^4)/24)*ddx(x0,y0,z0,t0)+(((i*h)^5)/120)*dddx(x0,y0,z0,t0)+
          (((i*h)^6)/720)*ddddx(x0,y0,z0,t0)+(((i*h)^7)/5040)*dddddx(x0,y0,z0,t0);
t(i)=t0+(i*h)*f0+(((i*h)^2)/2)*dx(x0,y0,z0,t0)+(((i*h)^3)/6)*ddx(x0,y0,z0,t0)+
          (((i*h)^4)/24)*dddx(x0,y0,z0,t0)+(((i*h)^5)/120)*ddddx(x0,y0,z0,t0)+
          (((i*h)^6)/720)*dddddx(x0,y0,z0,t0);
f(i)=g(x(i),y(i),z(i),t(i));
end
ypp1=t0+h*((95/288)*f0+(1427/1440)*f(1)-(133/240)*f(2)+(241/720)*f(3)-
        (173/1440)*f(4)+(3/160)*f(5));
```

```
ypp2=t0+h*((14/45)*f0+(43/30)*f(1)+(7/45)*f(2)+(7/45)*f(3)-(1/15)*f(4)+
    (1/90)*f(5));
ypp3=t0+h*((51/160)*f0+(219/160)*f(1)+(57/80)*f(2)+(57/80)*f(3)-
    (21/160)*f(4)+(3/160)*f(5));
ypp4=t0+h*((14/45)*f0+(64/45)*f(1)+(8/15)*f(2)+(64/45)*f(3)+(14/45)*f(4));
ypp5=t0+h*((95/288)*f0+(125/96)*f(1)+(125/144)*f(2)+(125/144)*f(3)+
    (125/96)*f(4)+(95/288)*f(5));
yp1=z0+h*t0+(h^2)*((1231/5040)*f0+(863/2016)*f(1)-(761/2520)*f(2)+
    (941/5040)*f(3)-(341/5040)*f(4)+(107/10080)*f(5));
yp2=z0+2*h*t0+(h^2)*((71/126)*f0+(544/315)*f(1)-(37/63)*f(2)+
    (136/315)*f(3)-(101/630)*f(4)+(8/315)*f(5));
yp3=z0+3*h*t0+(h^2)*((123/140)*f0+(3501/1120)*f(1)-(9/140)*f(2)+
    (87/112)*f(3)-(9/35)*f(4)+(9/224)*f(5));
yp4=z0+4*h*t0+(h^2)*((376/315)*f0+(1424/315)*f(1)+(176/315)*f(2)+
    (608/315)*f(3)-(16/63)*f(4)+(16/315)*f(5));
yp5=z0+5*h*t0+(h^2)*((1525/1008)*f0+(11875/2016)*f(1)+(625/504)*f(2)+
    (3125/1008)*f(3)+(625/1008)*f(4)+(275/2016)*f(5));
fr1=g(x(1),y(1),yp1,ypp1);
fr2=g(x(2),y(2),yp2,ypp2);
fr3=g(x(3),y(3),yp3,ypp3);
fr4=g(x(4),y(4),yp4,ypp4);
fr5=g(x(5),y(5),yp5,ypp5);
yr1=y0+h*z0+(1/2)*(h^2)*t0+(h^3)*((3929/40320)*f0+(995/8064)*fr1-
    (1931/20160)*fr2+(173/2880)*fr3-(883/40320)*fr4+(139/40320)*fr5);
yr2=y0+2*h*z0+(2)*(h^2)*t0+(h^3)*((317/630)*f0+(367/315)*fr1-(38/63)*fr2+
    (122/315)*fr3-(89/630)*fr4+(1/45)*fr5);
yr3=y0+3*h*z0+(9/2)*(h^2)*t0+(h^3)*((783/640)*f0+(16119/4480)*fr1-
    (2187/2240)*fr2+(423/448)*fr3-(1539/4480)*fr4+(243/4480)*fr5);
yr4=y0+4*h*z0+(8)*(h^2)*t0+(h^3)*((712/315)*f0+(2336/315)*fr1-
    (32/45)*fr2+(704/315)*fr3-(40/63)*fr4+(32/315)*fr5);
yr5=y0+5*h*z0+(25/2)*(h^2)*t0+(h^3)*((29125/8064)*f0+(101875/8064)*fr1+
    (625/4032)*fr2+(19375/4032)*fr3-(625/1152)*fr4+(1375/8064)*fr5);
frr1=g(x(1),yr1,yp1,ypp1);
```

```
frr2=g(x(2),yr2,yp2,ypp2);
frr3=g(x(3),yr3,yp3,ypp3);
frr4=g(x(4),yr4,yp4,ypp4);
frr5=g(x(5),yr5,yp5,ypp5);
yss1=(1/(h^2))*((419/12)*y0-(13213/186)*yr1+(143/31)*yr2+(11945/186)*yr3-
    (12169/372)*yr4+(h^3)*((37711/124992)*f0+(3313949/208320)*frr1+
    (10893919/312480)*frr2+(4777247/312480)*frr3+(3715/13888)*frr4+
    (571/624960)*frr5));
yss2=(1/(h^2))*((-61/12)*y0+(1120/93)*yr1-(287/62)*yr2-(608/93)*yr3+
    (1565/372)*yr4+(h^3)*((-1717/39060)*f0-(22807/9765)*frr1-
    (15866/3255)*frr2-(19574/9765)*frr3-(247/7812)*frr4-(1/3255)*frr5));
yss3=(1/(h^2))*((-61/12)*y0+(1667/186)*yr1+(143/31)*yr2-(2935/186)*yr3+
    (2711/372)*yr4+(h^3)*((-8903/208320)*f0-(1487977/624960)*frr1-
    (1834433/312480)*frr2-(20837/6944)*frr3-(48089/624960)*frr4+
    (571/624960)*frr5));
yss4=(1/(h^2))*((419/12)*y0-(6080/93)*yr1-(767/62)*yr2+(7552/93)*yr3-
    (14275/372)*yr4+(h^3)*((11513/39060)*f0+(53069/3255)*frr1+
    (378926/9765)*frr2+(37556/1953)*frr3+(2879/4340)*frr4-(59/9765)*frr5));
yss5=(1/(h^2))*((-2653/12)*y0+(78947/186)*yr1+(1679/31)*yr2-
    (92503/186)*yr3+(89207/372)*yr4+(h^3)*((-1166197/624960)*f0-
    (64461097/624960)*frr1-(25086091/104160)*frr2-(6997805/62496)*frr3-
    (212729/624960)*frr4+(57833/208320)*frr5));
ys1=(1/h)*((-197/20)*y0+(8752/465)*yr1+(21/155)*yr2-(2842/155)*yr3+
    (3433/372)*yr4+(h^3)*((-88541/1041600)*f0-(1568843/347200)*frr1-
    (5081257/520800)*frr2-(320591/74400)*frr3-(26283/347200)*frr4-
    (253/1041600)*frr5));
ys2=(1/h)*((113/60)*y0-(64/15)*yr1+(0)*yr2+(64/15)*yr3-(113/60)*yr4+
    (h^3)*((17/1050)*f0+(152/175)*frr1+(961/525)*frr2+(152/175)*frr3+
    (17/1050)*frr4-(0)*frr5));
ys3=(1/h)*((-113/60)*y0+(113/31)*yr1-(21/155)*yr2-(1921/465)*yr3+
    (1553/620)*yr4+(h^3)*((-791/49600)*f0-(914143/1041600)*frr1-
    (1067159/520800)*frr2-(188733/173600)*frr3-(26303/1041600)*frr4-
    (253/1041600)*frr5));
```

```
ys4=(1/h)*((197/20)*y0-(8576/465)*yr1-(507/155)*yr2+(3328/155)*yr3-
        (17869/1860)*yr4+(h^3)*((2707/32550)*f0+(988/217)*frr1+(5092/465)*frr2+
           (17912/3255)*frr3+(339/2170)*frr4-(26/16275)*frr5));
ys5=(1/h)*((-695/12)*y0+(52238/465)*yr1+(1857/155)*yr2-(60596/465)*yr3+
           (118873/1860)*yr4+(h^3)*((-509161/1041600)*f0-(28134557/1041600)*frr1-
           (32579461/520800)*frr2-(14624261/520800)*frr3+(28983/49600)*frr4+
           (20413/347200)*frr5));
fss1=g(x(1),yr1,ys1,yss1);
fss2=g(x(2),yr2,ys2,yss2);
fss3=g(x(3),yr3,ys3,yss3);
fss4=g(x(4),yr4,ys4,yss4);
fss5=g(x(5),yr5,ys5,yss5);
yrr1=y0+(20041/29850)*h*z0+(9809/29850)*h*yp1+(2558/14925)*(h^2)*t0+
            (h^3)*((10340809/601776000)*f0-(297137/17193600)*fss1+
             (1038421/300888000)*fss2-(386363/300888000)*fss3+
             (200701/601776000)*fss4-(24551/601776000)*fss5);
m1=toc;
err1=abs(gr(x(1))-yrr1);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(1),gr(x(1)),yrr1,err1,m1)

yrr2=y0-(1574/14925)*h*z0+(31424/14925)*h*yp1-(1574/14925)*(h^2)*t0+
          (h^3)*((-104143/9402750)*f0+(248029/940275)*fss1+(153458/4701375)*fss2-
           (27274/4701375)*fss3+(1589/1343250)*fss4-(599/4701375)*fss5);
m2=toc;
err2=abs(gr(x(2))-yrr2);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(2),gr(x(2)),yrr2,err2,m2)
yrr3=y0-(21909/9950)*h*z0+(51759/9950)*h*yp1-(3492/4975)*(h^2)*t0+
         (h^3)*((-1049949/22288000)*f0+(6112179/4457600)*fss1+
          (6625719/11144000)*fss2-(301257/11144000)*fss3+
           (187839/22288000)*fss4-(21789/22288000)*fss5);
m3=toc;
err3=abs(gr(x(3))-yrr3);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(3),gr(x(3)),yrr3,err3,m3)
```

```
yrr4=y0-(85708/14925)*h*z0+(145408/14925)*h*yp1-(26008/14925)*(h^2)*t0+
    (h^3)*((-80104/671625)*f0+(3051488/940275)*fss1+(10488736/4701375)*fss2+
    (1955392/4701375)*fss3+(114008/4701375)*fss4-(8608/4701375)*fss5);
m4=toc;
err4=abs(gr(x(4))-yrr4);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(4),gr(x(4)),yrr4,err4,m4)
yrr5=y0-(12655/1194)*h*z0+(18625/1194)*h*yp1-(1850/597)*(h^2)*t0+
    (h^3)*((-136325/687744)*f0+(28672625/4814208)*fss1+
     (11712025/2407104)*fss2+(4556425/2407104)*fss3+
       (2469025/4814208)*fss4+(23725/4814208)*fss5);
m5=toc;
err5=abs(gr(x(5))-yrr5);
fprintf('%2.4f %3.16f %3.16f %1.6e %2.4f \n',x(5),gr(x(5)),yrr5,err5,m5)
x0=x(5);y0=yr5;z0=yp5;t0=ypp5;
end
end*
```

APPENDIX 6 (ABSTRACTS OF PUBLICATIONS FROM THIS THESIS)

***American Journal of Computational Mathematics*, 2012, 2, 341-344**
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(http://www.SciRP.org/journal/ajcm)

A New Block-Predictor Corrector Algorithm for the Solution of $y''' = f(x, y, y', y'')$

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ABSTRACT

We consider direct solution to third order ordinary differential equations in this paper. Method of collection and interpolation of the power series approximant of single variable is considered to derive a linear multistep method (LMM) with continuous coefficient. Block method was later adopted to generate the independent solution at selected grid points. The properties of the block viz: order, zero stability and stability region are investigated. Our method was tested on third order ordinary differential equation and found to give better result when compared with existing methods.

Keywords: Collection; Interpolation; Power Series; Approximant; Linear Multistep; Continuous Coefficient; Block Method

APPENDIX 7

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Four Steps Continuous Method for the Solution of $y'' = f(x, y, y')$

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ABSTRACT

This paper proposes a continuous block method for the solution of second order ordinary differential equation. Colloca-tion and interpolation of the power series approximate solution are adopted to derive a continuous implicit linear multistep method. Continuous block method is used to derive the independent solution which is evaluated at selected grid points to generate the discrete block method. The order, consistency, zero stability and stability region are investi-gated. The new method was found to compare favourably with the existing methods in term of accuracy.
Keywords: Predictor; Corrector; Collocation; Interpolation; Power Series Approximant;

Continuous Block Method

APPENDIX 8

Four steps Block Predictor- Block Corrector Method for the solution of $y'' = f(x, y, y')$

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ABSTRACT
A method of collocation and interpolation of the power series approximate solution at some selected grid points is considered to generate a continuous linear multistep method with constant step size. Predictor-corrector method was adopted where the predictors and the correctors considered two and three interpolation points implemented in block method respectively. The efficiency of the proposed method was tested on some numerical examples and found to compete favorably with the existing methods.

Keywords: Collocation; interpolation; power series approximation; block method; step size; grid points; efficiency.

AMS Subject Classification (2010): 65L05, 65L06,65D30

APPENDIX 9

Five Steps Block Predictor-Block Corrector Method for the Solution of $y'' = f(x, y, y')$

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Abstract
Theory has it that increasing the step length improves the accuracy of a method. In order to affirm this we increased the step length of the concept in [1] by one to get $k = 5$. The technique of collocation and interpolation of the power series approximate solution at some selected grid points is considered so as to generate continuous linear multistep methods with constant step sizes. Two, three and four interpolation points are considered to generate the continuous predictor-corrector methods which are implemented in block method respectively. The proposed methods when tested on some numerical examples performed more efficiently than those of [1]. Interestingly the concept of self starting [2] and that of constant order are reaffirmed in our new methods.

Keywords
Step Length, Power Series, Block Predictor, Block Corrector, Constant Order, Step Size, Grid Points, Self Starting, Efficiency