PARALLEL ALGORITHM FOR SOLVING SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEMS

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Declaration

I hereby declare that this work is the product of my research effort undertaken under the supervision of Dr. Ma'aruf Shehu Minjibir and has not been presented anywhere for the award of a degree or certificate. All sources have been duly acknowledge.

Abdulhadi Mohd Hamza (SPS/12/MMT/00007) ——————————-

Certification

This is to certify that the research work for this dissertation and the subsequent write-up were carried out by Abdulhadi Mohammad Hamza (SPS/12/MMT/00007) under my supervision.

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Dedication

This work is dedicated **To My Entire Family.**

Contents

Abstract

In this research work, the notion of split variational inequality problem and split monotone variational inclusion problems were studied in a real Hilbert space. The problems entail finding a solution of one problem (Variational inequality problem (VIP) or monotone variational inclusion problem (MVIP)) the image of which under a given linear bounded transformation is a solution of another problem such as a Variational inequality problem (VIP) or Monotone variational inclusion problem (MVIP). A parallel algorithm for solving split monotone variational inclusion problem for the class of monotone map was formulated and analysed using product space approach in a real Hilbert space.

CHAPTER ONE

INTRODUCTION

1.1 Preamble

The notion of equilibrium is a central concept in numerous disciplines including economics, management science, operations research, and engineering. Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included systems of equations, optimization theory, complementarity theory, and fixed point theory. Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems. In general variational inequality problem can be formulated on any finite or infinite - dimensional Banach space. A *variational inequality problem* (VIP) in a Hilbert space H is of the form

$$(VIPH) \quad \begin{cases} \text{find } x^* \in C \text{ such that} \\ \langle f(x^*),\ y - x^* \rangle \geq 0, \forall y \in C, \end{cases} \tag{1.1.1}$$

where C is nonempty closed convex subset of H, and $f : C \longrightarrow H$ is a map .
In Banach space X, variational inequality problem can be expressed as

$$(VIPB) \quad \begin{cases} \text{find } x^* \in C,\ j(x - x^*) \in J(x - y) \text{ such that} \\ \langle f(x^*),\ j(x - x^*) \rangle \geq 0, \forall x \in C, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between the elements of X and those of its dual X^*, C a subset of a Banach space X, $f : C \longrightarrow X$ is a map and J the normalized duality map. The solution of VIP is represented by $SOL(C, f)$.

Variational inequality theory provides us with a simple, natural, general and unified framework for studying a wide class of unrelated problems arising in mechanics, physics, optimization, nonlinear programming, economics, transportation, equilibrium, and engineering sciences, for more details, see for example [3, 7, 8] and [10]-[13]. In recent years, variational inequalities have been extended and generalized in different directions using new, novel and innovative techniques, both for the sake of the applications. We present here examples of applications of variational inequality problem.

Example 1.1.1 *Consider the problem of finding the minimizer of a differentiable function f over a closed and bounded interval $I = [a,b]$. Let $x^* \in I$ be a minimizer of f. Then 3 cases can occur : $x^* = a$, $x^* = b$ or $a < x^* < b$.*

- *if $a < x^* < b$, then $f^{'}(x^*) = 0$.*

- *if $x^* = a$, then $f^{'}(x^*) \geq 0$.*

- *if $x^* = b$, then $f^{'}(x^*) \leq 0$.*

Each of these 3 cases satisfies the inequality $f^{'}(x^*)(y - x^*) \geq 0,\ \forall y \in I$. Therefore every minimizer is a solution of the variational inequality

$$\begin{cases} \text{find } x^* \in [a,b] \text{ such that} \\ \langle f^{'}(x^*),\ y - x^* \rangle \geq 0\ \forall y \in [a,b]. \end{cases}$$

Thus, the set of minimizers of f over $[a,b]$ is a subset of $SOL([a,b], f^{'})$, the solution set of the variational inequality.

Example 1.1.2 *Let f be a Gateaux differentiable real-valued function defined on closed convex subset K of euclidean N dimensional space. Again we shall give a link between VIP and the point $x_0 \in K$ such that*

$$f(x_0) = \min_{x \in K} f(x).$$

Assume x_0 is a point where the minimum is achieved and let $x \in K$. Since K is convex, the segment $(1-t)x_0 + tx = x_0 + t(x - x_0)$, $0 \le t \le 1$, lies in K. The function $\psi(t) = f(x_0 + t(x - x_0))$, $0 \le t \le 1$, attains its minimum at $t = 0$ as $f(x_0) = \min_{x \in K} f(x)$ and $\psi(0) = f(x_0)$. So $\frac{\psi(t)-\psi(0)}{t-0} \ge 0$, $\forall t \in (0,1]$. Thus taking limit as $t \longrightarrow 0^+$, we have

$$\psi'(0) = \nabla f(x_0)(x - x_0) \ge 0 \text{ for any } x \in K.$$

Consequently, the point x_0 satisfies the variational inequality

$$\begin{cases} \text{find } x^* \in K \text{ such that} \\ \langle \nabla f(x^*), (x - x^*) \rangle \ge 0 \;\; \forall\, x \in K. \end{cases}$$

Here too, the set of minimizers of f over K is a subset of $SOL(K, \nabla f)$, the solution set of the variational inequality.

A useful and significant generalization of variational inequalities is variational inclusion. Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in many fields of pure and applied sciences. A *variational inclusion problem* (VI) is of the form:

$$(VI) \qquad \begin{cases} \text{find } x^* \in C \text{ such that} \\ 0 \in f(x^*) + B(x^*), \end{cases} \tag{1.1.2}$$

where C is a closed convex nonempty subset of a real Hilbert space H, $f : C \longrightarrow H$ is a map and $B : H \longrightarrow 2^H$ is a set-valued map.

Essentially using the resolvent technique, one can show that variational inclusions are equivalent to fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving variational inclusions and related optimization problems, see for example [14, 15].

In an excellent paper [2], Censor, Gibali and Reich introduced the following split variational inequality problem (SVIP) in Hilbert space as

$$(SVIP) \qquad \begin{cases} \text{find } x^* \in C \text{ such that } \ \langle f(x^*),\ y - x^* \rangle \geq 0, \forall y \in C \\ y^* = Ax^* \in Q \text{ such that } \ \langle g(y^*),\ z - y^* \rangle \geq 0, \forall z \in Q, \end{cases} \tag{1.1.3}$$

where C is a closed convex subset of a real Hilbert space H_1, Q a closed convex subset of a real Hilbert space H_2, $A : H_1 \longrightarrow H_2$ a bounded linear map and $f : H_1 \longrightarrow H_1$, $g : H_2 \longrightarrow H_2$ are two given maps. When looked at separately, the equations above are a pair of classical variational inequality problems. The SVIP is quite general and permits split minimization between two spaces so that the image of a minimizer of a given function, under a bounded linear map, is a minimizer of another function. Other special cases are, for instance, split zero problems and the split feasibility problem (SFP) which have already been used in practice as a model in the intensity-modulation radiation therapy (IMRT) treatment planning. For more on this see [21]- [23], [27]. This formalism is also at the core of the modeling of many inverse problems and has been used to model significant real-life problems. For instance, in sensor networks, in computerized tomography and data compression, see, for example, [25, 26].

Moudafi in [1] was the first to give a generalization of split variational inequality problem where he introduced split monotone variational inclusion problem (SMVIP) as follows:

$$(SMVIP) \qquad \begin{cases} \text{find } x^* \in H_1 \text{ such that } \quad 0 \in f(x^*) + B_1(x^*) \\ \text{and } y^* = Ax^* \in H_2 \text{ such that } \ 0 \in g(y^*) + B_2(y^*), \end{cases} \tag{1.1.4}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ are two given maps and $B_i : H_i \longrightarrow 2^{H_i}$ are maximal monotone maps, $i = 1, 2$.

In this research we focus on split monotone variational inclusion problem for which we develop a parallel algorithm for split monotone variational inclusion as hinted by Moudafi in [1].

1.2 Research Problem

In [20] Censor *et al.* discussed parallel algorithm for split common fixed piont problem. Their technique was used by Censor *et al.* in [2] to develop a parallel algorithm which generates a sequence that converges weakly to a common solution of multi-set split variational inequality problem (SVIP). Moudafi in [1] provided a generalization of SVIP by considering split monotone variational inclusion problem (SMVIP). He used same algorithm of Censor *et al.* and proved that the sequence generated by the algorithm converges weakly to a solution of SMVIP. Our research interest is the following:

To develope a parallel algorithm for a multi-set split monotone variational incluson problem (MSSMVIP) and prove that the sequence generated by the algorithm converges weakly to a common solution of the MSSMVIP.

1.3 Aim and objectives

The aim of this work is to develop a parallel algorithm for the SMVI which give a generalize multi-sets split monotone variatonal inclusion problem and prove the convergence of the sequence generated by the algorithm. We also elaborate on the work of Moudafi in [1] and that of Censor *et al.* in [2] which contain the origin of our work.

1.4 Scope and limitation

In this work, parallel algorithm for the MSSMVIP is developed in real Hilbert space. However, only weak convergence has been obtained in infinite dimensonal real Hilbert spaces.

1.5 Research Methodology

Our methodology was based on a split monotone variational inclusion problem for the class of monotone map using product space approach in a real Hilbert space. The papers [1, 2] and [20] were duly acknowledged in this work.

1.6 Definition of Some Basic Terms

Here, we give basic definitions of some terms which are used in our work. These definitions are standard and can be found in, for example, [61].

Definition 1.6.1 (Nonexpansive map) *Let H be Hilbert space and let $T : H \longrightarrow H$ be a map. T is said to be nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H.$$

Definition 1.6.2 (Lipschitz continuous map) *Let H be Hilbert space and let $T : H \longrightarrow H$ be a map. T is called Lipschitz continuous on C subset of H with constant $L > 0$ if*

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

Definition 1.6.3 (Firmly nonexpansive map) *Let H be Hilbert space and let $T : H \longrightarrow H$ be a map. T is called firmly nonexpansive if*

$$\langle Tx - Ty,\ x - y\rangle \geq \|Tx - Ty\|^2 \text{ for all } x, y \in H.$$

Definition 1.6.4 (Averaged map) *Let H be Hilbert space and let $T : H \longrightarrow H$ be a map. Then T is said to be averaged if it can be written as the average of the identity map and a nonexpansive map, i.e.,*

$$T = (1 - \alpha)I + \alpha N,$$

where $\alpha \in (0, 1)$ and $N : H \longrightarrow H$ is a nonexpansive map.

Definition 1.6.5 (Monotone map) *Let H be Hilbert space and let $T : H \longrightarrow H$ be a map. Then T is monotone if*

$$\text{for all } x, y \in H, \quad \langle T(x) - T(y),\ x - y\rangle \geq 0.$$

The notion of monotonicity is extended to multi-valued maps as follows: $T : H \longrightarrow 2^H$ *is called monotone if*

$$\text{for all } x,y \in H,\ \langle u-v,\ x-y\rangle \geq 0\ \ \forall u \in T(x),\ \ \forall v \in T(y).$$

Definition 1.6.6 (α- Inverse strongly monotone map ($\alpha - ism$)) *Let H be Hilbert space and let* $T : H \longrightarrow H$ *be a map.* T *is called* α*- inverse strongly monotone if* $\alpha > 0$

$$\langle Tx - Ty,\ x-y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \text{for all } x,y \in H.$$

Remark 1.6.7 *Every* α*-ism is* $\frac{1}{\alpha}$*- Lipschitz. Indeed,* T *is* α*-ism implies*

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &\leq \langle Tx - Ty, x-y\rangle\ \ \forall x,y \in H\\ &\leq \|Tx - Ty\|\|x-y\|\ \ \forall x,y \in H\\ &\leq \|x-y\| \quad \forall x,y \in H.\end{aligned}$$

So

$$\|Tx - Ty\| \leq \frac{1}{\alpha}\|x-y\|, \text{ for all } x,y \in H.$$

Definition 1.6.8 (Maximal monotone map) *Let H be Hilbert space and let* $T : H \longrightarrow 2^H$ *be a multi-valued map.* T *is called maximal monotone map if* T *is monotone, i.e.,*

$$\text{for all } x,y \in H, \quad \langle u-v,\ x-y\rangle \geq 0 \quad \forall u \in T(x),\ \forall v \in T(y)$$

and the graph $G(T)$ *of* T*,*

$$G(T) := \{(x,u) \in H \times H : u \in T(x)\},$$

is not properly contained in the graph of any other monotone map.

Definition 1.6.9 (Resolvent map) *Let* H *be Hilbert space and let* $T : H \longrightarrow 2^H$ *be a maximal monotone map. Suppose* I *is the identity map on* H*. The map* $J_\lambda^T : H \longrightarrow H$ *defined by*

$$J_\lambda^T = (I + \lambda T)^{-1}$$

is called a resolvent map of T, where $\lambda > 0$.

Remark 1.6.10 *(i) We recall that given any set-valued map $T; X \longrightarrow 2^Y$, its inverse exists as a set-valued map. In fact $T^{-1} : Y \longrightarrow 2^X$, the inverse map, is defined as $T^{-1}y = \{x \in X : y \in Tx\}$. Of course it is possible for $T^{-1}y$ to be empty for some $y \in Y$ just as Tx can be empty for some $x \in X$. However, for y in $R(T)$, the range of T, i.e., $R(T) = \bigcup_{x \in X} Tx$, $T^{-1}y \neq \emptyset$.*

(ii) We shall denote J_λ^T by J_λ when there is no confusion about the map T.

(iii) Althrough the map T is set-valued, its resolvent J_λ is single-valued. Moreover it is nonexpansive. We defer discussion on why J_λ^T is single-valued till in Chapter three of this dissertation.

Definition 1.6.11 (Adjoint map) *Let H be Hilbert space and let $A : H \longrightarrow H$ be a bounded linear map, A map $A^* : H \longrightarrow H$ defined by*

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \quad \forall x, y \in H,$$

is called the adjoint map of A.

Remark 1.6.12 *Riesz representation theorem guarantees the existence and uniqueness of the adjoint map.*

Definition 1.6.13 (Spectral radius) *Let H be a Hilbert space and $A : H \longrightarrow H$ be a bounded linear map. Spectral radius L of A is a supremum of the set of magnitudes of the elements in its spectrum, i.e.,*

$$L = sup\{|\lambda| : \ \lambda \in \sigma(T)\},$$

where $\sigma(T)$, the spectrum of T is defined as $\sigma(T) := \{\lambda \in \mathbb{C} : I - \lambda T \text{ is not bijective}\}$.

Remark 1.6.14 *The spectral radius $L \in \mathbb{R}$ since $\emptyset \neq \sigma(T) \subset B(0, \|A\|)$ (see, e.g.,[63]). We observe that the spectral radius of A is the smallest radius of the closed disc which contains $\sigma(T)$.*

Definition 1.6.15 (Fixed point) *Let $T : H \longrightarrow H$ be a map. The fixed point set of T is defined by*

$$Fix(T) := \{x \in H \mid T(x) = x\},$$

and if $T : H \longrightarrow 2^H$ is a multi-valued map, the fixed point set of T is defined by

$$Fix(T) := \{x \in H \mid x \in T(x)\}.$$

Definition 1.6.16 (Projection map) *Let H be a Hilbert space and K be a non-empty closed and convex subset of H. For $x \in H$, by projection of x on K, we mean the unique element $z \in K$ denoted by $P_K(x)$ such that*

$$\|x - P_K(x)\| \leq \|x - y\| \;\; \forall y \in K,$$

that is,

$$\|x - P_K(x)\| = min\{\|x - y\| : \;\; y \in K\}.$$

This immediately defines a map $P_K : H \longrightarrow K$. The map P_K is called the projection map on K.

Remark 1.6.17 *The existence and uniqueness of $P_K(x)$ $\forall x \in H$, is guaranteed in every Hilbert space (see [50], for example). Also, the projection has many properties which make it very useful. We mention some here.*

(i) A projection map is nonexpansive.

(ii) If K is a subspace, then P_K is linear and bounded.

Definition 1.6.18 (Normal cone map) *Let C be a nonempty closed and convex subset of H. The normal cone of C is a multi-valued map $N_C : H \longrightarrow 2^H$ defined by*

$$N_C(x) = \begin{cases} \{u \in H : \langle \, u \, , \, y \text{ - } x \, \rangle \leq 0 \; \forall y \in C\}, & if\, x \in C \\ \emptyset, & if\, x \notin C. \end{cases}$$

Remark 1.6.19 *N_C is maximal monotone. It follows from [62] the fact that N_C is nothing but the subdifferential of a convex, proper and lower semi continuous functional.*

Definition 1.6.20 *Let H be a Hilbert space K a closed and convex subset of H, and let $M : K \longrightarrow H$ be a map. Then M is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\}_{k=0}^{\infty}$ in K such that $x_k \rightharpoonup x^* \in K$ and $M(x_k) \longrightarrow y$, we have $M(x^*) = y$.*

Lemma 1.6.21 (Demiclosedness principle, see, e.g., [2]) *Let H be a Hilbert space and let K be a closed convex subset of H. Suppose $T : K \longrightarrow H$ is a nonexpansive map. Then $I - T$ (I the identity map on H) is demiclosed at $y \in H$.*

Remark 1.6.22 *The symbol "$\rightharpoonup$" indicates weak convergence and "$\longrightarrow$" denotes strong convergence in the definition above. In fact, throughout this dissertation, we use these notations with the same meanings.*

Definition 1.6.23 (Fejer-monotone) *Let H be a Hilbert space and let K be a nonempty closed and convex subset of H. A sequence $\{x_k\}_{k=1}^{\infty} \in H$ is Fejer-monotone with respect to K, if for every $u \in K$,*

$$\|x_{k+1} - u\| \leq \|x_k - u\| \ \ \forall k \geq 0, \text{ i.e.,}$$

the sequence $\{\|x_k - u\|\}_{k=0}^{\infty}$ is monotone decresing for every $u \in K$.

Definition 1.6.24 *Let X be a normed linear space. A sequence $\{x_n\}$ in X is said to be Convergent if $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for $n > N$ and some $x \in X$.*

Definition 1.6.25 *Let X be a normed linear space. A sequence $\{x_n\}$ in X is called weakly convergent in X, in symbols $x_n \rightharpoonup x$, if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$ for all $f \in X^*$ i.e., for $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that $\|f(x_n) - f(x)\| \leq \varepsilon$ for $n > N$ and $\forall f \in X^*$.*

CHAPTER TWO

LITERATURE REVIEW

2.1 Literature review

Numerous problems in mathematics and physical science can be cast as convex feasibility problem. Convex feasibility problem (CFP) is the problem of finding a point x^* satisfying

$$x^* \in \bigcap_{i=1}^{p} C_i, \tag{2.1.1}$$

where $p \geq 1$ is an integer and C_i is nonempty closed convex subset of real Hilbert space H for each $i \in \{1, 2, \cdots, p\}$. This problem was studied in [51] and it has a broad applicability in different disciplines. They include partial differential equations (Dirichlet problems), control theory (controlled lineal systems), image reconstruction and signal processing (computerized tomography) see, e.g., [7], [11], [13] . CFP is a special case of finding a common fixed point problem for nonlinear mappings:

$$x^* \in \bigcap_{i=1}^{p} Fix(T_i), \tag{2.1.2}$$

where each $T_i : H \longrightarrow H$ is a (nonlinear) mapping. Indeed, if we take $T_i = P_{C_i}$ the projection map from H onto C_i, then the CFP is reduced to a common fixed point problem. It is an interesting problem to find out for what kind of mappings T_i one can solve (2.1.2) iteratively (assuming existence of solutions). In the literature, there exists quite a lot of work for solving (2.1.2) for the class of nonexpansive mappings. In [51] Bregman proved that the sequence generated by method of alternating projection, i.e.,

$$\begin{cases} x_0 \in H \text{ arbitrary}, \\ x_{k+1} = \ (P_{C_p} P_{C_{p-1}} \ldots P_{C_1} \ )x_0, \ \forall k \geq 0, \end{cases} \tag{2.1.3}$$

converges weakly to a solution $x^* \in \bigcap_{i=1}^{p} C_i$ if $\bigcap_{i=1}^{p} C_i \neq \emptyset$.

A special case of CFP is the so-called split feasibility problem SFP initially introduced by Censor and Elfving in [12] , which can be mathematically formulated as

$$\begin{cases} \text{find } x^* \in C \text{ such that} \\ Ax^* \in Q, \end{cases} \tag{2.1.4}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, and A is an $M \times N$ matrix. Also, they proposed an algorithm for solving such problem as

$$x_{k+1} = A^{-1}(I + AA^T)^{-1}(AP_C(x_k) + AA^T P_Q(Ax_k)).$$

However, the algorithm involves the complicated computations of matrix inverses. A new iterative algorithm for solving the (SFP) problem was presented by Byrne in [29], namely CQ-algorithm, which is defined and considered by the following iterative scheme

$$\begin{cases} \text{select abitrary starting point } x_0 \in C, \\ x_{k+1} = P_C(x_k + \gamma A^T (P_Q - I)Ax_k), \end{cases}$$

P_C and P_Q denote the metric projections onto C and Q, respectively and he proved the convergence of sequences generated by this algorithm to a solution. Qu and Xiu in [52, 53] studied the CQ-method in which the metric projections are replaced by subgradient projections, and proved the convergence of the method in the consistent case. The split feasibility problem has been considered by many authors see, for example, [23], [28], [29]. It is worth mentioning that the split feasibility problem in finite dimensional Hilbert spaces is used in practice as a model in the intensity-modulation radiation therapy (IMRT) treatment planning, see [22], [19], [23]. In 2010, Xu in [30] extended the split feasibility problem to the case of infinite dimensional Hilbert spaces and proposed a modified algorithm called CQ-algorithm:

$$\begin{cases} \text{select abitrary starting point } x_0 \in H_1, \\ x_{k+1} = P_C(x_k + \gamma A^*(P_Q - I)Ax_k), \end{cases}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \longrightarrow H_2$ is a bounded linear map, and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. He proved that the iterative sequence above converges weakly to a solution of the split feasibility problem. Some versions of the method with an application of quasi-nonexpansive maps satisfying the demi-closedness principle were studied in 2010 and 2011 by Moudafi in [54, 55]. The split feasibility problem is a special case of the split common fixed point problem (SCFPP), introduced by Censor and Segal in [20]. The problem is to find a common fixed point of a finite family of maps defined on a real Hilbert space, whose image under a bounded linear transformation is a common fixed point of another family of maps defined on a real Hilbert space. The problem is also called the multiple-sets split feasibility problem (MSSFP) in [19]. The problem was studied by Censor *et al.* in 2005, by Xu, H.K, in 2006, by Reich, S and Masad, E, in 2007, by Censor,Y and Segal in 2009 and by Wang, F and Xu, H.K, in 2011, where various methods were proposed for solving it.

Related to CFP is the equilibrium problem theory which has emerged as an interesting branch of applied mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization and operational research in a general and unified way. The equilibrium problem for functional f is of the form

$$\begin{cases} \text{find } x^* \in C \text{ such that} \\ f(x^*, y) \geq 0, \ \forall y \in C, \end{cases} \tag{2.1.5}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $f : C \times C \longrightarrow \mathbb{R}$ is a real-valued functional with $f(x,x) = 0$ for all $x \in C$. The set of solutions of the equilibrium problem is denoted by $EP(f)$. Combettes and Hirstoaga in [31] introduced an iterative scheme of finding the best approximation to the initial data when $EP(f)$ is nonempty and proved a strong convergence theorem. The split equilibrium problem is of the form

$$\begin{cases} \text{find } x^* \in C \text{ such that } \ f_1(x^*,x) \geq 0, y \in C \ , \forall y \in C \\ y^* = Ax^* \in Q \text{ and } f_2(y^*,y) \geq 0, y \in C \ , \forall z \in Q, \end{cases} \tag{2.1.6}$$

where C and Q are closed convex subset of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f_1 : C \times C \longrightarrow \mathbb{R}$ and $f_2 : Q \times Q \longrightarrow \mathbb{R}$ are bifunctions.

A special case of equalibrium problem is the variational inequality problem on Hilbert spaces. Recall that variational inequality problem in Hilbert space (see, e,g, [16], [32], [33], [34]) has the form

$$(VIPH) \qquad \begin{cases} \text{find } x^* \in C \text{ such that} \\ \langle f(x^*),\ y - x^* \rangle \geq 0, \forall y \in C, \end{cases}$$

where C is nonempty closed convex subset H, and a map $f : C \longrightarrow H$. If we define $A : C \times C \longrightarrow \mathbb{R}$ by $A(x,y) = \langle f(x),\ y - x \rangle$, then we see that $x^* \in C$ is a solution of (VIPH) if and only if x^* is a solution of the equilibrium

$$\begin{cases} \text{find } x^* \in C \text{ such that} \\ A(x^*,y) \geq 0, \forall y \in C. \end{cases}$$

So far, this problem has been studied under a variety of settings. In 2003, W. Takahashi and M. Toyoda in [4] introduced an iteration process of finding common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for an inverse strongly monotone mapping. They proved the following theorem.

Theorem 2.1.1 *Let K be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$. Let A be an α- inverse strongly-monotone mapping of K into H, and let S be a nonexpansive mapping of K into itself such that $F(S) \cap VI(K,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the algorithm*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + (\alpha_n - 1) SP_K(x_n - \lambda_n A x_n), \end{cases}$$

for every n= 0, 1, 2, . . . , where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, the sequence $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(K,A)$, $z = \lim_{n\to\infty} P_{(F(S)\cap VI(C,A))} x_n$.

Also in 2006, N. Nadezhkina and W. Takahashi in [59] introduced an iterative process of finding common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz-continuous mapping. The iterative process is based on the so-called extragradient method which is given in the theorem below.

Theorem 2.1.2 *Let C be a closed convex subset of a real Hilbert space H. Let A be monotone k- Lipschitz-continuous mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the algorithm*

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (\alpha_n - 1) SP_C(x_n - \lambda_n A y_n), \end{cases}$$

for every n= 0, 1, 2, . . . , where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/k)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to $z \in F(S) \cap VI(C,A)$, where $z = \lim_{n\to\infty} P_{(F(S)\cap VI(C,A))} x_n$.

In 2010, Y. Censor, A. Gibali, S. Reich in [60] present a subgradient extragradient method for solving variational inequalities in Hilbert space. In addition, they propose a modified version of their algorithm that finds a solution of a variational inequality which is also a fixed point of a given nonexpansive mapping. The problem has been studied under a variety of settings. Some examples are also as follows:

(a) C is a Cartesian product and f is not monotone but such that for some $\lambda > 0$ the mapping $I - \lambda f$ is block-contractive, where I is the identity mapping of H see [35], [36], [37]. (b) C is not a Cartesian product but the intersection of the fixed point sets of nonexpansive mappings in H and f is strongly monotone and Lipschitz continuous see [38], [39]. The problem under the setting (a) includes important problems in network engineering, such as traffic assignment problem (see, e.g., [36], [37]) and power control problem (see ,e.g., [41]). For the setting (a), Pang [40] and Bertsekas and Tsitsiklis [36], [37] developed parallel algorithms by using the metric projection P_C of H onto C. However, in practice, it may be hard or expensive to compute P_C because the closed form expression of P_C is rarely known. On the other hand, to solve VIP under the setting (b), Yamada in [39] developed a remarkably simple algorithm named the *hybrid steepest descent method*, by extending ideas of Yamada et al in [42] and [43] and Deutsch and Yamada in [44]. The hybrid steepest descent method enables one to solve VIP by using computable nonexpansive mappings instead of P_C. (see e.g., [39], [45], [46], [47]) for some examples of such nonexpansive mappings.

Very recently, Censor, Gibali and Reich, in [2], introduced a concept of Split Variational Inequality Problem (SVIP) which is formulated as follows:

$$\begin{cases} \text{find } x^* \in C \text{ such that } \langle f(x^*), y - x^* \rangle \geq 0, \forall y \in C \\ y^* = Ax^* \in Q \text{ such that } \langle g(y^*), z - y^* \rangle \geq 0, \forall z \in Q, \end{cases} \tag{2.1.7}$$

where C and Q are closed convex subset of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ are two given maps. To solve (SVIP) problem, they proposed the following algorithm

$$\begin{cases} \text{select abitrary starting point } \mathrm{x_0 \in H_1}, \\ x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), k \geq 0, \end{cases}$$

where $\gamma \in \left(0, \frac{1}{L}\right)$, $\lambda > 0$ and $U = P_C(I - \lambda f)$ and $T = P_Q(I - \lambda g)$. Under some suitable conditions imposed upon the maps f and g, they proved the weak convergence of the sequence $\{x_k\}_{k=0}^{\infty}$ generated from the algorithm to a solution of the split variational inequality problem.

Moudafi in [1] introduced a generalization of SVIP as Split Monotone Variational Inclusion SMVI as follows:

$$\begin{cases} \text{find } x^* \in H_1 \text{ such that } \quad 0 \in f(x^*) + B_1(x^*) \\ \text{and } y^* = Ax^* \in H_2 \ \text{ such that } 0 \in \mathrm{g(y^*) + B_2(y^*)}, \end{cases} \tag{2.1.8}$$

where $A : H_1 \longrightarrow H_2$: is a bounded linear map, $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ are two given single-valued maps and $B_i : H_i \longrightarrow 2^{H_i}$ are set-valued maximal montone maps $i = 1, 2$.

Monotone maps appear in modern optimization and analysis (see e.g., [17, 18]). Due to their set-valuedness, there has always been considerable interest to describe and study monotone maps from many points of view. A key tool is the so-called resolvent map associated with a given monotone map. This resolvent is not only always single-valued, but also firmly nonexpansive (and thus Lipschitz continuous). Moreover, the resolvent has full domain precisely when the map is maximal monotone.

To see how split monotone variational inclusion problem (2.1.8) generalizes the (SVIP) we use the normal cone map. We recall that N_C is maximal monotone. Also

$$\begin{aligned} x^* \in SOL(C, f) &\Longleftrightarrow \langle f(x^*),\, x - x^* \rangle \geq 0, \forall x \in C \\ &\Longleftrightarrow -\langle f(x^*),\, x - x^* \rangle \leq 0, \forall x \in C \\ &\Longleftrightarrow \langle -f(x^*),\, x - x^* \rangle \leq 0, \forall x \in C \\ &\Longleftrightarrow -f(x^*) \in N_C(x^*) \quad (\text{from definition of } N_C) \\ &\Longleftrightarrow 0 \in f(x^*) + N_C(x^*). \end{aligned}$$

Thus $x^* \in SOL(C,f)$ if and only if $0 \in f(x^*) + N_C(x^*)$. Therefore the (SVIPH) can be written as

$$\begin{cases} \text{find } x^* \in C \text{ such that} \quad 0 \in f(x^*) + N_C(x^*) \\ \text{and } y^* = Ax^* \in Q \ \text{ such that } 0 \in \mathrm{g(y^*) + N_Q(y^*)} \end{cases} \tag{2.1.9}$$

which is a special case of the (SMVIP) with $B_1 = N_C$ and $B_2 = N_Q$. Moudafi in [1] proposed an iterative method for solving (SMVIP) as follows:

$$\begin{cases} \text{select abitrary starting point } \ \mathrm{x_0 \in H_1}, \\ x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \end{cases}$$

where $\gamma \in \left(0, \frac{1}{L}\right)$, $\lambda > 0$, $U = J_\lambda^{B_1}(I - \lambda f)$ and $T = J_\lambda^{B_2}(I - \lambda g)$. Under some suitable conditions imposed upon the maps f and g, he proved the weak convergennce of the generated sequence $\{x_k\}_{k=0}^{\infty}$ to a solution of split monotone variational inclusion problem.

In this dissertation, we give a parallel algorithm for the multi-set split monotone variational inclusion problem (MSSMVIP) as hinted by the author in [1] and prove weak convergence to a common solution of finitely family of (SMVIP) of the sequence generated therefrom.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, detailed accounts of the works in [1] and [2] are given. These two works contain the origin of the work in this dissertation. We first take the following preliminaries which will be used in this Chapter and sudsequently.

Lemma 3.1.1 (see, e.g., [2]) *Let K be a nonempty, closed and convex subset of Hilbert space H. For any $x \in H$, $z = P_K(x)$ if and only if*

$$\langle x - z,\ y - z \rangle \leq 0 \quad \forall y \in K$$

or

$$\langle x - P_K(x),\ y - P_K(x) \rangle \leq 0 \quad \forall y \in K. \tag{3.1.1}$$

Lemma 3.1.2 *Let H be a Hilbert space and let K be a nonempty, closed and convex subset of H. Suppose $f : C \longrightarrow H$ is a map. Then for any $x^* \in C$, $x^* \in SOL(K, f) \Longleftrightarrow x^* \in Fix(P_K(I - \lambda f))$.*

Proof

$$
\begin{aligned}
x^* \in SOL(K,f) &\iff \langle f(x^*),\, x-x^*\rangle \geq 0 \quad \forall x \in K\\
&\iff -\lambda\langle f(x^*),\, x-x^*\rangle \leq 0 \quad \forall x \in K\\
&\iff \langle -\lambda f(x^*),\, x-x^*\rangle \leq 0 \quad \forall x \in K\\
&\iff \langle x^* - \lambda f(x^*) - x^*,\, x-x^*\rangle \leq 0 \quad \forall x \in K\\
&\iff x^* = P_K(x^* - \lambda f(x^*)) \ \ (by\ Lemma\ 3.1.1)\\
&\iff x^* \in Fix(P_K(I-\lambda f).
\end{aligned}
$$

Lemma 3.1.3 (see, e.g., [2]) *Every nonexpansive map $T: H \longrightarrow H$ satisfies $\forall (x,y) \in H \times H$*

$$\langle (x-T(x)) - (y-T(y)),\, T(x)-T(y)\rangle \leq 1/2\|(T(x)-x)-(T(y)-y)\|^2.$$

Lemma 3.1.4 (see, e.g., [2]) *Let K be a nonempty, closed and convex subset of H and let $h: H \longrightarrow H$ be an α-ism map on H. If $\lambda \in [0,2\alpha]$ then the map $P_K(I-\lambda h)$ is nonexpansive on K. If, in addition for all $x^* \in SOL(K,h)$*

$$\langle h(x),\, P_K(I-\lambda h)(x) - x^*\rangle \geq 0, \quad for\ \ all \quad x \in H, \tag{3.1.2}$$

then the following hold:

(a) For all $x \in H$ and $q \in Fix(P_K(I-\lambda h))$,

$$\langle P_K(I-\lambda h)(x) - x,\, P_K(I-\lambda h)(x) - q\rangle \leq 0;$$

(b) For all $x \in H$ and $q \in Fix(P_K(I-\lambda h))$,

$$\|P_K(I-\lambda h)(x) - q\|^2 \leq \|x-q\|^2 - \|P_K(I-\lambda h)(x) - x\|^2.$$

Proof

Let $x, y \in H$. Then

$$\begin{aligned}
\|P_K(I-\lambda h)(x) - P_K(I-\lambda h)(y)\|^2 &= \|P_K(x-\lambda h(x)) - P_K(y-\lambda h(y))\|^2 \\
&\leq \|(x-\lambda h(x)) - (y-\lambda h(y))\|^2 \quad (P_K \text{ is nonexpansive}) \\
&= \|(x-y) - \lambda(h(x)-h(y))\|^2 \\
&= \|x-y\|^2 - 2\lambda\langle x-y, h(x)-h(y)\rangle \\
&\quad + \lambda^2\|(h(x)-h(y)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\alpha\|h(x)-h(y)\|^2 \\
&\quad + \lambda^2\|(h(x)-h(y)\|^2 \\
&= \|x-y\|^2 + \lambda(\lambda-2\alpha)\|h(x)-h(y)\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

The result follows.

(a) Let $x \in H$ and $q \in Fix(P_K(I-\lambda h))$. Then

$\langle P_K(x-\lambda h(x)) - x, P_K(x-\lambda h(x)) - q\rangle$

$$\begin{aligned}
&= \langle P_K(x-\lambda h(x)) - x + \lambda h(x) - \lambda h(x), P_K(x-\lambda h(x)) - q\rangle \\
&= \langle P_K(x-\lambda h(x)) - (x-\lambda h(x)), P_K(x-\lambda h(x)) - q\rangle \\
&\quad -\lambda\langle h(x), P_K(x-\lambda h(x)) - q\rangle \\
&\leq 0 \ \ (from\ (3.1.1),\ Lemma\ 3.1.2\ and\ (3.1.2)).
\end{aligned}$$

(b) Let $x \in H$ and $q \in Fix(P_K(I-\lambda h))$. Then

$$\begin{aligned}
&\|q-x\|^2 \\
&\quad = \|(P_K(I-\lambda h)(x) - x) - (P_K(I-\lambda h)(x) - q)\|^2 \\
&\quad = \|P_K(I-\lambda h)(x) - x\|^2 + \|P_K(I-\lambda h)(x) - q\|^2 \\
&\qquad - 2\langle P_K(I-\lambda h)(x) - x, P_K(I-\lambda h)(x) - q\rangle.
\end{aligned}$$

By (a), we get

$$-2\langle P_K(I-\lambda h)(x)-x, P_K(I-\lambda)h(x)-q\rangle \geq 0.$$

Thus,

$$\|q-x\|^2 \geq \|P_K(I-\lambda h)(x)-x\|^2+\|P_K(I-\lambda h)(x)-q\|^2$$

or

$$\|P_K(I-\lambda h)(x)-q\|^2 \leq \|q-x\|^2-\|P_K(I-\lambda h)(x)-x\|^2.$$

The result follows.

Lemma 3.1.5 (see, e.g., [64]) *Suppose the sequence* $\{x_k\}_{k=1}^{\infty}$ *is Fejer monotone with respect to K. Then* $\{x_k\}_{k=1}^{\infty}$ *has at most one weak cluster point in K. Consequently,* $\{x_k\}_{k=1}^{\infty}$ *converges weakly to some point in K if and only if all weak cluster points of* $\{x_k\}_{k=1}^{\infty}$ *lie in* K.

Lemma 3.1.6 (see, e.g., [62]) *Let* H *be a real Hilbert space and let* $A: H \longrightarrow 2^H$ *be a maximal monotone map. Then the range of* $(I+A)$ *is* H, *i.e.,* $R(I+A)=H$, *where I is the identity map of* H.

Remark 3.1.7 *Since for every maximal monotone map* A, λA *is also maximal monotone,* $\lambda > 0$, *we infer that for every maximal monotone map* $A: H \longrightarrow 2^H$, *the range of* $(I+\lambda A)$ *is* H, *i.e.,* $R(I+\lambda A)=H$.

Lemma 3.1.8 *Let* $A: D(A) \subset H \longrightarrow 2^H$ *be a maximal monotone map and* $J_\lambda = (I+\lambda A)^{-1}$ *be a resolvent map of* A. *Then*

(i) J_λ *is single-valued.*

(ii) J_λ *is nonexpansive.*

Proof

(i) A is maximal monotone implies $R(I+\lambda A)=H \ \ \forall\lambda > 0$. (By Remark 3.1.7)

Let $y \in H$. Then $y \in R(I+\lambda A)$. Therefore there exists $z \in D(A)$ such that $y \in (I+\lambda A)z$. So

$$z \in (I+\lambda A)^{-1}y = J_\lambda(y).$$

Thus

$$J_{\lambda.}(y) \neq \emptyset \ \ \forall y \in H.$$

We now show that $J_\lambda(y)$ is singleton for every $y \in H$. Let $u, v \in J_\lambda(y)$. Then $y \in (I+\lambda A)u$ and $\mathrm{y} \in (\mathrm{I}+\lambda \mathrm{A})\mathrm{v}$. Therefore,

$$y \in u+\lambda A(u) \ \text{ and } \ \mathrm{y} \in \mathrm{v}+\lambda \mathrm{A}(\mathrm{v}).$$

So

$$y = u+\lambda \overline{u} \ \text{ and } \ \mathrm{y} = \mathrm{v}+\lambda \overline{\mathrm{v}} \ \text{ for some } \ \overline{\mathrm{u}} \in \mathrm{A}(\mathrm{u}) \text{ and } \overline{\mathrm{v}} \in \mathrm{A}(\mathrm{v}).$$

Thus,

$$u+\lambda \overline{u} = v+\lambda \overline{v}$$

or

$$u-v = \lambda(\overline{v}-\overline{u}). \tag{3.1.3}$$

Since A is monotone and $\overline{u} \in Au$, $\overline{v} \in Av$, we have

$$\langle u-v,\ \overline{v}-\overline{u}\rangle \geq 0.$$

Hence

$$\begin{aligned} 0 &\leq \langle u-v,\ \overline{v}-\overline{u}\rangle \\ &= \langle u-v,\ -\frac{1}{\lambda}(u-v)\rangle \quad \text{(from (3.1.3))} \\ &= -\frac{1}{\lambda}\langle u-v,\ u-v\rangle \\ &= -\frac{1}{\lambda}\|u-v\|^2 \leq 0. \end{aligned}$$

It follows that

$$\|u-v\| = 0, \ \text{ i.e., } \ \mathrm{u} = \mathrm{v}.$$

Hence $J_\lambda(y)$ contains only one member.

(ii) Let $x, y \in H$ then there exists $u \in J_\lambda(x),\ v \in J_\lambda(y)$ such that $x \in (I+\lambda A)u$ and $\mathrm{y} \in (\mathrm{I}+\lambda \mathrm{A})\mathrm{v}$. Therefore, $x \in u+\lambda A(u)$ and $\mathrm{y} \in \mathrm{v}+\lambda\mathrm{A}(\mathrm{v})$. So $x = u+\lambda\overline{u}$ and $\mathrm{y} = \mathrm{v}+\lambda\overline{\mathrm{v}}$ for some $\overline{\mathrm{u}} \in \mathrm{A(u)}$ and $\overline{\mathrm{v}} \in \mathrm{A(v)}$. Now

$$\begin{aligned}
\|x-y\|^2 &= \|u+\lambda\overline{u}-v+\lambda\overline{v}\|^2 \\
&= \|u-v+\lambda(\overline{u}-\overline{v})\|^2 \\
&= \|u-v\|^2 + 2\lambda\langle u-v,\ \overline{u}-\overline{v}\rangle + \|\overline{u}-\overline{v}\|^2 \\
&\geq \|u-v\|^2 \\
&= \|J_\lambda(x)-J_\lambda(y)\|^2.
\end{aligned}$$

Therefore,

$$\|J_\lambda(x)-J_\lambda(y)\| \leq \|x-y\|.$$

Hence J_λ is nonexpansive.

Lemma 3.1.9 *Let H be a Hilbert space and* $\alpha \in (0,1)$, *then*

$$\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \ \ \forall x,y \in H.$$

Lemma 3.1.10 *Let* H *be a Hilbert space and* $T : C \subset H \longrightarrow H$ *be a continuous map. Then* $FixT$ *is closed.*

Proof Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in $FixT$ such that $x_k \longrightarrow x^* \in H$. Since T is continuous

$$Tx_k \longrightarrow Tx^* \in H.$$

Since $\{x_k\}_{k=0}^{\infty} \subset FixT$, then

$$Tx_k = x_k \longrightarrow x^* \in H.$$

Therefore,

$$Tx_k \longrightarrow x^* \in H$$

Thus $Tx_k \longrightarrow Tx^* \in H$ and $Tx_k \longrightarrow x^* \in H$, and uniqueness of limit we have $Tx^* = x^*$. Hence

$$x^* \in FixT.$$

■

Lemma 3.1.11 *Let H be a Hilbert space and $T : C \subset H \longrightarrow H$ be nonexpansive, then $FixT$ is convex.*

Proof Let $x, y \in FixT$ and $\alpha \in (0,1)$. Set $u = \alpha x + (1-\alpha)y$. Then,

$$\begin{aligned}
\|\alpha x + (1-\alpha)y - Tu\|^2 &= \|\alpha(x - Tu) + (1-\alpha)(y - Tu)\|^2 \\
&= \alpha\|x - Tu\|^2 + (1-\alpha)\|y - Tu\|^2 - \alpha(1-\alpha)\|x - y\|^2 \ \text{(by Lemma 3.1.9)} \\
&= \alpha\|Tx - Tu\|^2 + (1-\alpha)\|Ty - Tu\|^2 - \alpha(1-\alpha)\|x - y\|^2 \ (x, y \in FixT) \\
&\leq \alpha\|x - u\|^2 + (1-\alpha)\|y - u\|^2 - \alpha(1-\alpha)\|x - y\|^2 \ (T \text{ nonexpansive}) \\
&= \|\alpha(x - u) + (1-\alpha)(y - u)\|^2 \\
&= \|\alpha x + (1-\alpha)y - u\|^2 \\
&= 0
\end{aligned}$$

Thus, $\|u - Tu\| \leq 0$ and so $u = Tu$, i.e., $u = \alpha x + (1-\alpha)y \in FixT$. Hence $FixT$ is convex.

■

3.2 The Split Variational Inequality Problem

We recall that the split variational inequality problem (SVIP) is a problem of the form

$$\begin{cases} \text{find } x^* \in C \text{ such that } \ \langle f(x^*), y - x^* \rangle \geq 0, \forall y \in C \\ y^* = Ax^* \in Q \text{ such that } \ \langle g(y^*), z - y^* \rangle \geq 0, \forall z \in Q, \end{cases} \tag{3.2.1}$$

where H_1, H_2 are two real Hilbert spaces, and $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ are two given maps and C, Q are nonempty closed convex subsets of H_1 and H_2 respectively (see [2]).

Generally the solution of classical variational inequality is represented by $SOL(C,f)$. In the case of SVIP we look for $x^* \in SOL(C,f)$ such that $Ax^* \in SOL(Q,g)$. The solution set of the SVIP is denoted

$$\Gamma = \Gamma(C,Q,f,g,A) := \{z \in SOL(C,f) : Az \in SOL(Q,g)\}.$$

Censor *et al.* in [2] proposed the following algorithm for solving SVIP :

Algorithm 3.2.1 Initialization : *Let $\lambda > 0$ and select an arbitrary starting piont $x_0 \in H_1$.*
Iterative step : *Given the current iterate x_k, compute*

$$x_{k+1} = U(x_k + \gamma A^*(T-I)Ax_k),$$

where $\gamma \in (0, 1/L)$, L is the spectral radius of the map A^*A , and A^* is the adjoint map of A.

The following lemma, which asserts Fejer-monotonicity, is crucial for the convergence theorem.

Lemma 3.2.2 ([2]) *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \longrightarrow H_2$ be a bounded linear map. Let $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ be α_1-ism and α_2-ism maps on H_1 and H_2 respectively, and set $\alpha := min\{\alpha_1, \alpha_2\}$. Assume that $\Gamma \neq \emptyset$ and that $\gamma \in (0, 1/L)$. Consider the maps $U = P_C(I-\lambda f)$, $T = P_Q(I-\lambda g)$ with $\lambda \in [0, 2\alpha]$. Then any sequence $\{x_k\}_{k=1}^{\infty}$ generated by Algorithm 3.2.1 is Fejer-monotone with respect to the solution set Γ.*

Proof.
Let $z \in \Gamma$. Then by Lemma 3.1.2, $z = P_C(I-\lambda f)z$, $Az = P_Q(I-\lambda g)Az$, i.e., $z = U(z)$, $Az = T(Az)$. From the choise of λ and Lemma 3.1.4 the map $P_C(I-\lambda f)$ is nonexpansive on C. Therefore,

$$\begin{aligned}
\|x_{k+1} - z\|^2 &= \|U(x_k + \gamma A^*(T-I)Ax_k) - z\|^2 \\
&= \|U(x_k + \gamma A^*(T-I)Ax_k) - U(z)\|^2 \quad \text{(since } z \text{ is a fixed point of } U) \\
&\leq \|(x_k + \gamma A^*(T-I)Ax_k) - z\|^2 \quad (U \text{ is nonexpansive}) \\
&= \|(x_k - z) + \gamma A^*(T-I)Ax_k)\|^2. \qquad (3.2.2)
\end{aligned}$$

We have

$$\begin{aligned}\|(x_k - z) + \gamma A^*(T-I)Ax_k)\|^2 &= \|x_k - z\|^2 + \gamma^2\|A^*(T-I)Ax_k)\|^2 \\ &+ 2\gamma\langle x_k - z,\, A^*(T-I)Ax_k)\rangle \qquad (3.2.3)\end{aligned}$$

Since A^* is the adjoint map of A, we obtain $\|A^*y\|^2 = \langle A^*y,\, A^*y\rangle = \langle AA^*y,\, y\rangle$. Therefore, for every $k \geq 0$,

$$\begin{aligned}\|A^*(T-I)Ax_k)\|^2 &= \langle A^*(T-I)Ax_k,\, A^*(T-I)Ax_k)\rangle \\ &= \langle AA^*(T-I)Ax_k,\, (T-I)Ax_k)\rangle. \qquad (3.2.4)\end{aligned}$$

Thus, using (3.2.2), (3.2.3) and (3.2.4) we have

$$\begin{aligned}\|x_{k+1} - z\|^2 &\leq \|x_k - z\|^2 + \gamma^2\langle AA^*(T-I)Ax_k,\, (T-I)Ax_k)\rangle \\ &+ 2\gamma\langle x_k - z,\, A^*(T-I)Ax_k)\rangle. \qquad (3.2.5)\end{aligned}$$

From definition of L (the spectral radius of AA^*), we have for all $k \geq 0$,

$$\begin{aligned}\gamma^2\langle AA^*(T-I)Ax_k,\, (T-I)Ax_k)\rangle &\leq \gamma^2 L\langle (T-I)Ax_k,\, (T-I)Ax_k)\rangle \\ &= L\gamma^2\|(T-I)Ax_k)\|^2. \qquad (3.2.6)\end{aligned}$$

Also,

$$2\gamma\langle x_k - z,\, A^*(T-I)Ax_k)\rangle = 2\gamma\langle A(x_k - z),\, (T-I)Ax_k)\rangle \quad \forall k \geq 0.$$

By linearity of A,

$$\begin{aligned}
2\gamma\langle A(x_k - z),\ (T-I)Ax_k)\rangle &= 2\gamma\langle Ax_k - Az),\ (T-I)Ax_k)\rangle \\
&= 2\gamma\langle Ax_k - Az + (T-I)Ax_k) - (T-I)Ax_k),\ (T-I)Ax_k)\rangle \\
&= 2\gamma\langle Ax_k - Az + T(Ax_k) - Ax_k - (T-I)Ax_k),\ (T-I)Ax_k)\rangle \\
&= 2\gamma\langle T(Ax_k) - Az - (T-I)Ax_k),\ (T-I)Ax_k)\rangle \\
&= 2\gamma\big(\langle T(Ax_k) - Az,\ (T-I)Ax_k)\rangle - \|(T-I)Ax_k)\|^2\big) \quad \forall k \geq 0.
\end{aligned}$$

By Lemma 3.1.3 $\forall (x,y) \in H \times Fix(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq 1/2\|(T(x) - x)\|^2. \tag{3.2.7}$$

Since H is a real inner product space,

$$\begin{aligned}
\langle T(Ax_k) - Az,\ (T-I)Ax_k)\rangle &= \langle T(Ax_k) - Az,\ T(Ax_k) - Ax_k\rangle \\
&= \langle T(Ax_k) - Ax_k,\ T(Ax_k) - Az\rangle \\
&= \langle Ax_k - T(Ax_k),\ Az - T(Ax_k)\rangle \quad \forall k \geq 0.
\end{aligned}$$

From (3.2.7), since $Az \in Fix(T)$ as z $\in \Gamma$, we have

$$\begin{aligned}
\langle T(Ax_k) - Az,\ (T-I)Ax_k)\rangle &= \langle Ax_k - T(Ax_k),\ Az - T(Ax_k)\rangle \\
&\leq 1/2\|T(Ax_k) - Ax_k\|^2 \\
&= 1/2\|(T-I)(Ax_k)\|^2.
\end{aligned}$$

So,

$$\begin{aligned}
2\gamma(\langle T(Ax_k) - Az, (T-I)Ax_k)\rangle - \|(T-I)Ax_k)\|^2) &\leq 2\gamma\big(1/2\|(T-I)(Ax_k)\| - \|(T-I)(Ax_k)\|^2\big) \\
&= -\gamma\|(T-I)(Ax_k)\|^2. \qquad (3.2.8)
\end{aligned}$$

Using (3.2.6) and (3.2.8) in (3.2.5) we have

$$\begin{aligned} \|x_{k+1} - z\|^2 &\leq \|x_k - z\|^2 + L\gamma^2 \|(T-I)Ax_k)\|^2 - \gamma \|(T-I)Ax_k)\|^2 \\ &= \|x_k - z\|^2 + \gamma(L\gamma - 1)\|(T-I)Ax_k)\|^2 \\ &\leq \|x_k - z\|^2 \quad (\text{since}\gamma \in (0, 1/\text{L})). \end{aligned} \tag{3.2.9}$$

Therefore,

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 \quad \forall k \geq 0.$$

This completes the proof.

Theorem 3.2.3 *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \longrightarrow H_2$ be a bounded linear map. Let $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ be α_1-ism and α_2-ism maps on H_1 and H_2, respectively, and set $\alpha := min\{\alpha_1, \alpha_2\}$. Assume that $\Gamma \neq \emptyset$ and that $\gamma \in (0, 1/L)$ where L is the spectral radius of AA^*. Consider the maps $U = P_C(I - \lambda f)$, $T = P_Q(I - \lambda g)$ with $\lambda \in [0, 2\alpha]$. Suppose for all $x^* \in SOL(C, f)$,*

$$\langle f(x),\ P_C(I - \lambda f)(x) - x^* \rangle \geq 0 \ \ \textit{for all} \ \ x \in H_1.$$

Then any sequence $\{x_k\}_{k=0}^{\infty}$ generated by the algorithm 3.2.1 above converges weakly to a solution $x^* \in \Gamma$.

Proof.

Let $z \in \Gamma$. Then from Lemma 3.2.2 we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 \quad \forall k \geq 0.$$

So the sequence $\{\|x_k - z\|\}_{k=0}^{\infty}$ is monotonically decreasing and bounded below by 0. Hence $\lim_{k\to\infty} \|x_{k+1} - z\|$ exists in $\mathbb{R}$. From (3.2.9) we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + \gamma(L\gamma - 1)\|(T-I)Ax_k)\|^2 \quad \forall k \geq 0.$$

Since $\gamma \in (0,1/L)$, $\gamma(L\gamma-1)\|(T-I)Ax_k)\|^2 \le 0$ and so $-\gamma(L\gamma-1) \ge 0$. Therefore,

$$0 \le -\gamma(L\gamma-1)\|(T-I)Ax_k)\|^2 \le \|x_k - z\| - \|x_{k+1} - z\|.$$

Therefore (by sandwich theorem of sequences),

$$\lim_{k\to\infty} \|(T-I)Ax_k)\| = 0. \tag{3.2.10}$$

Since $\{\|x_k - z\|\}_{k=0}^{\infty}$ is convergent, there exists $M > 0$ such that $\|x_k - z\| \le M \quad \forall k \ge 0$. So,

$$\|x_k\| - \|z\| \le |\|x_k\| - \|z\|| \le \|x_k - z\| \le M, \ \forall k \ge 0.$$

Thus,

$$\|x_k\| \le M + \|z\| \ \ \forall k \ge 0.$$

Hence the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded. By Eberlein Smulyan Theorem (see, e.g., [50]),$\{x_k\}_{k=0}^{\infty}$ has a weakly convergent subsequence $\{x_{k_j}\}_{j=0}^{\infty}$ such that $x_{k_j} \rightharpoonup x^*$. This implies, using Riesz representation theorem, that $\langle x, x_{k_j}\rangle \to \langle x, x^*\rangle \ \ as\, j \longrightarrow \infty \quad \forall x \in H_1$. Observe that

$$Ax_k \rightharpoonup Ax^* \Longleftrightarrow g(Ax_k) \longrightarrow g(Ax^*) \ \ \forall g \in H_2^* = H_2.$$

Therefore using Riesz representation theorem again, we have

$$Ax_k \rightharpoonup Ax^* \Longleftrightarrow \langle y, Ax_{k_j}\rangle \longrightarrow \langle y, Ax^*\rangle \quad \forall y \in H_2.$$

Now from every $y \in H_2$, $\langle y, Ax_{k_j}\rangle = \langle A^*y, x_{k_j}\rangle \xrightarrow{j\longrightarrow\infty} \langle A^*y, x^*\rangle = \langle y, Ax^*\rangle$, by the fact that $x_k \rightharpoonup x^*$. Hence

$$Ax_{k_j} \rightharpoonup Ax^* \ \text{ as } \ j \longrightarrow \infty.$$

Since T is nonexpansive on H_2 by Lemma 1.6.21 , $I-T$ is demiclosed at 0. Therefore, using $Ax_{k_j} \rightharpoonup Ax^*$, and (3.3.4) we have $(I-T)Ax^* = 0$. But

$$(I-T)Ax^* = 0 \Longleftrightarrow (Ax^*) - T(Ax^*) = 0 \Longleftrightarrow T(Ax^*) = Ax^*.$$

Hence

$$Ax^* \in Fix(T).$$

It follows that

$$(Ax^*) \in SOL(Q,g).$$

We define

$$v_k := x_k + \gamma A^*(T-I)(Ax_k) \quad \forall k \geq 0.$$

Then

$$v_{k_j} := x_{k_j} + \gamma A^*(T-I)(Ax_{k_j}) \quad \forall j \geq 0.$$

Since $\lim_{k\to\infty} \|(T-I)Ax_k)\| = 0$, we have $(T-I)Ax_k \rightharpoonup 0$ as $k \longrightarrow \infty$ and so $(T-I)Ax_{k_j} \rightharpoonup 0$ as $j \longrightarrow \infty$. Thus $x_{k_j} \rightharpoonup x^*$ and $(T-I)Ax_{k_j} \rightharpoonup 0$ as $j \longrightarrow \infty$. It therefore follows that $x_{k_j} + (T-I)Ax_{k_j} \longrightarrow x^*$ as $j \longrightarrow \infty$, i.e.,

$$v_{k_j} \rightharpoonup x^* \text{ as } j \longrightarrow \infty.$$

Next we show that

$$x^* \in SOL(C,f).$$

By contradiction, suppose $x^* \notin SOL(C,f)$, i.e., $U(x^*) \neq x^*$. Since U is nonexpansive on H_1, by Lemma 1.6.21, $I-U$ is demiclosed at 0. Hence

$$U(x^*) \neq x^* \Longrightarrow \|(U-I)v_{k_j}\| \nrightarrow 0 \text{ as j} \longrightarrow \infty.$$

Therefore $\|(U-I)v_{k_j}\| \nrightarrow 0$ as j $\longrightarrow \infty$. This implies that there exists an $\varepsilon > 0$ and a subsequence $\{v_{k_{j_s}}\}_{s=1}^{\infty}$ of $\{v_{k_j}\}_{j=1}^{\infty}$ such that

$$\|U(v_{k_{j_s}}) - v_{k_{j_s}}\| > \varepsilon, \quad \forall s \geq 0. \tag{3.2.11}$$

The condition in the theorem that $\forall u \in SOL(C,f), \quad \langle f(x), P_C(I-\lambda f)(x)-u\rangle \geq 0 \quad \forall x \in H_1$ justifies the use of Lemma 3.1.4(b). This yields for all $s \geq 0$, for all $z \in Fix(U)$,

$$\begin{aligned} \|U(v_{k_{j_s}}) - U(z)\|^2 &= \|U(v_{k_{j_s}}) - z\|^2 \\ &\leq \|v_{k_{j_s}} - z\|^2 - \|U(v_{k_{j_s}}) - v_{k_{j_s}}\|^2 \\ &< \|v_{k_{j_s}} - z\|^2 - \varepsilon^2. \end{aligned} \tag{3.2.12}$$

By the similar argument to that of the proof of Lemma 3.2.2 we have, for $z \in Fix(U)$,

$$\|v_k - z\| = \|(x_k + \gamma A^*(T-I)Ax_k) - z\| = \|(x_k - z) + \gamma A^*(T-I)Ax_k)\| \leq \|x_k - z\| \;\; \forall k \geq 0.$$

Since U is nonexpansive

$$\|x_{k+1} - z\| = \|U(v_k) - z\| \leq \|v_k - z\| \quad \text{for all z} \in \text{Fix(U)}$$

From the two inequalities above we have

$$\|x_{k+1} - z\| \leq \|v_k - z\| \leq \|x_k - z\| \quad \forall k \geq 0. \tag{3.2.13}$$

This implies that the sequence $\{x_1, u_1, x_2, u_2, x_3, u_3, x_4, u_4, x_5, \cdots\}$ is Fejer monotone with respect to Γ. Since

$$x_{k_{j_s+1}} = U(v_{k_{j_s}}), \quad s \geq 0,$$

we obtain

$$\|v_{k_{j_{s+1}}} - z\|^2 \leq \|v_{k_{j_s}} - z\|^2 \;\; \text{(since U is nonexpansive.)}$$

Hence $\{v_{k_{j_s}}\}_{s=0}^{\infty}$ is Fejer monotone with respect to Γ and so $\lim_{k\to\infty} \|v_{k_{j_s}} - z\|$ exist in $\mathbb{R}$ for all $z \in \Gamma$. From (3.2.12) and (3.2.13) and the fact that $x_{k_{j_s+1}} = U(v_{k_{j_s}})$, we have

$$\|v_{k_{j_{s+1}}} - z\|^2 < \|v_{k_{j_s}} - z\|^2 - \varepsilon^2 \;\; \forall s \geq 0.$$

Therefore $\varepsilon^2 \leq \|v_{k_{js}} - z\|^2 - \|v_{k_{js+1}} - z\|^2 \longrightarrow 0$ as s $\longrightarrow \infty$.
Thus, $\varepsilon^2 \leq 0$ which is a contradiction. Therefore, $Ux^* = x^*$, i.e.,

$$x^* \in SOL(C, f).$$

So

$$x^* \in \Gamma.$$

Next we show that $\{x\}_{k=0}^{\infty}$ converges weakly to x^*. Let $\{x_{k_j}\}_{j=0}^{\infty}$ be arbitrary subsequence of $\{x_k\}_{k=1}^{\infty}$ which converges weakly to u. Following similar argument, we have $u \in \Gamma$. Hence Γ contains all the weak limit points of $\{x\}_{k=0}^{\infty}$. By the Lemma 3.1.5 we have

$$x_k \rightharpoonup x^* \text{as k} \longrightarrow \infty.$$

This completes the proof.

3.3 Parallel Algorithm For Solving Split Variational Inequality Problem

The extension of SVIP to the multi-set split variational inequality problem (MSSVIP), is formulated as follows:

$$\begin{cases} \text{find } x^* \in C := \bigcap_{i=1}^{p} C_i \text{ such that } \langle f_i(x^*),\ x - x^* \rangle \geq 0, \forall x \in C_i, i = 1, 2, ...p \\ y^* = Ax^* \in Q = \bigcap_{j=1}^{p} Q_j \text{ such that } \langle g_j(y^*),\ y - y^* \rangle \geq 0, \forall y \in Q_j, j = 1, 2, ...r, \end{cases}$$

where H_1 and H_2 are two real Hilbert spaces, $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f_i : H_1 \longrightarrow H_1$, $i = 1, 2, ...p$ and $g_j : H_2 \longrightarrow H_2$, $j = 1, 2, ...r$, are maps, $C_i \subseteq H_1$, $Q_i \subseteq H_2$ are nonempty, closed and convex subsets for $i = 1, 2, ...p$ and $j = 1, 2, ...r$.

The authors in [2] used a product space approach to the MSSVIP. We present here an algorithm for solving MSSVIP which is carried out in certain product space.

Let Ψ be the solution set of multi-set split variational inequality problem, i.e.,

$$\Psi := \{z \in \bigcap_{i=1}^{p} SOL(C_i, f_i) \mid Az \in \bigcap_{j=1}^{r} SOL(Q_j, g_j)\}.$$

Recall that

$$x^* \in SOL(C_i, f_i) \Leftrightarrow x^* \in Fix(U_i) \text{ and } Ax^* \in SOL(Q_j, g_j) \Leftrightarrow Ax^* \in Fix(T_j),$$

where $U_i = P_{C_i}(I - \lambda f_i)$, $T_j = P_{Q_j}(I - \lambda g_j)$.
Thus, the solution set of multi-set split variational inequality problem is equivalent to

$$\Psi = \{z \in \bigcap_{i=1}^{p} Fix(U_i) \mid Az \in \bigcap_{j=1}^{r} Fix(T_j)\}.$$

Product space formulation is used, following [9], to derive and analyze a simultaneous algorithm for the MSSVIP as above.

We define the spaces $\mathbf{W_1} := H_1$ and $\mathbf{W_2} := H_1^p \times H_2^r$, where p and r are as in the indexes above, and we denote objects in the product space by boldface letters. We define $U_i := P_C(I - \lambda f_i)$ and $T_j := P_Q(I - \lambda g_j)$ for $i = 1, 2, ...p$ and $j = 1, 2, ...r$, respectively,

$$\mathbf{C} := H_1$$

and

$$\mathbf{Q} := \prod_{i=1}^{p} \sqrt{\alpha}_i Fix(U_i) \times \prod_{j=1}^{r} \sqrt{\beta}_j Fix(T_j).$$

The map $\mathbf{A} : H_1 \longrightarrow \mathbf{W_2}$ is defined by

$$\mathbf{A} = (\sqrt{\alpha_1} I, \sqrt{\alpha_2} I, \cdots, \sqrt{\alpha_p} I, \sqrt{\beta_1} A^*, \sqrt{\beta_2} A^*, \cdots, \sqrt{\beta_r} A^*)^*,$$

where A^* is the adjoint of A and $\{\alpha_i\}_{i=1}^{p}$, $\{\beta_j\}_{j=1}^{r}$ are positive real numbers. Define the map

$$\mathbf{T} : \mathbf{W_2} \longrightarrow \mathbf{W_2}$$

by

$$\mathbf{T}(\mathbf{y}) = \mathbf{T}\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{p+r} \end{bmatrix} = (U_1(y_1), U_2(y_2), ., ., ., U_p(y_p), T_1(y_{p+1}), T_2(y_{p+2}), ., ., ., T_r(y_{p+r}))^*, \tag{3.3.1}$$

where $y_1, y_2, ...y_p \in H_1$ and $y_{p+1}, y_{p+2}, ...y_{p+r} \in H_2$.

This leads to a SVIP with just two maps **F** and **G** and two sets **C** and **Q** respectively, in product space taking

$$\mathbf{F} \equiv 0$$

$$\mathbf{G}(\mathbf{y}) = (f_1(y_1), f_2(y_2), ..., f_p(y_p), g_1(y_{p+1}), g_2(y_{p+2}), ...g_r(y_{p+r})),$$

and map $\mathbf{A} : H_1 \longrightarrow \mathbf{W_2}$, the identity map $\mathbf{I} : \mathbf{C} \longrightarrow \mathbf{C}$, and the map $\mathbf{T} : \mathbf{W_2} \longrightarrow \mathbf{W_2}$.

This problem can be solved using algorithm 3.2.1. It is easy to see that

$$x \in \Psi \Longleftrightarrow \mathbf{A}x \in \mathbf{Q}.$$

In deed,

$$\begin{aligned} (\mathbf{A}x) \in \mathbf{Q} &\Longleftrightarrow \sqrt{\alpha_i}x = \sqrt{\alpha_i}z_i, \quad z_i \in Fix(U_i) \\ &\qquad and \quad \sqrt{\beta_j}Ax = \sqrt{\beta_j}w_j, \quad w_j \in Fix(T_j) \\ &\Longleftrightarrow x \in Fix(U_i) \ \ and \ \ Ax \in Fix(T_j) \\ &\Longleftrightarrow x \in \bigcap_{i=1}^{p} Fix(U_i) \ \ and \ \ Ax \in \bigcap_{j=1}^{r} Fix(T_j) \\ &\Longleftrightarrow x \in \Psi. \end{aligned} \tag{3.3.2}$$

Therefore, if we apply algorithm 3.2.1 to this product space setting, we have

$$\begin{cases} x_0 \in H_1 \text{ arbitrary} \\ x_{k+1} = x_k + \gamma \mathbf{A}^*(\mathbf{T} - \mathbf{I})(\mathbf{A}x) \quad \text{for all } k \geq 0. \end{cases} \tag{3.3.3}$$

It is very cumbersome to use this scheme in the product space. It is desirable therefore to have equivalent scheme in the original spaces H_1, H_2. For this purpose, using the relation

$$\mathbf{T}(\mathbf{A}x) = (\sqrt{\alpha_1}U_1(x), \sqrt{\alpha_2}U_2(x), \cdots, \sqrt{\alpha_p}U_p(x), T_1(\sqrt{\beta_1}Ax), T_2(\sqrt{\beta_2}Ax), \cdots, T_r(\sqrt{\beta_r}Ax))^*,$$

We have

$$\begin{aligned} x_{k+1} &= x_k + \gamma \mathbf{A}^*(\mathbf{T}(\mathbf{A}x) - (\mathbf{A}x)) \\ &= x_k + \gamma \mathbf{A}^*(\sqrt{\alpha_1}(U_1(x_k) - x_k), \sqrt{\alpha_2}(U_2(x_k) - x_k), \cdots, \sqrt{\alpha_p}(U_p(x_k) - x_k), \\ &\quad \sqrt{\beta_1}(T_1(Ax_k) - Ax_k), \sqrt{\beta_2}(T_2(Ax_k) - Ax_k), \cdots, \sqrt{\beta_r}(T_r(Ax_k) - Ax_k)). \end{aligned}$$

Since

$$\mathbf{A}^* = (\sqrt{\alpha_1}I, \sqrt{\alpha_2}I, \cdots, \sqrt{\alpha_p}I, \sqrt{\beta_1}A^*, \sqrt{\beta_2}A^*, \cdots, \sqrt{\beta_r}A^*),$$

we have

$$\begin{aligned} x_{k+1} &= x_k + \gamma(\sqrt{\alpha_1}I, \sqrt{\alpha_2}I, \cdots, \sqrt{\alpha_p}I, \sqrt{\beta_1}A^*, \sqrt{\beta_2}A^*, \cdots, \sqrt{\beta_r}A^*)(\sqrt{\alpha_1}(U_1(x_k) - x_k), \\ &\quad \sqrt{\alpha_2}(U_2(x_k) - x_k), \cdots, \sqrt{\alpha_p}(U_p(x_k) - x_k), \sqrt{\beta_1}(T_1(Ax_k) - Ax_k), \sqrt{\beta_2}(T_2(Ax_k) - Ax_k), \\ &\quad \cdots, \sqrt{\beta_r}(T_r(Ax_k) - Ax_k)) \\ &= x_k + \gamma(\alpha_1(U_1(x_k) - x_k) + \alpha_2(U_2(x_k) - x_k) + \cdots + \alpha_p(U_p(x_k) - x_k) + \beta_1 A^*(T_1(Ax_k) - Ax_k) \\ &\quad + \beta_2 A^*(T_2(Ax_k) - Ax_k) + \cdots + \beta_r A^*(T_r(Ax_k) - Ax_k)) \\ &= x_k + \gamma\left(\sum_{i=1}^{p} \alpha_i(U_i(x_k) - x_k) + \sum_{j=1}^{r} \beta_j A^*(T_j(Ax_k) - Ax_k)\right). \end{aligned}$$

Thus, we obtain the following algorithm in the original spaces which is equivalent to the algorithm 3.2.1 in the product spaces.

Algorithm 3.3.1 Initialization : *Let $\lambda > 0$ and select an arbitrary starting piont $x_0 \in H_1$.*
Iterative step : *Given the current iterate x^k, compute*

$$x_{k+1} = x_k + \gamma \left(\sum_{i=1}^{p} \alpha_i (U_i(x_k) - x_k) + \sum_{j=1}^{r} \beta_j A^* (T_j(Ax_k) - Ax_k) \right),$$

where $\gamma \in (0, \frac{1}{L})$, with $L = \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{r} \beta_j \|A\|^2$.

Now we can present the convergence analysis of the following. Although it is more convenient to use the algorithm in the original spaces, the convergence proof of the scheme will be given in the product space. Before we take the theorem and its proof, we first have the following preliminary results.

Lemma 3.3.2 (Product) *Let $(X_i, \langle \cdot, \cdot \rangle_i), 1 \leq i \leq n, n \in \mathbb{N}$ be a finite collection of inner product spaces over the same field K ($= \mathbb{R}$ or $\mathbb{C}$). Let $\mathbf{X} = \prod_{i=1}^{p} X_i$. The map $\langle\langle \cdot, \cdot \rangle\rangle$: $\mathbf{X} \longrightarrow K$ defined by $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \sum_{i=1}^{n} \langle x_i, y_i \rangle_i$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{X}$ is an inner product space. Moreover, if each X_i is Hilbert, then $\mathbf{X}$ is also Hilbert.*

Proof.

(i)

$$\begin{aligned} \langle\langle \mathbf{x}, \mathbf{x} \rangle\rangle &= \sum_{i=1}^{n} \langle x_i, x_i \rangle_i \\ &= \sum_{i=1}^{n} \langle x, x \rangle_i \geq 0 \quad (\mathrm{since} \quad \langle \mathrm{x_i}, \mathrm{x_i} \rangle_\mathrm{i} \geq 0, \quad \mathrm{i} = 1, 2, \cdots, \mathrm{p}). \end{aligned}$$

Therefore $\quad \langle\langle \mathbf{x},\mathbf{x}\rangle\rangle \geq 0 \quad \forall \mathbf{x}\in \mathbf{X}$.

Also,

$$\begin{aligned}\langle\langle \mathbf{x},\mathbf{x}\rangle\rangle = 0 &\Longleftrightarrow \sum_{i=1}^{n}\langle x_i,x_i\rangle_i = 0\\ &\Longleftrightarrow \sum_{i=1}^{n}\|x_i\|_i^2 = 0\\ &\Longleftrightarrow \|x_i\|_i^2 = 0 \ \ \forall i\\ &\Longleftrightarrow x_i = 0 \ \ \forall i\\ &\Longleftrightarrow (x_1,x_2,\cdots,x_n) = (0,0,\cdots,0)\\ &\Longleftrightarrow \mathbf{x} = 0.\end{aligned}$$

(ii)

$$\begin{aligned}\langle\langle \mathbf{x},\mathbf{y}\rangle\rangle &= \sum_{i=1}^{n}\langle x_i,y_i\rangle_i,\\ &= \overline{\sum_{i=1}^{n}\langle y_i,x_i\rangle_i} \ \ (\text{since } \ \langle \mathrm{x_i},\mathrm{y_i}\rangle_\mathrm{i} = \overline{\langle \mathrm{y_i},\mathrm{x_i}\rangle_\mathrm{i}}, \quad \mathrm{i} = 1,2,\cdots,\mathrm{p})\\ &= \overline{\langle\langle \mathbf{y},\mathbf{x}\rangle\rangle} \ \ \forall \mathbf{x},\mathbf{y}\in \mathbf{X}.\end{aligned}$$

(iii)

$$\begin{aligned}\langle\langle \alpha\mathbf{x}+\beta\mathbf{y},\mathbf{z}\rangle\rangle &= \sum_{i=1}^{n}\langle \alpha x_i+\beta y_i,z_i\rangle_i,\\ &= \sum_{i=1}^{n}\left(\langle \alpha x_i,z_i\rangle_i + \langle \beta y_i,z_i\rangle_i\right) \ \ (\text{since } \ \langle\cdot,\cdot\rangle_\mathrm{i} \text{ is as inner product})\\ &= \alpha\sum_{i=1}^{n}\langle x_i,z_i\rangle_i + \beta\sum_{i=1}^{n}\langle y_i,z_i\rangle_i\\ &= \alpha\langle\langle \mathbf{x},\mathbf{z}\rangle\rangle + \beta\langle\langle \mathbf{y},\mathbf{z}\rangle\rangle \ \ \forall \mathbf{x},\mathbf{y}\in \mathbf{X}.\end{aligned}$$

Hence $\langle\langle\cdot,\cdot\rangle\rangle$ is an inner product on $\mathbf{X}$.

We shall denote by $|||\cdot|||$ the norm generated by $\langle\langle\cdot,\cdot\rangle\rangle$. That is for $\mathbf{x}\in\mathbf{X}$, $|||\mathbf{x}||| = (\langle\langle \mathbf{x},\ \mathbf{x}\rangle\rangle)^{\frac{1}{2}} = (\sum_{i=1}^{n}\|x_i\|_i^2)^{1/2}$, where $\|\cdot\|_i$ corresponds to $\langle\cdot,\cdot\rangle_i, 1\leq i\leq p$.

Now suppose X_i is Hilbert space, $i = 1, 2, \cdots, p$. To show $\mathbf{X} = \prod_{i=1}^{p} X_i$ is a Hilbert space, let $\{\mathbf{x}_m\}_{m=1}^{\infty} \subset \mathbf{X}$ be Cauchy with $\mathbf{x}_m = (x_1^m, x_2^m, \cdots, x_p^m)$, $x_i^m \in X_i$, $i = 1, 2, \cdots, p$, $m \in \mathbb{N}$. Then for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|||\mathbf{x}_l - \mathbf{x}_m||| < \varepsilon \quad \forall l, m \geq N(\varepsilon)$. Thus,

$$\sum_{i=1}^{p} \|x_i^l - x_i^m\|^2 < \varepsilon^2 \ \ \forall l, m \geq N(\varepsilon). \qquad (*)$$

This implies

$$\|x_i^l - x_i^m\|^2 < \varepsilon^2 \ \ \forall\, l, m \geq N(\varepsilon), \quad i = 1, 2, \cdots, p,$$

therefore

$$\|x_i^l - x_i^m\| < \varepsilon \ \ \forall\, l, m \geq N(\varepsilon), \quad i = 1, 2, \cdots, p.$$

This implies $\{x_i^m\}_{i=1}^{\infty}$ is a Cauchy sequence in X_i, $i = 1, 2, \cdots, p$. Since X_i is Hilbert, there exists $x_i^* \in X_i$ such that $x_i^m \longrightarrow x_i^*$ as $m \longrightarrow \infty$, $i = 1, 2, \cdots, p$. Let $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_p^*) \in \mathbf{X}$.

Claim: $\quad \mathbf{x}_m \xrightarrow{|||\cdot|||} \mathbf{x}$ as $m \longrightarrow \infty$.

From (*) we have

$$\sum_{i=1}^{p} \|x_i^l - x_i^m\|^2 < \varepsilon^2 \quad \forall l, m \geq N(\varepsilon).$$

Therefore for every fixed m, by allowing $l \longrightarrow \infty$ we have

$$\sum_{i=1}^{p} \|x_i^* - x_i^m\|^2 \leq \varepsilon^2 \quad \forall m \geq N(\varepsilon).$$

The inequality above means that

$$|||\mathbf{x}^* - \mathbf{x}_m|||^2 < \varepsilon^2 \quad \forall m \geq N(\varepsilon)$$

showing that $\mathbf{x}_m \longrightarrow \mathbf{x}^*$ in $(\mathbf{X}, |||\cdot|||)$. Thus $\mathbf{X}$ is Hilbert.

Lemma 3.3.3 *Let X_i, $i = 1, 2, \cdots, p$, be a real vector spaces. Let C_i, $i = 1, 2, \cdots, p$, be a convex subsets of X_i, $i = 1, 2, \cdots, p$, respectively. Then $\prod_{i=1}^{p} C_i$ is convex in $\mathbf{X} = \bigcap_{i=1}^{p} X_i$.*

Proof.

Let $(x_1, x_2, \cdots, x_p), (y_1, y_2, \cdots, y_p) \in \prod_{i=1}^{p} C_i$ and $\lambda \in (0,1)$.

$$(1-\lambda)(x_1, x_2, \cdots, x_p) + \lambda(y_1, y_2, \cdots, y_p) = ((1-\lambda)x_1 + \lambda y_1, \cdots, (1-\lambda)x_p + \lambda y_p).$$

Since every C_i is convex, $i = 1, 2, \cdots, p$, then

$$(1-\lambda)x_1 + \lambda y_1 \in C_1, \cdots, (1-\lambda)x_p + \lambda y_p \in C_p.$$

Therefore,

$$((1-\lambda)x_1 + \lambda y_1, \cdots, (1-\lambda)x_p + \lambda y_p) \in \prod_{i=1}^{p} C_i.$$

Hence

$$(1-\lambda)(x_1, x_2, \cdots, x_p) + \lambda(y_1, y_2, \cdots, y_p) \in \prod_{i=1}^{p} C_i.$$

It follows that $\prod_{i=1}^{p} C_i$ is convex.

Lemma 3.3.4 *Let $(X_i, \|\cdot\|),\ i = 1, 2, \cdots, p$ be a normed linear space and C_i be a closed subset of $X_i,\ i = 1, 2, \cdots, p$. Then $\prod_{i=1}^{p} C_i$ is closed in $(\mathbf{X}, |\|\cdot\||)$, where $\mathbf{X} = \prod_{i=1}^{p} X_i$ and $|\|\mathbf{x}\|| = \left(\sum_{i=1}^{p} \|x\|^2\right)^{\frac{1}{2}}$, $\mathbf{x} = (x_1, x_2, \cdots, x_p) \in \mathbf{X}$.*

Proof.

Let $\{(x_1^m, x_2^m, \cdots, x_p^m)\}_{m=0}^{\infty}$ be a sequence in $\prod_{i=1}^{p} C_i$ such that

$$(x_1^m, x_2^m, \cdots, x_p^m) \longrightarrow (x_1^*, x_2^*, \cdots, x_p^*) \in \mathbf{X} \text{ as } m \longrightarrow \infty.$$

This implies $x_i^m \longrightarrow x_i^*$ in $(X_i, \|\cdot\|),\ i = 1, 2, \cdots, p$. Since C_i is closed, $x_i^* \in C_i,\ i = 1, 2, \cdots, p$. Hence

$$(x_1^*, x_2^*, \cdots, x_p^*) \in \prod_{i=1}^{p} C_i.$$

Remark 3.3.5 *(i) Using Lemma 3.3.4 $\mathbf{W_2}$ is Hilbert as a finite product of Hilbert spaces.*

(ii) Also, $\mathbf{A}$ is a bounded linear map.

Indeed, consider the map $\Phi : \mathbf{W_2} \longrightarrow \mathbf{W_2}$ define by

$$\Phi(\mathbf{x}) = (\sqrt{\alpha_1}x_1, \sqrt{\alpha_2}x_2, \cdots, \sqrt{\alpha_p}x_p, \sqrt{\beta_1}A^*x_{p+1}, \sqrt{\beta_2}A^*x_{p+2}, \cdots, \sqrt{\beta_r}A^*x_{p+r})$$

for $\mathbf{x} = x_1, x_2, \cdots, x_p, x_{p+1}, x_{p+2}, \cdots, x_{p+r} \in \mathbf{W_2}$. Then $\mathbf{A} = \Phi^*$, it suffices, therefore, to show Φ is linear and bounded.

$$\begin{aligned}
\Phi(\mathbf{x}+\lambda\mathbf{y}) = (&\sqrt{\alpha_1}(x_1+\lambda y_1), \cdots, \sqrt{\alpha_p}(x_p+\lambda y_p), \sqrt{\beta_1}A^*(x_{p+1}+\lambda y_{p+1}), \\
&\cdots, \sqrt{\beta_r}A^*(x_{p+1}+\lambda y_{p+1})) \\
= (&\sqrt{\alpha_1}x_1, \cdots, \sqrt{\alpha_p}x_p, \sqrt{\beta_1}A^*x_{p+1}, \cdots, \sqrt{\beta_r}A^*x_{p+r}) \\
&+\lambda(\sqrt{\alpha_1}y_1, \cdots, \sqrt{\alpha_p}y_p, \sqrt{\beta_1}A^*y_{p+1}, \cdots, \sqrt{\beta_r}A^*y_{p+r}) \\
= \Phi&(\mathbf{x}) + \lambda\Phi(\mathbf{y})
\end{aligned}$$

Also,

$$\begin{aligned}
|||\Phi(\mathbf{x})|||^2 &= \sum_{i=1}^{p} \alpha_i \|x_i\|_{H_1}^2 + \sum_{j=1}^{r} \beta_j \|x_{p+j}\|_{H_2}^2 \\
&\leq K\left(\sum_{i=1}^{p} \|x_i\|_{H_1}^2 + \sum_{j=1}^{r} \|x_{p+j}\|_{H_2}^2\right), \; K = Max\{\alpha_i, \beta_j, 1 \leq i \leq p, \; 1 \leq j \leq r\} \\
&= K|||\mathbf{x}|||^2 \; \forall \mathbf{x} \in \mathbf{W_2}.
\end{aligned}$$

Thus $|||\Phi(\mathbf{x})||| \leq \overline{K}|||\mathbf{x}||| \; \forall \mathbf{x} \in \mathbf{W_2}$, $\overline{K} = \sqrt{K}$ and so Φ is bounded.

Lemma 3.3.6 *Let $\mathbf{W_2} := H_1^p \times H_2^r$ be Hilbert space. Let $\mathbf{T} : \mathbf{W_2} \longrightarrow \mathbf{W_2}$ be a map define by*

$$\mathbf{T}(y) = \mathbf{T}\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{p+r} \end{bmatrix} = (U_1(y_1), U_2(y_2), ., ., ., U_p(y_p), T_1(y_{p+1}), T_2(y_{p+2}), ., ., ., T_r(y_{p+r}))^*,$$

where $y_1, y_2, ...y_p \in H_1$ *and* $y_{p+1}, y_{p+2}, ...y_{p+r} \in H_2$, *then*

(i) $\mathbf{T}$ *is nonexpansive. (ii)* $\mathbf{I}-\mathbf{T}$ *is* α*-ism.*

Proof

(i) Let $\mathbf{x}, \mathbf{y} \in \mathbf{W_2}$.

$$\begin{aligned}|||\mathbf{T}(\mathbf{x})-\mathbf{T}(\mathbf{y})|||^2 &= \sum_{i=1}^{p}\|U_i(x_i)-U_i(y_i)\|^2+\sum_{j=p+1}^{p+r}\|T_j(x_i)-T_j(y_i)\|^2\\ &\leq \sum_{i=1}^{p+r}\|(x_i)-(y_i)\|^2 \ \ (\textit{since } U_i \textit{ and } T_j \textit{ are nonexpansive},\ 1\leq i\leq p,\ 1\leq j\leq r.)\\ &= |||\mathbf{x}-\mathbf{y}|||^2.\end{aligned}$$

(ii) Let $\mathbf{x}, \mathbf{y} \in \mathbf{W_2}$.

$\langle\langle(\mathbf{I}-\mathbf{T})\mathbf{x}-(\mathbf{I}-\mathbf{T})\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle\rangle$

$$\begin{aligned}&= \langle\langle(I-U_1)x_1-(I-U_1)y_1,\ldots,(I-U_p)x_p-(I-U_p)y_p,(I-T_1)x_{p+1}-(I-T_1)y_{p+1},\\ &\quad \ldots,(I-T_r)x_{p+r}-(I-T_r)y_{p+r},\ \mathbf{x}-\mathbf{y}\rangle\rangle\\ &= \sum_{i=1}^{p}\langle(I-U_i)x_i-(I-U_i)y_i,\ x_i-y_i\rangle+\sum_{j=1}^{r}\langle(I-T_j)x_{p+j}-(I-T_j)y_{p+j},\ x_{p+j}-y_{p+j}\rangle\\ &\geq \alpha_1\sum_{i=1}^{p}\|(I-U_i)x_i-(I-U_i)y_i\|^2+\alpha_2\sum_{j=1}^{r}\|(I-T_j)x_{p+j}-(I-T_j)y_{p+j}\|^2\\ &\geq \alpha\left(\sum_{i=1}^{p}\|(I-U_i)x_i-(I-U_i)y_i\|^2+\sum_{j=1}^{r}\|(I-T_j)x_{p+j}-(I-T_j)y_{p+j}\|^2\right)\\ &= \alpha|||(\mathbf{I}-\mathbf{T})\mathbf{x}-(\mathbf{I}-\mathbf{T})\mathbf{y}|||^2, \quad \alpha = min\{\alpha_1,\alpha_2\}.\end{aligned}$$

Theorem 3.3.7 *Let* H_1 *and* H_2 *be real Hilbert spaces and let* $A: H_1 \longrightarrow H_2$ *be a bounded linear map. Let* $f_i: H_1 \longrightarrow H_1$, $i = 1,2,...p$, *and* $g_j: H_2 \longrightarrow H_2$, $j = 1,2,...r$, *be* α*-ism maps on nonempty, closed and convex subsets* $C_i \subset H_1$, $Q_j \subset H_2$, *respectively. Assume that* $\gamma \in (0,1/L)$ *and* $\Psi \neq \emptyset$. *Set the maps* $U_i = P_{C_i}(I-\lambda f_i)$, $T_j = P_{Q_j}(I-\lambda g_j)$, $i = 1,2,...p$, $j = 1,2,...r$, *with* $\lambda \in [0,2\alpha]$. *If, in addition for each* $i = 1,2,...p$, $j = 1,2,...r$, *we have ,*

$$\langle f_i(x), P_{C_i}(I-\lambda f_i)(x)-x^*\rangle \geq 0 \ \textit{ for all } \ x \in H_1$$

for all $x^* \in SOL(C_i, f_i)$ *and*

$$\langle g_j(x), P_{Q_j}(I-\lambda g_j)(x) - x^*\rangle \geq 0 \text{ for all } x \in H_2$$

for all $x^* \in SOL(Q_j, G_j)$, *then any sequence* $\{x_k\}_{k=0}^{\infty}$ *generated by the algorithm 3.3.1 above converges weakly to a solution* $x^* \in \Psi$.

Proof

Following similar lines as we saw above, algorithm 3.3.1 is equivalent to 3.3.3. For this reason, we employ similar techniques as those in the proof of theorem 3.2.3.

$U = \mathbf{I} : H_1 \longrightarrow H_1$, $FixU = \mathbf{C}$, and $T = \mathbf{T} : \mathbf{W_2} \longrightarrow \mathbf{W_2}$, $FixT = \mathbf{Q}$. Let $z \in \Psi$ Then from Lemma 3.2.2 we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 \quad \forall k \geq 0.$$

So the sequence $\{\|x_k - z\|\}_{k=1}^{\infty}$ is monotonically decreasing and bounded below by 0. Hence $\lim_{k\to\infty} \|x_{k+1} - z\|$ exists in $\mathbb{R}$. From (3.2.9) we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + \gamma(L\gamma - 1)\|(\mathbf{T} - \mathbf{I})\mathbf{A}x_k)\|^2 \quad \forall k \geq 0.$$

Since $\gamma \in (0, 1/L)$, $\gamma(L\gamma - 1)\|(\mathbf{T} - \mathbf{I})\mathbf{A}x_k)\|^2 \leq 0$ and so $-\gamma(L\gamma - 1) \geq 0$. Therefore,

$$0 \leq -\gamma(L\gamma - 1)\|(\mathbf{T} - \mathbf{I})\mathbf{A}x_k)\|^2 \leq \|x_k - z\| - \|x_{k+1} - z\|.$$

Therefore (by sandwich theorem of sequences),

$$\lim_{k\to\infty} \|(\mathbf{T} - \mathbf{I})\mathbf{A}x_k)\| = 0. \tag{3.3.4}$$

Since $\{\|x_k - z\|\}_{k=0}^{\infty}$ is convergent, there exists $M > 0$ such that $\|x_k - z\| \leq M \quad \forall k \geq 0$. So,

$$\|x_k\| - \|z\| \leq |\|x_k\| - \|z\|| \leq \|x_k - z\| \leq M, \ \forall k \geq 0.$$

Thus,

$$\|x_k\| \leq M + \|z\| \ \ \forall k \geq 0.$$

Hence the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded. By Eberlein Smulyan Theorem (see, e.g., [50]),$\{x_k\}_{k=0}^{\infty}$ has a weakly convergent subsequence $\{x_{k_j}\}_{j=0}^{\infty}$ such that $x_{k_j} \rightharpoonup x^*$. This implies, using Riesz representation theorem, that $\langle x, x_{k_j}\rangle \to \langle x, x^*\rangle \;\; as\, j \longrightarrow \infty \quad \forall x \in H_1$. Observe that

$$\mathbf{A}x_k \rightharpoonup \mathbf{A}x^* \Longleftrightarrow g(\mathbf{A}x_k) \longrightarrow g(\mathbf{A}x^*), \forall g \in \mathbf{W_2^*} = \mathbf{W_2}.$$

Therefore using Riesz representation theorem, we have

$$\mathbf{A}x_k \rightharpoonup \mathbf{A}x^* \Longleftrightarrow \langle \mathbf{y}, \mathbf{A}x_{k_j}\rangle \longrightarrow \langle \mathbf{y}, \mathbf{A}x^*\rangle \quad \forall \mathbf{y} \in \mathbf{W_2}.$$

Now $\langle \mathbf{y}, \mathbf{A}x_{k_j}\rangle = \langle \mathbf{A}^*\mathbf{y}, x_{k_j}\rangle \xrightarrow{j\longrightarrow\infty} \langle \mathbf{A}^*\mathbf{y}, x^*\rangle = \langle \mathbf{y}, \mathbf{A}x^*\rangle$, by the fact that $x_k \rightharpoonup x^*$. Hence

$$\mathbf{A}x_k \rightharpoonup \mathbf{A}x^*$$

Since $\mathbf{T}$ is nonexpansive on $\mathbf{W_2}$ by Lemma 3.3.6 , $\mathbf{I}-\mathbf{T}$ is demiclosed at 0. Therefore, using $\mathbf{A}x_{k_j} \rightharpoonup \mathbf{A}x^*$ and (3.3.4) we have $(\mathbf{I}-\mathbf{T})\mathbf{A}x^* = 0$. But

$$(\mathbf{I}-\mathbf{T})\mathbf{A}x^* = 0 \Longleftrightarrow (\mathbf{A}x^*) - \mathbf{T}(\mathbf{A}x^*) = 0 \Longleftrightarrow \mathbf{T}(\mathbf{A}x^*) = \mathbf{A}x^*.$$

Hence

$$\mathbf{A}x^* \in Fix(\mathbf{T}).$$

It follows that

$$\mathbf{A}x^* \in SOL(Q_j, g_j) \;\; j = 1, 2, \cdots, r.$$

We define

$$\mathbf{v}_k := x_k + \gamma \mathbf{A}^*(\mathbf{T}-\mathbf{I})(\mathbf{A}x_k) \quad \forall k \geq 0.$$

Then

$$\mathbf{v}_{k_j} := x_{k_j} + \gamma \mathbf{A}^*(\mathbf{T}-\mathbf{I})(\mathbf{A}x_{k_j} \quad \forall j \geq 0.$$

Since $\lim_{k\to\infty} \|(\mathbf{T}-\mathbf{I})\mathbf{A}x_k)\| = 0$, we have $(\mathbf{T}-\mathbf{I})\mathbf{A}x_k \rightharpoonup 0$ as $k \longrightarrow \infty$ and so $(\mathbf{T}-\mathbf{I})\mathbf{A}x_{k_j} \rightharpoonup 0$ as $j \longrightarrow \infty$. Thus $x_{k_j} \rightharpoonup x^*$ and $(\mathbf{T}-\mathbf{I})\mathbf{A}x_{k_j} \rightharpoonup 0$ as $j \longrightarrow \infty$. It therefore follows that

$x_{k_j} + (\mathbf{T}-\mathbf{I})\mathbf{A}x_{k_j} \longrightarrow x^*$ as $j \longrightarrow \infty$, i.e.,

$$\mathbf{v}_{k_j} \rightharpoonup x^* \text{ as } j \longrightarrow \infty.$$

Next we show that

$$x^* \in SOL(C_i, f_i), \quad i = 1, 2, \cdots, p.$$

Now

$$\begin{aligned}
(\mathbf{A}x^*) \in \mathbf{Q} &\Longleftrightarrow \sqrt{\alpha_i}x^* = \sqrt{\alpha_i}z_i, \quad z_i \in Fix(U_i) \text{ and } \sqrt{\beta_j}Ax^* = \sqrt{\beta_j}w_j, \quad w_j \in Fix(T_j) \\
&\Longleftrightarrow x^* \in Fix(U_i) \;\; and \;\; Ax^* \in Fix(T_j) \\
&\Longleftrightarrow x^* \in \bigcap_{i=1}^{p} Fix(U_i) \;\; and \;\; Ax^* \in \bigcap_{j=1}^{r} Fix(T_j) \\
&\Longleftrightarrow x^* \in \Psi.
\end{aligned}$$

Since $(\mathbf{A}x^*) \in \mathbf{Q}$,

$$x^* \in \Psi.$$

Let $\{x_{k_j}\}_{j=0}^{\infty}$ be arbitrary subsequence of $\{x_k\}_{k=0}^{\infty}$ which converges weakly to u. Following similar arguments, we have $u \in \Gamma$. Hence Γ contains all its weak limit points of $\{x\}_{k=0}^{\infty}$. By the Lemma 3.1.5 we have

$$x_k \rightharpoonup x^* \text{as k} \longrightarrow \infty.$$

This completes the proof.

3.4 Split Monotone Variational Inclusion Problem

Modaufi was the first to introduce a generalization of split variational inequality problem in [1] and called it split monotone variational inclusion problem. Before giving the problem, we first take the following preliminaries.

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. Let $f : C \longrightarrow H$ be a map. We consider the variational inequality problem

$$\begin{cases} \text{find } x^* \in C \text{ such that} \\ \langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C. \end{cases}$$

It is immediately seen that

$$\begin{aligned} x^* \in SOl(C, f) \quad &\Longleftrightarrow \langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C \\ &\Longleftrightarrow -\langle f(x^*), x - x^* \rangle \leq 0 \ \forall x \in C \\ &\Longleftrightarrow \langle -f(x^*), x - x^* \rangle \leq 0 \ \forall x \in C \\ &\Longleftrightarrow -f(x^*) \in N_C(x^*) \\ &\Longleftrightarrow 0 \in f(x^*) + N_C(x^*). \end{aligned}$$

Thus, finding a solution of a variational inequality problem is equivalent to finding a zero of the multivalued map $f + N_C$. For the split variational inequality problem in (3.2.1), one easily sees that

$$x^* \in \Gamma \Longleftrightarrow \begin{cases} 0 \in f(x^*) + N_C(x^*) \ \text{ such that} \\ 0 \in g(Ax^*) + N_Q(Ax^*). \end{cases}$$

Therefore the split monotone variational inclusion problem is formulated as follows

$$\begin{cases} \text{find } x^* \in H_1 \text{ such that } \ 0 \in f(x^*) + B_1(x^*) \\ y^* = Ax^* \in H_2 \text{ such that } \ 0 \in g(y^*) + B_2(y^*), \end{cases}$$

where H_1, H_2 are two real Hilbert spaces, $A : H_1 \longrightarrow H_2$ is a bounded linear map, *Let* $f : H_1 \longrightarrow H_1$ and $g : H_2 \longrightarrow H_2$ are two given single-valued maps and $B_1 : H_1 \longrightarrow 2^{H_1}$ and $B_2 : H_2 \longrightarrow 2^{H_2}$ be two multi-valued mappings on H_1 and H_2, respectively.

Following the work of Censor *et al.* in [2], Moudafi in [1] proposed the following iterative algorithm.

Algorithm 3.4.1 **Initialization :** *Let $\lambda > 0$ and select an arbitrary starting piont $x_0 \in H_1$*
Iterative step : *Given the current iterate x^k, compute*

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k),$$

where $\gamma \in (0, 1/L)$, L is the spectral radius of the map A^*A , and A^* is the adjoint of A.

Before we present the convergence analysis of the iterative algorithm, we need to see some key properties of averaged maps.

Proposition 3.4.2 (see, e.g., [1]) *(i) If $T = (1-\alpha)S + \alpha V$, where $S : H \longrightarrow H$ is averaged, $V : H \longrightarrow H$ is nonexpansive and $\alpha \in (0,1)$, then T is also averaged.*
(ii) The composition of finitely many averaged mappings is averaged.
(iii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and they have a nonempty common fixed point set, then

$$\bigcap_{i=1}^{N} FixT_i = Fix(T_1 T_2 \cdots T_N).$$

(iv) If T is α- ism, then γT is $\frac{\alpha}{\gamma}$- ism for all $\gamma > 0$.
(v) T is averaged if and only if its complement $I - T$ is α- ism for some $\alpha > \frac{1}{2}$.

Averaged mappings are useful in the convergence analysis of iterative algorithms for fixed point problems due to the following result (see [48], [49]).

Theorem 3.4.3 (Krasnoselskii-Mann Theorem) *Let $M : H \longrightarrow H$ be an averaged map and assume $FixM \neq \emptyset$. Then for any starting point $x_0 \in H$, the sequence $\{M^k x_0\}$ converges weakly to a fixed point of M.*

Proof

Let $\{x_k\}_{k=0}^{\infty}$ be a sequence generated by M in the sence that $x_0 \in H$ and $x_k = M^k(x_0),\ k \geq 0$. Since M is averaged, there exist nonexpansive map T, and $\alpha \in (0,1)$ such that $M = (1-\alpha)I + \alpha T$. Therefore

$$x_{k+1} = M^{k+1}x_0 = MM^k x_0 = Mx_k = (1-\alpha)x_k + \alpha T x_k,\ \ k \geq 0.$$

Let z be a fixed point of M. Then z is a fixed point of T and

$$\begin{aligned}
\|x_{k+1} - z\|^2 &= \|(1-\alpha)x_k + \alpha T x_k - z\|^2 \\
&= \|(1-\alpha)(x_k - z) + \alpha(T x_k - z)\|^2 \\
&= (1-\alpha)\|(x_k - z)\|^2 + \alpha\|T x_k - z\|^2 - \alpha(1-\alpha)\|x_k - T x_k\|^2 \\
&= (1-\alpha)\|(x_k - z)\|^2 + \alpha\|T x_k - T z\|^2 - \alpha(1-\alpha)\|x_k - T x_k\|^2 \\
&\leq (1-\alpha)\|(x_k - z)\|^2 + \alpha\|x_k - z\|^2 - \alpha(1-\alpha)\|x_k - T x_k\|^2 \\
&= \|(x_k - z)\|^2 - \alpha(1-\alpha)\|x_k - T x_k\|^2 \qquad (3.4.1) \\
&\leq \|(x_k - z)\|^2.
\end{aligned}$$

Thus

$$\|x_{k+1} - z\|^2 \leq \|(x_k - z)\|^2 \ \ \forall k \geq 0,$$

i.e., $\{x_k\}_{k=0}^{\infty}$ is Fejer monotone with respect to $FixM$. So the sequence $\{\|x_k - z\|\}_{k=0}^{\infty}$ is monotonically decreasing and bounded below. Hence the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded. By Eberlein smulyan theorem (see, e.g., [50]), $\{x_k\}_{k=0}^{\infty}$ has a weakly convergent subsequence $\{x_{k_j}\}_{j=0}^{\infty}$ such that $x_{k_j} \rightharpoonup x^* \in H$. By (3.4.1),

$$\alpha(1-\alpha)\|x_k - T x_k\|^2 \leq \|(x_k - x^*)\|^2 - \|x_{k+1} - x^*\|^2 \longrightarrow 0 \ \ as\ k \longrightarrow \infty.$$

Since $\alpha(1-\alpha) > 0$, we have $\|x_k - T x_k\|^2 \longrightarrow 0$. This implies $\|x_k - T x_k\| \longrightarrow 0$ i.e., $(I-T)x_k \longrightarrow 0$ as $k \longrightarrow \infty$. Using Lemma 1.6.21 and the fact that T is nonexpansive, we have $(I-T)$ is demiclosed at 0, so that $(I-T)x^* = 0$. Thus $x^* \in FixT = FixM$. Let $\bar{x}$ be a weak cluster point of M, then there exists a subsequence x_{k_j} of x_k such that $x_{k_j} \rightharpoonup \bar{x}$ as $k \longrightarrow \infty$. Following similar arguments as above, $\bar{x} \in FixM$. By Lemma 3.1.10 and 3.1.11

we have $FixM$ is closed and convex. It follows from Lemma 3.1.5 that

$$x_k \rightharpoonup x^* \in FixM.$$

Remark 3.4.4 *[1] Let $\lambda > 0$ and B_1 a maximal monotone map. Then, it is easy to see*

$$x^* \in H_1 \ \ such\ that\ \ 0 \in f(x^*) + B_1(x^*) \Leftrightarrow x^* = J_\lambda^{B_1}(x^* - \lambda f(x^*)) \Leftrightarrow x^* \in Fix(J_\lambda^{B_1}(I - \lambda f)).$$

Indeed, when we conbined with definition of the resolvent and inverse maps we have

$$\begin{aligned}
0 \in f(x^*) + B_1(x^*) &\Leftrightarrow -f(x^*) \in B_1(x^*) \\
&\Leftrightarrow -\lambda f(x^*) \in \lambda B_1(x^*), \ \ for\ \ \lambda > 0 \\
&\Leftrightarrow x^* - \lambda f(x^*) \in x^* + \lambda B_1(x^*), \ \ \lambda > 0 \\
&\Leftrightarrow x^* - \lambda f(x^*) \in (I + \lambda B_1)(x^*), \ \ \lambda > 0 \\
&\Leftrightarrow (I + \lambda B_1)^{-1}(x^* - \lambda f(x^*)) = x^*, \ \ \lambda > 0 \\
&\Leftrightarrow J_\lambda^{B_1}(x^* - \lambda f(x^*)) = x^*, \ \ \lambda > 0 \\
&\Leftrightarrow x^* \in Fix(J_\lambda^{B_1}(I - \lambda f)) \ \ \lambda > 0.
\end{aligned}$$

(2) Let $\lambda > 0$, f, g be α_1-ism, α_2-ism, respectively and B_1, B_2 be maximal monotone maps. If $\lambda \in [0, 2\alpha]$, then the maps $J_\lambda^{B_1}(I - \lambda f)$ and $J_\lambda^{B_2}(I - \lambda g)$ are averaged. Indeed since f is $\alpha_1 - ism$, from proposition 3.4.2[iv], for $\lambda > 0$, λf is also $\frac{\alpha_1}{\lambda} - ism$ and therefore from proposition 3.4.2[v], $I - \lambda f$ is averaged. Since composition of finite averaged maps is averaged, from proposition 3.4.2[iii] and $J_\lambda^{B_1}$ is nonexpansive, $J_\lambda^{B_1}(I - \lambda f)$ is averaged.

We now present the main proof of Moudafi's theorem in [1]. We shall denote the solution set of the SMVIP by $\sum$.

Theorem 3.4.5 *Let H_1 and H_2 be two real Hilbert spaces and let $f : H_1 \longrightarrow H_1, g : H_2 \longrightarrow H_2$ be $\alpha_1 - ism$ on H_1 and $\alpha_2 - ism$ on H_2, respectively. Suppose B_1, B_2 are two maximal monotone maps and $A : H_1 \longrightarrow H_2$ is linear and bounded. Set $\alpha = min\{\alpha_1, \alpha_2\}$ and* *consider the two maps* $U := J_{\lambda}^{B_1}(I-\lambda f)$ *and* $T := J_{\lambda}^{B_2}(I-\lambda g)$ *with* $\lambda \in (0,2\alpha)$. *Then any sequence* $\{x_k\}_{k=1}^{\infty}$ *generated by the algorithm 3.4.1 above weakly converges to* $x^* \in \sum$.

Proof.

For any $z \in H_1$, $U(z) = z \Longleftrightarrow 0 \in f(z) + B_1(z)$. Indeed,

$$\begin{aligned}
0 \in f(z) + B_1(z) &\Leftrightarrow 0 = f(z) + u \ \text{ for some } \ u \in B_1(z)\\
&\Leftrightarrow z - \lambda f(z) = z + \lambda u \ \text{ for some } \ u \in B_1(z)\\
&\Leftrightarrow z - \lambda f(z) \in z + \lambda B_1(z)\\
&\Leftrightarrow (I-\lambda f)z \in (I+\lambda B_1)z\\
&\Leftrightarrow (I+\lambda B_1)^{-1}(I-\lambda f)z = z\\
&\Leftrightarrow U(z) = z.
\end{aligned}$$

Similarly,

$$\begin{aligned}
0 \in g(Az) + B_2(Az) &\Leftrightarrow 0 = g(Az) + u \ \text{ for some } \ u \in B_2(Az)\\
&\Leftrightarrow z - \lambda g(Az) = z + \lambda u \ \text{ for some } \ u \in B_2(Az)\\
&\Leftrightarrow z - \lambda g(z) \in Az + \lambda B_2(Az)\\
&\Leftrightarrow (I-\lambda g)Az \in (I+\lambda B_2)Az\\
&\Leftrightarrow (I+\lambda B_2)^{-1}(I-\lambda g)Az = Az\\
&\Leftrightarrow T(Az) = Az.
\end{aligned}$$

Let $z \in \sum$. Then $0 \in f(z) + B_1(z)$ and $0 \in g(Az) + B_2(Az)$. Thus, $U(z) = z$ and $T(Az) = Az$.

Let $V = I + \gamma A^*(T - I)A$. Then for any $z \in \sum$,

$$\begin{aligned} V(z) &= (I + \gamma A^*(T - I)A)z \\ &= z + \gamma A^*(T - I)Az \\ &= z + \gamma A^*(T(Az) - Az) \\ &= z + \gamma A^*(0) \\ &= z + 0 \\ &= z. \end{aligned}$$

Therefore, $\sum \subset Fix(V)$. Moreover, the map V is averaged. Indeed, for $x, y \in H_1$,

$$\begin{aligned} \langle A^*(I-T)Ax - A^*(I-T)Ay, x-y\rangle &= \langle A^*((I-T)Ax - (I-T)Ay),\ x-y\rangle \\ &= \langle (I-T)Ax - (I-T)Ay,\ A(x-y)\rangle \\ &= \langle (I-T)Ax - (I-T)Ay,\ Ax - Ay\rangle. \end{aligned}$$

Since T is averaged, $I - T$ is α-ism (from proposition 3.4.2[v]). Thus,

$$\begin{aligned} \langle (I-T)Ax - (I-T)Ay, Ax - Ay\rangle &\geq \alpha \|(I-T)Ax - A^*(I-T)Ay\|^2 \\ &\geq \frac{\alpha}{L}\|A^*(I-T)Ax - A^*(I-T)Ay\|^2 \\ &\quad \text{(since } L \text{ is the spectral radius of } AA^* \text{ see, e.g., [1])} \end{aligned}$$

for all $x, y \in H_1$. Hence, $\gamma A^*(I-T)A$ is $\frac{\alpha}{\gamma L}$-ism. Using proposition 3.4.2[v] again and using $0 < \gamma < \frac{1}{L}$ we have

$$V = I - \gamma A^*(I-T)A$$

is averaged. Therefore, the map $M := U(I + \gamma A^*(T - I)A)$ is a composition of two averaged maps and so is averaged. From Theorem 3.4.3, $\{M^k x_0\}_{k=0}^{\infty}$ converges weakly to a fixed point x^* of M, i.e., $\{x_k\}_{k=0}^{\infty}$ converges weakly to a fixed point x^* of M. This fixed points of M is also a fixed point of U and V by proposition 3.4.2[iii]. Now,

$$U(x^*) = x^* \Longleftrightarrow J_\lambda^{B_1}(x^* - \lambda f(x^*)) = x^* \Longleftrightarrow x^* \in Fix(J_\lambda^{B_1}(I - \lambda f)) \Longleftrightarrow 0 \in f(x^*) + B_1(x^*)$$

and

$$V(x^*) = x^* \Longleftrightarrow x^* + \gamma A^*(T-I)Ax^* = x^* \Longleftrightarrow \gamma A^*(T-I)Ax^* = 0 \Longleftrightarrow A^*(T-I)Ax^* = 0.$$

Setting $w = (T-I)Ax^*$, we have $A^*w = 0$ and $T(Ax^*) = Ax^* + w$. The fact that $T(Az) = Az$ yeilds

$$\begin{aligned}
\|T(Ax^*) - T(Az)\|^2 &= \|(Ax^* + w) - Az\|^2 \\
&= \|Ax^* - Az\|^2 + 2\langle Ax^* - Az, w\rangle + \|w\|^2 \\
&= \|Ax^* - Az\|^2 + 2\langle A(x^* - z), w\rangle + \|w\|^2 \\
&= \|Ax^* - Az\|^2 + 2\langle x^* - z, A^*w\rangle + \|w\|^2 \\
&= \|Ax^* - Az\|^2 + 0 + \|w\|^2 \\
&= \|Ax^* - Az\|^2 + \|w\|^2.
\end{aligned}$$

Since T is nonexpansive,

$$\|Ax^* - Az\|^2 + \|w\|^2 = \|T(Ax^*) - T(Az)\|^2 \tag{3.4.2}$$

$$\leq \|Ax^* - Az\|^2 \tag{3.4.3}$$

which gives $\|w\|^2 = 0$. Thus, $w = 0$. By definition of w we have $T(Ax^*) = Ax^*$. i.e.,

$$Ax^* \in Fix(T) \text{ or } 0 \in g(Ax^*) + B_2(Ax^*).$$

It follows that,

$$0 \in f(x^*) + B_1(x^*) \text{ and } 0 \in g(Ax^*) + B_2(Ax^*),$$

i.e.,

$$x^* \text{solve the } SMVIP.$$

CHAPTER FOUR

MAIN RESULT

4.1 Introduction

This chapter presents the solution of the research problem, i,e., it presents the parallel algorithm for the multi-set split mononote variational inclusion problem and its convergence.

4.2 Multi-set split monotone variational inclusion problem

Now we introduce the multi-set split monotone variational inclusion problem. Let H_1, H_2 be two real Hilbert spaces. A multi-set split variational inequality is the following

$$(MSSMVIP) \quad \begin{cases} \text{find } x^* \in H_1 \text{ such that } \quad 0 \in f_i(x^*) + B_i^1(x^*),\ 1 \leq i \leq p \\ y^* = Ax^* \in H_2 \text{ such that } \quad 0 \in g_j(y^*) + B_j^2(y^*),\ 1 \leq j \leq r, \end{cases}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear map, $f_i : H_1 \longrightarrow H_1$ and $g_j : H_2 \longrightarrow H_2$ are two given singled valued maps and $B_i^1 : H_1 \longrightarrow 2^{H_1}$ and $B_j^2 : H_2 \longrightarrow 2^{H_2}$ are two multi valued maximal monotone maps, $i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, r$.

This problem is a generalization of split monotone variational inclusion problem considered in Chapter three where only two maps f and g and two multi-value maximal monotone maps B_1 and B_2 were considered. Here we have two families of single-valued maps f_i, g_j and two families of multi-valued maximal monotone maps B_i^1 and B_j^2. One recalls that the split monotone variational inclusion problem is a generalization itself, of the split variational inequality problem introduced in [2] by Censor *et al.* Throughout this Chapter, we shall denote by $\sum$ the solution set of multi-set split monotone variational inclusion problem, i.e.,

$$\sum := \{z \in H_1 : 0 \in f_i(z) + B_i^1(z) \text{ and } 0 \in g_j(Az) + B_j^2(Az), 1 \leq i \leq p, 1 \leq j \leq r\}.$$

4.3 Parallel algorithm for solving monotone variational inclusion problem

From the definition of the multi-set split monotone variational inclusion problem above, it is clear that the problem is that of finding a common solution to finitely many split monotone variational inclusion problems. We now present a parallel algorithm which generates a sequence and we prove that this sequence converges weakly to a common solution of this finite family of split monotone variational inclusion problems.

Theorem 4.3.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $f_i : H_1 \longrightarrow H_1, i = 1, 2, ...p$ and $g_j : H_2 \longrightarrow H_2$, $j = 1, 2, ...r$ be $\alpha_1 -$ ism and $\alpha_2 -$ ism maps on H_1 and H_2, respectively. Suppose $B_i^1 : H_1 \longrightarrow 2^{H_1}$, $B_j^2 : H_2 \longrightarrow 2^{H_2}$ are two maximal monotone maps, and let $A : H_1 \longrightarrow H_2$ be linear and bounded. Set $\alpha = min\{\alpha_1, \alpha_2\}$ and consider the family of maps $U_i := J_\lambda^{B_i^1}(I - \lambda f_i)$ and $T_j := J_\lambda^{B_j^2}(I - \lambda g_j)$ $1 \leq i \leq p$, $1 \leq j \leq r$, with $\lambda \in (0, 2\alpha)$. Define a sequence iteratively by*

$$\begin{cases} x_0 \in H_1 \;\; \textit{arbitrary}, \\ x_{k+1} = x_k + \gamma\Big(\sum_{i=1}^p \alpha_i(U_i(x_k) - x_k) + \sum_{j=1}^r \beta_j A^*(T_j(Ax_k) - Ax_k)\Big), \;\; k \geq 0, \end{cases} \tag{4.3.1}$$

where $\gamma \in \left(0, \frac{1}{L}\right)$, with $L = \sum_{i=1}^p \alpha_i + \sum_{j=1}^r \beta_j \|A\|^2$. Then the sequence $\{x_k\}_{k=0}^\infty$ generated by the algorithm 4.3.1 converges weakly to $x^* \in \Gamma$.

Proof.

$$\begin{aligned} x_{k+1} &= x_k + \gamma \mathbf{A}^*(\mathbf{T}(\mathbf{A}x) - (\mathbf{A}x)) \\ &= x_k + \gamma \mathbf{A}^*(\sqrt{\alpha_1}(U_1(x_k) - x_k) + \sqrt{\alpha_2}(U_2(x_k) - x_k) + \cdots + \sqrt{\alpha_p}(U_p(x_k) - x_k) \\ &\quad + \sqrt{\beta_1}(T_1(Ax_k) - Ax_k) + \sqrt{\beta_2}(T_2(Ax_k) - Ax_k) + \cdots + \sqrt{\beta_r}(T_r(Ax_k) - Ax_k)). \end{aligned}$$

Since

$$\mathbf{A}^* = (\sqrt{\alpha_1}I, \sqrt{\alpha_2}I, \cdots, \sqrt{\alpha_p}I, \sqrt{\beta_1}A^*, \sqrt{\beta_2}A^*, \cdots, \sqrt{\beta_r}A^*),$$

we have

$$\begin{aligned} x_{k+1} &= x_k + \gamma(\sqrt{\alpha_1}I, \sqrt{\alpha_2}I, \cdots, \sqrt{\alpha_p}I, \sqrt{\beta_1}A^*, \sqrt{\beta_2}A^*, \cdots, \sqrt{\beta_r}A^*)(\sqrt{\alpha_1}(U_1(x_k) - x_k) + \\ &\quad \sqrt{\alpha_2}(U_2(x_k) - x_k) + \cdots + \sqrt{\alpha_p}(U_p(x_k) - x_k) + \sqrt{\beta_1}(T_1(Ax_k) - Ax_k) + \sqrt{\beta_2}(T_2(Ax_k) - Ax_k) \\ &\quad + \cdots + \sqrt{\beta_r}(T_r(Ax_k) - Ax_k)) \\ &= x_k + \gamma(\alpha_1(U_1(x_k) - x_k) + \alpha_2(U_2(x_k) - x_k) + \cdots + \alpha_p(U_p(x_k) - x_k) + \beta_1 A^*(T_1(Ax_k) - Ax_k) \\ &\quad + \beta_2 A^*(T_2(Ax_k) - Ax_k) + \cdots + \beta_r A^*(T_r(Ax_k) - Ax_k)) \\ &= x_k + \gamma\left(\sum_{i=1}^{p} \alpha_i(U_i(x_k) - x_k) + \sum_{j=1}^{r} \beta_j A^*(T_j(Ax_k) - Ax_k)\right), k \geq 0. \end{aligned}$$

For any $z \in H_1$,

$$\begin{aligned} 0 \in \bigcap_{i=1}^{p} \left(f_i(z) + B_i^1(z)\right) &\Leftrightarrow 0 = f_i(z) + u \text{ for some } u \in B_i^1(z), \ \forall i \in \{1,2,\dots p\} \\ &\Leftrightarrow z - \lambda f_i(z) = z + \lambda u \text{ for some } u \in B_i^1(z), \ \forall i \in \{1,2,\dots p\} \\ &\Leftrightarrow z - \lambda f_i(z) \in z + \lambda B_i^1(z), \ \forall i \in \{1,2,\dots p\} \\ &\Leftrightarrow (I - \lambda f_i)z \in (I + \lambda B_i^1)z, \ \forall i \in \{1,2,\dots p\} \\ &\Leftrightarrow (I + \lambda B_i^1)^{-1}(I - \lambda f_i)z = z, \ \forall i \in \{1,2,\dots p\} \\ &\Leftrightarrow U_i(z) = z, \ \forall i \in \{1,2,\dots p\}. \\ &\Leftrightarrow z \in \bigcap_{i=1}^{p} Fix(U_i) \end{aligned}$$

Therefore, for any $z \in H_1$, $0 \in \bigcap_{i=1}^{p} \left(f_i(z) + B_i^1(z)\right) \Leftrightarrow z \in \bigcap_{i=1}^{p} Fix(U_i)$. Similarly, for any $z \in H_2$, $0 \in \bigcap_{i=1}^{r} \left(g_j(z) + B_i^2(z)\right) \Leftrightarrow z \in \bigcap_{j=1}^{r} Fix(T_j)$. Let $z \in \sum$. Then $0 \in f_i(z) +$

$B_i^1(z)$ $\forall i \in \{1,2,\cdots,p\}$ and $0 \in g_j(Az) + B_j^2(Az)$, $\forall j \in \{1,2,\cdots,r\}$. Thus, $U_i(z) = z$ and $T_j(Az) = Az$.

Let $\mathbf{V} = \mathbf{I} + \gamma \mathbf{A}^*(\mathbf{T} - \mathbf{I})\mathbf{A}$. Looking at the algorithm 4.3.1, $x_{k+1} = \mathbf{V}x_k,\ k \geq 0$. Then for any $z \in \sum$,

$$\begin{aligned}
\mathbf{V}(z) &= (\mathbf{I} + \gamma \mathbf{A}^*(\mathbf{T} - \mathbf{I})\mathbf{A})z \\
&= z + \gamma \mathbf{A}^*(\mathbf{T} - \mathbf{I})\mathbf{A}z \\
&= z + \gamma \mathbf{A}^*(\mathbf{T}(\mathbf{A}z) - \mathbf{A}z) \\
&= z + \gamma \mathbf{A}^*(0) \\
&= z + 0 \\
&= z.
\end{aligned}$$

Therefore, $\sum \subset Fix(\mathbf{V})$. Moreover, the map $\mathbf{V}$ is averaged. Indeed, for any $x, y \in H_1$,

$$\begin{aligned}
\langle \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}x - \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}y,\ x - y\rangle &= \langle \mathbf{A}^*((\mathbf{I} - \mathbf{T})\mathbf{A}x - (\mathbf{I} - \mathbf{T})\mathbf{A}y),\ x - y\rangle \\
&= \langle (\mathbf{I} - \mathbf{T})\mathbf{A}x - (\mathbf{I} - \mathbf{T})\mathbf{A}y,\ \mathbf{A}(x - y)\rangle \\
&= \langle (\mathbf{I} - \mathbf{T})\mathbf{A}x - (\mathbf{I} - \mathbf{T})\mathbf{A}y,\ \mathbf{A}x - \mathbf{A}y\rangle.
\end{aligned}$$

Since $\mathbf{I} - \mathbf{T}$ is α-ism from Lemma 3.3.6[ii], $\mathbf{T}$ is averaged (from proposition 3.4.2[v]). Thus,

$$\begin{aligned}
\langle (\mathbf{I} - \mathbf{T})\mathbf{A}x - (\mathbf{I} - \mathbf{T})Ay,\ \mathbf{A}x - \mathbf{A}y\rangle &\geq \alpha \|(\mathbf{I} - \mathbf{T})\mathbf{A}x - \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}y\|^2 \\
&\geq \frac{\alpha}{L} \|\mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}x - \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}y\|^2.
\end{aligned}$$

Hence $\gamma \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}$ is $\frac{\alpha}{\gamma L}$-ism. Using proposition 3.4.2 and $0 < \gamma < \frac{1}{L}$, we have

$$\mathbf{V} = \mathbf{I} - \gamma \mathbf{A}^*(\mathbf{I} - \mathbf{T})\mathbf{A}$$

is averaged.

Therefore, from Theorem 3.4.3, $\{\mathbf{V}^k x_0\}_{k=1}^{\infty}$ converges weakly to a fixed point x^* of $\mathbf{V}$. Now,

$$\mathbf{V}(x^*) = x^* \Longleftrightarrow x^* + \gamma \mathbf{A}^*(\mathbf{T}-\mathbf{I})\mathbf{A}x^* = x^* \Longleftrightarrow \gamma \mathbf{A}^*(\mathbf{T}-\mathbf{I})\mathbf{A}x^* = 0 \Longleftrightarrow \mathbf{A}^*(\mathbf{T}-\mathbf{I})\mathbf{A}x^* = 0.$$

Setting $\mathbf{w} = (\mathbf{T}-\mathbf{I})\mathbf{A}x^*$, we have $\mathbf{A}^*\mathbf{w} = 0$ and $\mathbf{T}(\mathbf{A}x^*) = \mathbf{A}x^* + \mathbf{w}$. From the fact that $\mathbf{T}(\mathbf{A}z) = \mathbf{A}z$, we obtain

$$\begin{aligned}
|||\mathbf{T}(\mathbf{A}x^*) - \mathbf{T}(\mathbf{A}z)|||^2 &= |||(\mathbf{A}x^* + \mathbf{w}) - \mathbf{A}z|||^2 \\
&= |||\mathbf{A}x^* - \mathbf{A}z|||^2 + 2\langle\langle \mathbf{A}x^* - \mathbf{A}z,\ \mathbf{w}\rangle\rangle + |||\mathbf{w}|||^2 \\
&= |||\mathbf{A}x^* - \mathbf{A}z|||^2 + 2\langle\langle \mathbf{A}(x^* - z),\ \mathbf{w}\rangle\rangle + |||\mathbf{w}|||^2 \\
&= |||\mathbf{A}x^* - \mathbf{A}z|||^2 + 2\langle\langle x^* - z, \mathbf{A}^*\mathbf{w}\rangle\rangle + |||\mathbf{w}|||^2 \\
&= |||\mathbf{A}x^* - \mathbf{A}z|||^2 + 0 + |||\mathbf{w}|||^2 \\
&= |||\mathbf{A}x^* - \mathbf{A}z|||^2 + |||\mathbf{w}|||^2.
\end{aligned}$$

Since $\mathbf{T}$ is nonexpansive from Lemma 3.3.6[i],

$$\begin{aligned}
|||\mathbf{A}x^* - \mathbf{A}z|||^2 + |||\mathbf{w}|||^2 &= |||\mathbf{T}(\mathbf{A}x^*) - \mathbf{T}(\mathbf{A}z)|||^2 \\
&\leq |||\mathbf{A}x^* - \mathbf{A}z|||^2
\end{aligned}$$

which gives $|||\mathbf{w}|||^2 = 0$. Thus, $\mathbf{w} = 0$. By definition of $\mathbf{w}$ we have $\mathbf{T}(\mathbf{A}x^*) = \mathbf{A}x^*$, i.e.,

$$\mathbf{A}x^* \in Fix(\mathbf{T}).$$

This gives $\mathbf{T}(\mathbf{A}x^*) = \mathbf{A}x^*$, i.e.,
$(\sqrt{\alpha_1}U_1(x^*), \sqrt{\alpha_2}U_2(x^*), \ldots, \sqrt{\alpha_p}U_p(x^*), \sqrt{\beta_1}T_1(Ax^*), \sqrt{\beta_2}T_2(Ax^*), \ldots, \sqrt{\beta_r}T_r(Ax^*))^* =$
$(\sqrt{\alpha_1}x^*, \sqrt{\alpha_2}x^*, \ldots, \sqrt{\alpha_p}x^*, \sqrt{\beta_1}Ax^*, \sqrt{\beta_2}Ax^*, \ldots, \sqrt{\beta_r}Ax^*)^*$
$U_i(x^*) = x^*, \forall i \in \{1,2,\ldots,p\}$ and $T_j(x^*) = x^*, \forall j \in \{1,2,\ldots,r\}$.
Thus,

$$0 \in f_i(x^*) + B_i^1(x^*),\ \forall i \in \{1,2,\ldots,p\}\ \text{ and }\ 0 \in g_j(Ax^*) + B_j^2(\mathbf{A}x^*),\ \forall j \in \{1,2,\ldots,r\}.$$

It follows that $x^* \in \sum$. We conclude that $\{x_k\}_{k=0}^{\infty}$ converges weakly to a solution of the MSSMVIP.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Introduction

This chapter gives summary and conclusion of the dissertation, it also contains recommendation.

5.2 Summary

In this reseach, we presented a generalization of split monotone variational inclusion problem as multi-set split monotone variational inclusion problem. As we have seen, the split monotone variational inclusion problem gives a generalization of split variational inequality problem which is, itself, a generalization of classical variational inequality problem.

Chapter one of this dissertation gives a brief preamble on the concept of variational inequality problem, statement of our research problem, its scope and limitation and definition of some basic terms.

Literarature review on some published works that are related to our work was given in chapter two.

In Chapter three, we presented in elaborate manner the techniques of proof of the two papers [1] and [2] from which our work was formulated.

In chapter four, we present the main work of the research that is, generalization of SMVIP and established its weak convergence as proposed by moudafi in [1].

5.3 Conclusion

Censor *et al.* in [2] introduced the notion of split variational inequality problem successfully obtained weak convergence result for the split variational inequality problem. Furthermore, using similar techniques as in the work Censor and Segal in [20], weak convergence results for finite family of split variational inequality problem was obtained by Censor *et al.* in [2] . Following Moudafi in [1], we have been able to provide a parallel algorithm which generates sequence that converges weakly to a common solution of finite family of a split monotone variational inequality problem.

5.4 Recommendation

The convergence of the sequence in all the theorems considered are weak (in infinite dimensional real Hilbert space). It is desirable to obtain strong convergence theorem with respect to both split variational inequality problems and split monotone variational inclusion problems.

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