GEOMETRICAL APPROACH FOR CONSTRUCTING BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

BY

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DECLARATION

I, Akra, Ukeme Paulinus with Registration number STA/Ph.D/17/003, hereby declare that this thesis on "Geometrical Approach for Constructing Balanced Incomplete Block Design (BIBD)" is original, and has been written by me. It is a record of my research work and has not been presented before in any previous publication.

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CERTIFICATION

This is to certify that this thesis titled **'GEOMETRICAL APPROACH FOR CONSTRUCTING BALANCED INCOMPLETE BLOCK DESIGN (BIBD)'** and carried out by AKRA, UKEME PAULINUS with Registration Number STA/PhD/17/003 has been examined and found worthy of the award of the degree of Doctor of Philosophy Degree (PhD) in Statistics.

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ABSTRACT

Geometrical approach for constructing balanced incomplete block design (BIBD) is explored in this research work. Various construction techniques have been proposed by many authors to build the elements of BIBDs for specific parameters. In the search for more robust methods for construction of balanced incomplete block design(BIBD), the concept of group algebra were adopted to find a structure for BIBDs. Finite Projective Geometry of order PG(N,s), and Euclidean Geometry of order EG(N,s) were used to construct a design structure with elements specifying the parameters that satisfy BIBD. PG(3,2) and EG(3,2) is used to construct BIBD with the parameters (15,15,7,7,3) and (8,14,7,4,3). The concept of algebra was introduced to elucidate whether the newly constructed BIBDs satisfied the postulates of group, ring and field theory. Some theorems were presented and proved based on the result of the constructed balanced incomplete block designs. The results show that the technique used in this research proved a better solution than other methods, when compared in terms of estimation of parameters to build the design structure. Based on the aforementioned postulates. It was established that BIBD with varieties and blocks of (X, B) does not form a group under multiplicative binary operation (X, *) but under additive binary operation (X, +) does. Also a ring with the design (X, +,*) is form under the two binary operations. Hence, the balanced incomplete block design with the parameters (t,b,r,k,λ) of the form (X, B) formed a semigroup, commutative group, semiring, commutative ring and subfield with both binary operations. **(Word count: 249)**

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ABBREVIATIONS

BIBD	Balanced Incomplete Block Design
PG(N,s)	Projective Geometry of N-dimensional order (N,s)
EG(N,s)	Euclidean Geometry of N-dimensional order of (N,s)
GF(s)	Galois Field with finite number of elements (s)

SYMBOLS

X	Treatment
B	Blocks
R	Ring
G	Group
F	Field
$o(G)$	Order of a group
$(*,+)$	Binary operations
$\forall$	For all
$\exists$	There exist
$\in$	Belong to
Ψ	Shi
Φ	Phi
Ω	Omega
Σ	Summation
M	Matrix
Z	Integer
$p^n = s$	Finite number of elements
$\subset$	Subset
N	Dimension
e	Identity element

CHAPTER ONE
INTRODUCTION

1.1 Background of the study

Block designs originated from the statistical framework of design of experiments. The generation of block designs is a well-known combinatorial problem, which is very hard to solve (Colbourn and Dinitz, 1996). Block design is a type of combinatorial design defined as a pair (V, B) such that V is a finite set and B is a collection of nonempty subsets of V, the elements in V are called points while subsets in B are called blocks. This design has wider applications from many areas, including experimental design, finite geometry, physical chemistry, software testing, cryptography and algebraic units (Stinson, 2004). Blocking is the arranging of experimental units in groups (blocks) that are similar to one another. A block design in which all the blocks have the same size is called uniform. The incidence matrix of block designs provides a natural source of interesting block codes that are used as error correcting codes. The rows of their incidence matrices are also used as the symbols in a form of pulse – position modulation (Noshad and Brandt-Pearce, 2012).

The origin of incomplete block designs dated back to (Yates 1936) who introduced the concept of balanced incomplete block designs and their analysis using both intra-block and inter - block information. Other incomplete block designs were also proposed by Yates (1940), who referred to these designs as quasi-factorial or lattice designs. When the number of treatments to be compared is large, a large number of blocks to accommodate all the treatments will be needed for the experiments. This requires more experimental material and so the cost of experimentation becomes very high. The completely randomized design and randomized block design may not be suitable in such situations because they will require large number of experimental units to accommodate all the treatments. When sufficient numbers of homogeneous experimental units are not available to accommodate all the treatments in a block, then incomplete block designs can be used (Neil, 2010). In incomplete block designs, each block receives only some of the selected treatments and not all the treatments. A block design is incomplete if the number of varieties is greater than the block size ($k < v$), then the design is incomplete block design. Smith and Street (2003) investigated the best balanced incomplete block design used to estimate the parameter rates for a smaller variance than estimates obtained using the similar design.

The idea of balancing a design with incomplete blocks was introduced by (Fisher, 1940) in his theoretical study of the design of experiments in agriculture. A regular block design with V- varieties and b blocks is balanced and is called (v,b,r,k,λ)-design or (v,k,λ) - design if each pair of elements x_i and x_j has the same covalence called index of the design. A balanced design is complete if $k = v$, so that each block contains all of X. A balanced incomplete block design (BIBD) is an incomplete block design which consists of a set of v points that is divided into b subsets in such a way that each point in v is contained in r different subsets and any couple of points in v is contained in $\lambda < b$ subsets with $k < v$ points in each subset. A BIBD is specified by five parameters (v,b,r,k,λ). The five parameters defined as (v,b,r,k,λ)- BIBD are related and satisfy the following two relations: $bk = vr$ and $\lambda(v-1) = r(k-1)$. In fact, the corresponding instance can be defined by just three parameters (v, k, λ) since b and r are given in terms of the other parameters. Clearly, these relations restrict the set of admissible parameters for a BIBD; however, the parameter admissibility is a necessary condition but it is not sufficient to guarantee the existence of a BIBD (Cochran and Cox, 1957). Since then, the generation of block designs remains an unsolved problem in combinatorial mathematics (Colbourn and Dinitz, 2007).The problem has a number of variants, among which is the popular so-called Balanced

Incomplete Block Designs (BIBDs). The construction of BIBDs was initially attracted in the area of experimental design (Mead, 2016). Yu-pei, Chia-an, Yaotsu, and Chong-Dao (2018) proposed a new family of group divisible designs(GDDs), which is also involved with a well-known balanced incomplete block design (BIBD). However, a balanced incomplete block design is an incomplete block design in which any two varieties appear together an equal number of times. This design can be constructed by taking $\binom{v}{k}$ blocks and assigning a different combination of varieties to each block (Montgomery 2013).

A standard way of representing a BIBD is in terms of its incidence matrix $M \equiv \{m_{ij}\}_{v \times b}$, which is a $v \times b$ binary matrix with exactly r ones per row, k ones per column, a scalar product of λ between any pair of distinct rows, and where $m_{ij} \in \{0,1\}$ is equal to I if the i^{th} object is contained in the j^{th} block, and 0otherwise. In this context, m_{ij} represents the incidence of object i in block j of M. The construction of such design depends on the actual arrangement of the treatments into blocks and this problem is handled in combinatorial design.

Combinatorial design (incidence structure) consists of a domain of set X and another set B, commonly represented as subsets of that domain. Combinatorial design theory is a branch of combinatorial mathematics which underlines the study of existence, construction and properties of finite sets whose arrangements satisfy some concepts of balance and symmetry BIBD (Parathy, 2015). Combinatorial designs can be applied in different areas like: error-correcting codes, statistical design of experiments, cryptography and information security, mobile and wireless communications, group testing algorithms in DNA screening, software and hardware testing, interconnection networks, engineering, and life sciences (Mariusz, 2017).

In algebra, finite geometry and existence of combinatorial design are closely related which can be used to construct a particular design. In constructing BIBD using algebraic approach, basic knowledge of algebra shall be discussed such as finite group, rings, field, Isomorphism, automorphism and finite geometry. A combinatorial design on a finite set of elements is called a finite geometry. The elements of their domains are called points of the geometry and their distinguished subsets are called lines of the geometry. The two disjoint lines of geometry are often said to be parallel lines. Hence, this research adopts the combinatorial approach (finite geometries) of constructing balanced incomplete block design (BIBD).

1.2 Statement of the problems

Different techniques such as Fisher's inequality (1956), Local search algorithm (Pretwish, 2003), etc, have been advanced for testing the existence or non-existence of BIBDs for given parameters, still there is no suitable method to determine the existence of these designs. Also various construction methods like Graph theory and computing (Mills, 1979), complete set of orthogonal squares (Muller, 1965), unreduced method (Hsiao-Lih, Tai-Chang and Babul, 2007), cyclic shift (Fariha, Rashid and Munir, 2015) and Galois field (Janardan, 2018) have been introduced to build the elements of BIBDs for specific parameters, no general technique has been presented to find the structure of BIBDs. This research invokes geometrical approach to construct balanced incomplete block design to fill the existing gap above which is the major concern which advances this research.

1.3 Aim and objectives of the study

The aim of this research work is to explore geometrical approach to construct BIB-Design and the specific objectives are:

1. To construct a BIB-Design using Projective Geometry {PG(3,2)} and Euclidean Geometry{EG(3,2)}.
2. To examine whether the design constructed in (1) form a finite group, ring and field algebra.
3. To show that two BIB-Designs are isomorphic and automorphic to each other.
4. To state some theorems with proofes as it relate to (2) and (3) above.

1.4 Significance of the study

It is well-known that proper blocking reduces experimental error, thereby making it more sensitive to detecting significance of effects which in turn leads to fewer number of experiments. This design will help in eliminating the effect of heterogeneity to a greater extent than is possible with randomized block design and Latin square design. The approach used in this work to construct balanced incomplete block design will be significant for estimation of parameters and will also help in easy detection whether or not a particular design is a BIBD. The construction of balanced incomplete block designs using finite projective geometry and Euclidean geometry will also provide possible relationships in the parameters of the resulting BIBDs.

1.5 Scope of the study

This research is limited to balanced incomplete block design. The concept of finite group, rings, finite field, isomorphism and automorephism related to balanced incomplete block design will be considered. In constructing balanced incomplete block design, this research shall be restricted to only PG (3,2) and EG (3,2) designs.

CHAPTER TWO

LITERATURE REVIEW

2.1 Balanced Incomplete Block Designs

The origins of incomplete block designs go back to (Yates, 1936) who introduced the concept of balanced incomplete block designs and their analysis utilizing both intra-block and inter - block information. Other incomplete block designs were also proposed by (Yates, 1940) who referred to these designs as quasi-factorial or lattice designs. The notion of balanced incomplete block design was generalized to that of partially balanced incomplete block designs (Bose and Nair 1939), which encompass some of the lattice designs. Further extensions of the balanced incomplete block designs and lattice designs were made by introducing balanced incomplete block designs for eliminating heterogeneity in two directions (generalizing the concept of the Latin square design) and rectangular lattices some of which are more general designs than partially balanced incomplete block designs (Harshbarger, 2010). After this, there has been a very rapid development in this area of experimental design.

In order to eliminate heterogeneity; a concept of Balanced Incomplete Block Design was introduced, which reduce heterogeneity to a greater extent than is possible with randomized block design and Latin square design. The history of BIB designs probably dates back to the 19th Century. The solution of the famous Kirkman's School girl problem has one-one correspondence with the solution of BIB design (Kirkman 2017). Goud and Charyulu (2016) propose two new balanced incomplete block design construction methods using two associate class partially balanced incomplete block designs. Different methods of construction of balanced incomplete block designs have been given in the literature, like, (Agrawal and Prasad, 2016, 2017; Calinski,2017; Alltop, 2018; Hanani,2016; Majinder,2011;Shrikhande, 2013) etc. Bayrak and Bulut (2006) constructed orthogonal balanced incomplete block design. Pachamuthu (2011) showed construction of mutually orthogonal Latin square and check parameter relationship of balanced incomplete block design. There are three main concepts of balancing in incomplete block designs, namely; Variance Balanced, Efficiency Balanced and Neighbour Balanced (Puri and Nigam, 2016). BIBD can be classified into families; Hedayat and Hwang (2015) divided the collection of all BIB designs with $t > 2$ into three mutually exclusive and exhaustive families:

Family 1 consists of all (v, b, r, k, λ) - BIBD whose parameters (b, r, λ) have a common integer divisor, $t > 1$ and there exists one or more $(v, b/t, r/t, k, \lambda/t)$ - BIB designs.

Family 2 consists of all (v, b, r, k, λ) - BIBD whose parameters (b, r, λ) have one or more common integer divisors greater than one but there is no $(v, b/t, r/t, k, \lambda/t)$ - BIB design if $t > 1$ is one of the divisors of b, r, and λ. No member of this family can be obtained by collecting or taking copies of smaller size BIB designs of type $(v, b/t, r/t, k, \lambda/t)$ - BIBD.

Family 3 consists of all (v, b, r, k, λ) - BIBD whose parameters (b, r, λ) are relatively prime, thus the parameters v, b, r, k, λ for the members of this family are such that the great integer divisor of (b, r, λ) is one. As in family two, in family three no member can be obtained by collecting or taking copies of smaller size BIB designs of type $(v, b/t, r/t, k, \lambda/t)$ - BIBD.

BIB Designs are optimal for a number of criteria under the usual homocedastic linear additive model and this optimality is not affected if the design admits block repetition. Raghavarao and Padgett (2005) explain the significance and connections of BIB Designs to the following statistical areas: Controlled Sampling and Finite Sample Support, Balanced Incomplete Cross Validation, Box-Behnken Designs, Randomized

Response Procedure, Intercropping Experiments, Group Testing, Validation Studies, Tournament, Fractional Plans, Balanced Half-Samples and Lotto Designs.

2.2 Balance Incomplete Block designs with Repeated Blocks

Several authors discussed various properties of the balanced incomplete block design. From the point of view of application there is no reason to exclude the possibility that a BIB design would contain repeated blocks. Indeed, the statistical optimality of BIB designs is unaffected by the presence of repeated blocks. Consider a balanced incomplete block design with parameters: v, b, r, k and λ. Block of balanced incomplete block design is the set of distinct blocks and denotes the cardinality of support block by b*. The first published BIBDs with repeated blocks were those in series 1 and series 2 of (Bose, 1942b). Several study of necessary or sufficient conditions for the existence of a BIBD with repeated blocks and given parameters, for the multiplicity of a block in BIBD. For a (v, b, r, k, λ)-BIBD with m the maximum multiplicity of a block proved that m = b/v (Mann, 2016). Van Lint and Ryser (2014) proved that in addition, if m = b/v, then m divides gcd(b, r, λ). They also gave constructions for BIBDs with repeated blocks, usually with gcd(b, r, λ) > 1. Another reason for using BIBDR in a big number of applications is that, besides the variance expression for comparisons for each design being the same, the number of comparisons of block effects with the same variance is different for BIB Designs considering or not block repetition. BIB designs with repeated blocks have some block contrasts with minimum variance (Oliveira et al., 2006 and Mandal et al., 2008).

Van Lint (2006) has pointed out that many of the balanced incomplete block design constructed by Hanani (2016) have repeated blocks. Parker (2011) proved that there is no balanced incomplete block design with repeated blocks with parameters: v= 2x+2,b= 4x+2 and k= x+1. Stanton and Sprott (2017) showed that if s blocks of a balanced incomplete block design are identical, then sv – (s-2). Van Lint and Ryser (2014) thoroughly studied the problem of construction of balanced incomplete block design with repeated blocks. Their basic interest was in constructing a BIB design with repeated blocks with parameters v, b, r, k and λ such that b,r and λ are relatively prime. Wynn (2016) constructed a BIB design with v* = 8, b = 56, k = 3 and b* = 24 with repeated blocks. He also discussed the selection of a sample of k distinct elements from a set of λ elements (varieties). He was led in particular to consider balanced incomplete block designs in which some of the blocks are repeated. Foody and Hedayat (2014) presented some potential applications of the balanced incomplete block designs with repeated blocks to experimental designs and controlled sampling. They also provided some necessary and sufficient conditions for the existence of these designs and some algorithms for their constructions. A necessary and sufficient condition under which a set of blocks can be support of a BIB design were also found and a table of BIB designs with 22 = v*, b*= 56 for v=8 and k=3 was included. Designs with repeated blocks with the equireplications and with equal size of each block are discussed in the literature (Hedayat and Li, 2015, Khosrovshahi and Mahmoodian, 2012). The construction of BIB (v, b, r, k, λ) designs with repeated blocks becomes complicated whenever the three parameters b, r and λ are relatively prime. BIB (8, 56, 21, 3, 6) and BIB (10, 30, 9, 3, 2) designs are examples of such designs with the smallest number of varieties and blocks. Hedayat and Hwang (2015) made an interesting observation about BIB (8, 56, 21, 3, 6) designs and give a table of such designs with 30 different support sizes. The authors proved this fact by constructing a BIB (10, 30, 9, 3, 2) design that exists if and only if the size belongs to {21, 23, 24, 25, 26, 27,28, 29, 30}. Khosrovshahi and Mahmoodian (2012) studied the family of BIB designs with v =9 and k=3 from the view of possible support sizes b*'s. More recently different methods of constructing variance balanced and efficiency balanced block designs with repeated blocks have been given in the literature (Ghosh and Shrivastava, 2001, Ceranka and Graczyk, 2007, 2008, 2009).

Ghosh and Shrivastava (2001) developed the methods of construction of BIB designs with repeated blocks so as to distinguish the usual BIBD with repeated blocks. Also, a class of BIB design with parameters v=7, b=28, r=12, k=3, λ=4 has been constructed where, out of 15, 14 BIB designs have repeated blocks. Those 15 BIB designs, which have the same parameters, are compared on the basis of number of distinct blocks (d) and the multiplicities of variance of elementary contrasts of the block effect. Ceranka and Graczyk (2007) developed same new construction methods of the variance balanced block designs with repeated blocks. However from the practical point of view it may not possible to construct the design with equalize blocks accommodating the equireplication of each treatment in all the blocks. They also considered a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for varying replications. Ceranka and Graczyk (2008) developed some new construction methods of the variance balanced block designs with repeated blocks, which are based on the specialized product of incidence matrices of the balanced incomplete block designs. From a practical point of view, it may not be possible to construct a design with equiblock sizes accommodating the equireplication of each treatment in all the blocks. In this paper researcher consider a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for equal replications. Also Ceranka and Graczyk (2009) presented some new construction schemes of Efficiency Balanced block designs with repeated blocks for k treatments and some ways of admitting given design structures to construct new designs for other number of treatments. Awad and Banerjee (2013) a review of the available literature on balance incomplete block designs with repeated blocks.

2.3 Construction of Balanced Incomplete Block Design

Alabi, (2018) construct balanced incomplete block designs using method of lattice or orthogonal designs of series I and II due to Yates Algorithm and Khare and Federer. When the number of treatments is very large and blocking is must, the Incomplete Block Designs are generally used. The origins of incomplete block designs go back to Yates (1936) who introduced the concept of balanced incomplete block designs and their analysis utilizing both intra- and inter-block information Yates (1940). Other incomplete block designs were also proposed by Yates (1936, 1940), who referred to these designs as quasi-factorial or lattice designs. Further contributions in the early history of incomplete block designs were made by Bose (1939 and 1942b) and Fisher (1940) concerning the structure and construction of balanced incomplete block designs. In order to eliminate heterogeneity; a concept of Balanced Incomplete Block (BIB) Design was introduced, which reduce heterogeneity to a greater extent than is possible with randomized block design and Latin square design. The importance of BIB designs in statistical design of experiments for varietal trials was, however, realized only in 1936 when Yates (1936) discussed these designs in the context of biological experiments. F. Yates (1936) introduced these designs in his paper, a new method of arranging variety trials, involving a large number of varieties.

Sharma and Kumar (2014) developed balanced incomplete block design using Hadamardrhotrices. Bayrak and Bulut (2006) constructed orthogonal balanced incomplete block design. Arunachalamet al. (2016) constructed of efficiency-balanced design. Pachamuthu (2011) showed construction of mutually orthogonal Latin square and check parameter relationship of balanced incomplete block design. More recently different methods of constructing variance balanced and efficiency balanced block designs with repeated blocks have been given, such, Ghosh and Shrivastava (2001), Ceranka and Graczyk (2007, 2008 & 2009). Ghosh and Shrivastava (2001) developed the methods of construction of BIB designs with repeated blocks so as to distinguish the usual BIBD with repeated blocks. Ceranka and Graczyk (2007) developed same new construction methods of the variance balanced block designs with repeated blocks. However from the practical point of view it may not possible to construct the design with equalize blocks accommodating the equireplication of each treatment in all the blocks. Ceranka and Graczyk (2008) developed some new construction methods of the variance balanced block designs with repeated blocks, which are based on the specialized product of incidence matrices of the balanced incomplete block designs. From a practical point of view, it may not be possible to construct a design with equiblock sizes accommodating the equireplication of each treatment in all the blocks. Also Ceranka and Graczyk (2009) some new construction schemes of Efficiency Balanced block designs with repeated blocks for treatments and some ways of admitting given design structures to construct new designs for other number of treatments.

2.4 Construction of Balanced Incomplete Block Design by Galois Field

In Incomplete block designs, as their name implies, the block size is less than the number of treatments to be tested. These designs were introduced by Yates in order to eliminate heterogeneity to a greater extent than is possible with randomized blocks and Latin squares when the number of treatments is large. When the number of treatments to be compared is large, then a large number of blocks to accommodate all the treatments are needed. This requires more experimental material and so the cost of experimentation becomes high which may be in terms of money, labor, time etc. The completely randomized design and randomized block design may not be suitable in such situations because they will require large number of experimental units to accommodate all the treatments. In such situations, sufficient numbers of homogeneous experimental units are not available to accommodate all the treatments in a block, then incomplete block designs can be used. In incomplete block designs, each block receives only some of the selected treatments and not all the treatments. Sometimes it is possible that the available blocks can accommodate only a limited number of treatments due to several reasons. The precision of the estimate of a treatment effect depends on the number of replications of the treatment. When the number of replications of all pairs of treatments in a design is the same, then we have an important class of designs called Balanced Incomplete Block (BIB) designs. It was first devised by (Yates,1936) for agricultural experiments. These design have evidently some constructional problems because the allotments of k of the v treatments in different blocks, so that each pair of treatments is replicated a constant number of times is not straight - forward. Bose (1942) constructed more on the methods of construction of the balanced and their Incomplete Block Designs and was not necessarily constructed by the consideration of practical utility. Bayrak and Bulut (2006) constructed orthogonal balanced incomplete block design. Arunachalam and Ghosh (2016) constructed of efficiency-balanced design. Pachamuthu (2011) showed construction of mutually orthogonal Latin square and check parameter relationship of balanced incomplete block design. To construct mutual orthogonal Latin square, Galois field theory has been used. Janardan (2018) applied Galois field to construct balanced incomplete block design. He uses $GF(7)$ to generate the element of $GF(7)$ and construct mutual orthogonal Latin square (MOLS). Using mutual orthogonal Latin square, balanced incomplete block design has been made. Galois Field, named after Evariste Galois, also known as finite field, refers to a field in which there exist finitely many elements. It is particularly useful in translating computer data as they are represented in binary forms. That is, computer data consist of combination of two numbers, 0 and 1, which are the components in Galois field whose number of elements is two. Representing data as a vector in a Galois field allows mathematical operations to scramble data easily and effectively (Sharma and Kumar, 2014).

2.5 Combinatorial design

Combinatorial design is one of the fastest growing areas of modern mathematics focusing on a major part of introduction to Combinatorial Designs which provides a solid foundation in the classical areas of design theory as well as in more contemporary designs based on applications in a variety of fields (Wallis, 2007). Combinatorial design theory is the study of arranging elements of a finite set into patterns (subsets, words, arrays) according to specified rules. Probably the main object under consideration is a balanced incomplete block design (BIBD).One of the most powerful techniques for determining the existence of combinatorial designs is the idea of Partial balanced design (PBD) - closure, introduced by (Dukes, 2008). The main idea is to break up blocks of a pairwise balanced design, using small examples of designs to create larger ones. Partial balanced design (PBD)-closure underlines many existence results including the asymptotic result on edge-colored graph decompositions. Difference sets afford another pervasive method for construction of combinatorial designs. Qing (2011) included an update on some conjectures on difference sets. Lamken (2009) gave a survey on the state of the art for designs with sets of d mutually orthogonal resolutions. Techniques in this area include combinatorial recursions combined with direct constructions as well as using edge-r-colored decompositions of graphs. The majority of the known existence results are for balanced incomplete block designs. Luc (2014) pointed out a connection between mutually orthogonal resolutions for a (v, 2, 1)-BIBD and some nice structures in finite geometry.

Combinatorial designs have played an important role in cryptology. Wakaha, Kaoru, Douglas, and Hajime (2017) introduced three types of new combinatorial designs, external difference families (EDF), external BIBDs (EBIBD) and splitting BIBDs and show their applications to splitting authentication codes and secret sharing schemes secure against cheaters. An EDF can be considered as an extension of difference sets and difference families. An EBIBD is a generalization of a balanced incomplete block design (BIBD). Two of these combinatorial designs, external difference families (EDF) and external BIBDs (EBIBD), are to show that EDF is equivalent to EBIBD with a particular automorphism. As one of the fundamental discrete structures, combinatorial designs are used in fields as diverse as error-correcting codes, statistical design of experiments, cryptography and information security, mobile and wireless communications, group testing algorithms in DNA screening, software and hardware testing, and interconnection networks (Mohammad, 2013). Many applications of combinatorial designs have been proposed in communications, cryptography, statistics, lottery, quantum computing, and many other areas. Combinatorial designs, specially partial balanced designs can be useful in the construction of perfect authentication structures (Pei, 2006).Group testing is a combinatorial scheme developed for the purpose of efficient identification of infected individuals in a given pool of subjects. The main idea behind the approach is that if a small number of individuals are infected, one can test the population in groups, rather than individually, thereby saving in terms of the number of tests conducted. The coding schemes that are used for this testing are called superimposed codes. A family of superimposed codes is constructed based on block designs and Latin squares (Kim and Lebedev, 2017). Combinatorial design has been use to find the minimum number of tickets necessary to ensure that at least one ticket will intersect the winning numbers in more numbers, for different t's. BIBDs (Li, 2004).

2.6 Combinatorial design theory

The central problem in design theory is by determining the existence of the designs. Determining the full spectrum for a class of designs usually requires a combination of techniques, combinatorial, algebraic/geometric, and computational. Design theory is a field of combinatorics with close ties to several other areas of mathematics including group theory, the theory of finite fields, the theory of finite geometries, number theory, combinatorial matrix theory, and graph theory, and with a wide range of applications in areas such as information theory, statistics, computer science, biology, and engineering. The field of combinatorics has developed a subfields and groups depending on the main techniques used: combinatorial, algebraic, and algorithmic/computational. As design theory grown, researchers have become increasingly specialized and focused in subfields. In recent years, design theory has also become quite interdisciplinary with researchers found in mathematics and computer science departments as well as occasionally in engineering or applied mathematics groups and in industrial groups (Dukes, Lamken and Richard, 2008)

2.7 Combinatorial design in Graph decompositions

A large number of combinatorial design problems can be described in terms of decompositions of graphs into pre-specified subgraphs. Wilson (1975) proved necessary and sufficient conditions on n for the existence of a G-decomposition of K_n where G is a simple digraph on k vertices and K_n denotes the complete directed graph on n vertices. He also described applications and connections in design theory. A family F of subgraphs of a graph K will be called a decomposition of K if every edge $e \in E(K)$ belongs to exactly one member of F. Given a family Ω of edge-r-colored digraphs, a Ω-decomposition of K is a decomposition F such that every graph $f \in F$ is isomorphic to some graph $G \in \Omega$.Esther & Rick (2000) established a general result for the more colorful edge r – digraph and proved necessary and sufficient conditions on n for G - decompositions of $K_n^{(r)}$ where G is a family of simple edge-r-colored digraphs. These authors also provide new proofs for the asymptotic existence of resolvable designs, near resolvable designs, group divisible designs, and grid designs and proved the asymptotic existence of skew Room-d-cubes and the asymptotic existence of (v, k, 1)-BIBDs with any group of order k − 1 as an automorphism group. Wilson (2002) establishes existence results for Steiner systems that admit automorphisms with large cycles using graph decomposition. In combinatorial design, different designs have been established such as; designs with mutually orthogonal resolutions (Lamken, 2008), resolvable graph designs (Dukes and Ling, 2007), group divisible designs with block sizes in any given set K (Liu, 2007) using graph decompositions. These are graph decompositions in which every point appears as a vertex of exactly the same number of G-blocks. Although BIBDs trivially enjoy this property, where G is regarded as the complete graph K_k, graphs G which are not regular require additional necessary conditions to admit equireplicate G-decompositions. Extending this work to a family of graphs, or to edge-colored graphs, remain interesting open problems. Alex Rosa described the state of the art for decompositions of the complete 3-uniform hypergraph into Hamiltonian cycles. The problem of decomposing the complete k-uniform hypergraph into Hamiltonian cycles remains open. The problem of finding the best possible embedding for partial odd cycle systems for cycle length greater than or equal to 5 is completely open.

2.8 Combinatorial design in factorial experiment

The Connection between Galois Fields and Factorial Experiments In a factorial experiment, contrasts belonging to main effects and interactions are important, and in a s^n experiment, there are in all $(s-1)^m$ linearly independent contrasts belonging to a m-factor factorial effect, m=l,. ,no. Consider a Galois field (finite field) $GF(s)$ with s elements. An ordered set of n elements $x_1, x_2, \ldots, x_n$, where the xi's are elements of $GF(s)$, is called a point of the finite Euclidean Geometry $EG(N,s)$. There are s^n points in $EG(N,s)$. By keeping aI, $a_1, a_2, \ldots, a_n$ constant and varying over the elements of $GF(s)$, we generate s parallel (n - I)-flats that have no common point. These s flats constitute a pencil, denoted by $P(a_1, a_2, \ldots, a_n)$. Mausumi (2003) considered an s^n factorial and identified the s levels of each factor with the s elements of $GF(s)$ and the s^n treatment combinations with the points in $EG(N,s)$. Using the definitions of main effects and interactions, he showed that the pencil $P(a_1, a_2, \ldots, a_n)$ would represent the interaction of the $i_1^{th}, i_2^{th}, \ldots, i_n^{th}$ factors if and only if $a_{i1}, a_{i2}, \ldots, a_{ir}$ are non-zero and the other coordinates in the pencil are zero. He showed that the $(s-1)^{m-1}$ distinct pencils belonging to any m-factor effect give a representation for the treatment contrasts belonging to this effect in terms of $(s-1)^{m-1}$ mutually orthogonal sets of contrasts with s-llinearly independent contrasts in each set. Then, using these pencils, he developed a method for constructing (s^n, s^k) designs, i.e., designs for sn experiments in s^k blocks of equal size. Combinatorial design in partially confounded designs is practically useful. Mausumi (2003) introduced the notion of balancing in (s^n, s^k) designs and called a balanced design if there is the same L.1. on all pencils belonging to effects involving the same number of factors. So, balanced designs are intuitively appealing and it can be shown that they also have good statistical properties. Using properties of projective geometry, Bose showed how designs could be constructed where all main effects have zero L.I. and balance is achieved over interactions of all orders. He also gave expressions for the

L.I. on these interactions and the results are very useful for the users of factorial designs.

2.9 Combinatorial design in finite geometry

Arshaduzzaman (2014) summarized the work of Bose (1938) by presenting a paper that dealt with Latin squares, orthogonal Latin squares, mutually orthogonal Latin squares and the close connections between Latin squares and finite geometries. He also provided a historical background of Latin squares. Tang (2009) noted that Latin squares were relatively unknown aspect of mathematics and he also introduced the Kronecker Product method of constructing MOLS in addition to the Finite (Galois) Field method. Pachamuthu (2011) studied the construction of 2^2 and 3^2 Mutually Orthogonal Latin Squares by using Galois Field theory.

Vanpoucke (2012) discussed the important connection between Latin squares and projective planes especially when it came to mutually orthogonal Latin squares (MOLS). He proved the conjecture that there are (p -2)! distinct sets of (p - 1)MOLS(p), for prime p, describing PG(2; p) and extended his results to prime powers of p. Muhammad and Wahida (2018) apply the methods of modern algebra to construct a BIBD. BIBD is also called a (v, b, r, k, λ) configuration or 2- 2-(v, k, λ) tactical configuration or design (Silva, 2013). Andrew (2016) reviewed the methods of constructing Balanced Incomplete Block Designs (BIBDs) by means of Mutually Orthogonal Latin squares (MOLS) of prime powers order arising from Finite Geometries and Finite Fields.

CHAPTER THREE

METHODOLOGY

In this chapter, emphases are based on the techniques that will aid in analyzing the entire work to achieve the aim and objectives of this research. Construction of balanced incomplete block designs using finite projective geometry ($PG(N,P^n)$), finite Euclidean geometry ($EG(N,P^n)$) and the concept of finite group algebra, ring and finite field algebra, isomorphism and automorphism are also discussed in this chapter.

3.1 Basic concept of balanced incomplete block design

An incomplete block design (k < t) is said to be a balanced incomplete block design (t, b, r, k, λ) or (t, k, λ) - BIBD in a design (X, A) if the following postulates are satisfied;

(i) $|X| = t$

(ii) Each block contains exactly k points

(iii) Every pair of distinct contained points exactly λ blocks

The quantities $(t, b, r, k\ and\ \lambda)$ are called the parameters of the balanced incomplete block (BIB) design. The parameter t is the number of treatment, b is the number of block, r is the number of replicate observation per treatment, k is the number of observation per block and λ is the number of times any two treatments occur together in a block. The following relations exist among the parameters $(t, b, r, k\ and\ \lambda)$. The necessary conditions for the existence of (t, b, r, k, λ) - design are stated below:

(i) $rt = bk$ (1)

(ii) $r(k-1) = \lambda(t-1)$ (2)

(iii) $r > \lambda$ (3)

(iv) $b = t$ (4)

The BIBD cannot exist unless (1) and (2) are satisfied. This however, does not mean that whenever (1) and (2) occurs, and then a BIBD exists. Thus λ. must be positive integers. The relationship (1) follows from the fact that the total number of observation for the balanced incomplete block design is:

$$N = \sum_{i}\left(\sum_{j} n_{ij}\right) = \sum_{j}\left(\sum_{i} n_{ij}\right) \quad (5)$$

Proposition 3.1: For every non – empty (t, b, r, k, λ) - BIBD;

(a) $\lambda > 1$

(b) $k < t$

Proof

(a) Let b be the block in (t, b, r, k, λ)- BIBD with at least two elements, then $\exists$ at least one block and some pair has at least once occurrence. Therefore, all pairs occur equally, it follows that $\lambda \geq 1$.

(b) Let X be a domain and k is the size, then $b \subset X$. This follows that the block size cannot exceed the size of the domain. Thus, $k \leq t$. Since a BIBD is incomplete, it follows that $k < t$.

Proposition 3.2: The parameters of a BIBD on $X = \{x_1, x_2, \ldots, x_t\}$ satisfy the following two conditions:

(i) $bk = rt$

(ii) $\lambda(t-1) = r(k-1)$

Proof

(i) First, consider the $t \times b$ incidence matrix; $I_{ij} = \begin{cases} 1 & if\ x_i \in B_j \\ 0 & otherwise \end{cases}$

	$B_1, \ldots\ldots, B_b$
x_1	$I_{1,1} \ldots\ldots I_{1,b}$
.	. . . .
.	. .
.	. .
x_t	$I_{t,1} \ldots\ldots L_{t,b}$

$I =$ (the matrix above)

There are t rows each with row – sum r, and there are b columns each with column – sum k. therefore, $bk = rt$

(ii) Consider $\binom{t}{2} \times b$ pair incidence matrix

	$B_1, \ldots\ldots, B_b$
x_1x_2	$I'_{12,1} \ldots\ldots I_{12,b}$
.	. . . .
.	. .
.	. .
$x_{t-1}x_t$	$I'_{(t-1),t,1} \ldots\ldots L_{(t-1),t,b}$

$I' =$ (the matrix above)

With $I'_{ij,l} = \begin{cases} 1 & if\ x_i x_j \in B_l \\ 0 & otherwise \end{cases}$

There are $\binom{t}{2}$ rows each with row – sum λ and there are b columns each with column –sum $\binom{k}{2}$. Therefore, $\lambda\binom{t}{2} = b\binom{k}{2}$

Accordingly

$\lambda t(t-1) = bk(k-1)$

$\Rightarrow \lambda t(t-1) = tr(k-1)$

$\Rightarrow \lambda(t-1) = r(k-1)$, since $tr = bk$

Proposition 3.3

In a (t,k,λ) - BIBD, every point occurs in exactly $r = \dfrac{\lambda(t-1)}{k-1}$ blocks; where r is called the replication number of BIBD.

Proof

Let (E,ψ)be a (t,k,λ) - BIBD. Let $v \in E$ and r_v be the number of blocks containing E. Define a set $F = \{(y,\psi) : y \in E,\ y \neq v,\ W \in \psi, (v,y)\} \subseteq W$

We can compute F in two ways;

There are $(t-1)$ to choose $y \neq v \in E$, and for each y, there is exactly λ blocks such that $\{v,y\} \subseteq W$

Secondly, there are r_v block $v \in W$. For each W, there are $k-1$ ways to chose $y \in W$ and $y \neq v$. Hence $|F| = r_v(k-1)$

Combining these two equations of $|F|$, we have;

$\lambda(t-1) = r_v(k-1)$

Hence $r_v = \dfrac{\lambda(t-1)}{(k-1)}$

The r_v is independent of v and any element is combine in exactly r number of blocks.

Therefore, $r = \dfrac{\lambda(t-1)}{k-1}$ as required.

Proposition 3.4

For every non – void design, a (t,k,λ)- BIBD has exactly $b = \dfrac{tr}{k} = \dfrac{\lambda(t^2-t)}{k^2-k}$ blocks.

Proof

Let (Y,C) be a (t,k,λ) - BIBD and let $b = |C|$. Define a set $S = \{(y,C) : y \in Y,\ A \in C,\ y \in A\}$, then $|S|$ can be computed in two ways.

Firstly, there are t ways of choosing an element $y \in Y$. For each y, there are exactly r blocks such that $y \in A$. Hence $|S| = t\,r$

Secondly, there are b blocks $A \in C$. For each block A, there are k ways to choose $y \in A$. Hence $|S| = b\,k$.

Combining the two equations, we get;

$$b\,k = t\,r \Rightarrow b = \frac{t\,r}{k}$$

Substituting the value of r from the above theorem (3.1) in $b = \frac{t\,r}{k}$ yields

$$b = \frac{t\,\lambda(t-1)}{k(k-1)} = \frac{\lambda(t^2 - t)}{k^2 - k} \text{as required}$$

Corollary 3.1

For every non – empty BIBD, $\lambda < r$

Proof

From the proposition 3.2 (ii), $\lambda.(t-1) = r(k-1)$ and from proposition 3.1 (b), $k < t$, it follows that $\lambda < r$ as required.

3.2 Incidence matrices

According to Parvathy (2015), incidence matrices are convenient way of expressing balanced incomplete block designs in a matrix form.

Let (X, ψ) be a design where $X = (x_1, x_2, \ldots, x_t)$ and $\psi = \{\psi_1, \psi_2, \ldots, \psi_b\}$. The incidence matrix of (X, ψ) is the $t \times b_{0-1}$ matrix $M = (m_{ij})$ defined as:

$$m_{ij} = \begin{cases} 1 \text{ if } x_i \in \psi_j \\ 0 \text{ if } x_i \notin \psi_j \end{cases} \qquad (6)$$

From the definition and the properties of a BIBD, it is clear that the incidence matrix M of a (t,b,r,k,λ) -BIBD satisfies the following properties

(1) Every column of M contain exactly K'1's

(2) Every row of M contain exactly r'1's

(3) Two distinct rows of M contain both 1's in exactly λ columns

Proposition 3.5

In any balanced incomplete block design (BIBD), $b \geq t$

Proof

According to Fisher's inequality, let H be the incidence matrix of the BIBD. Then

$$HH^T = \begin{pmatrix} r & \lambda & \lambda & \lambda & . & . & \lambda \\ \lambda & r & \lambda & \lambda & . & . & \lambda \\ \lambda & \lambda & r & \lambda & . & . & \lambda \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & \lambda & \lambda & \lambda & . & . & r \end{pmatrix}$$

Subtracting the first column of a matrix from the other columns does not change the determinant. Hence, the above incidence matrix becomes;

$$\det(HH^T) = \begin{vmatrix} r & \lambda - r & \lambda - r & \lambda - r & . & . & \lambda - r \\ \lambda & r - \lambda & 0 & 0 & . & . & 0 \\ \lambda & 0 & r - \lambda & 0 & . & . & 0 \\ . & . & . & r - \lambda & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & 0 & 0 & 0 & . & . & r - \lambda \end{vmatrix}$$

Adding the other rows of a matrix $|HH^T|$ to the first row does not change the determinant. Hence

$$\det(HH^T) = \begin{pmatrix} r+(t-1)\lambda & 0 & 0 & 0 & . & . & 0 \\ \lambda & r-\lambda & 0 & 0 & . & . & 0 \\ \lambda & 0 & r-\lambda & 0 & . & . & 0 \\ . & . & . & r-\lambda & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & 0 & 0 & 0 & . & . & r-\lambda \end{pmatrix}$$

Since the upper triangle of this matrix is all zeros, the determinant is the product of the diagonal entries. Thus;

$\det(HH^T) = [r + (t-1)\lambda](r-\lambda)^{t-1}$

By corollary (3.1), $r-\lambda > 0$. Then $r+(t-1)\lambda$ is positive and $\det(HH^T)$ is not trival. Also the rank of $t \times t$ - matrix HH^T is t. Since the rank of the $t \times b$ incidence matrix H is almost b, and since the rank t of the product matrix HH^T cannot exceed the rank of the matrix H, it follows that $t \leq b$ as required.

Proposition 3.6

Let $H_{t \times b}$ be an incidence matrix and let $2 \leq k \leq t$, then $H_{t \times b}$ is the incidence matrix of a (t, b, r, k, λ) - BIBD iff $HH^T = \lambda J_t + (r-\lambda)I_t$ and $U_t H = KU_b$.

Proof

Let I_t denote the $t \times t$ identity matrix, J_t denote an $t \times t$ matrix in which every entry is 1. Let U_t denote a vector of length t in which every coordinate is 1. Let (Y,W) be a (t,b,r,k,λ) - BIBD, where $Y = (y_1, y_2, \ldots, y_t)$ and $W = (w_1, w_2, \ldots, w_b)$. Let H be its incidence matrix, then if $HH^T = B$, then the element b_{ij} of matrix B is the inner product of ith row of H with jth row of H.

Every element on the leading diagonal b_{ii} of B counts the number of 1's in the rth row of H, which gives in how many blocks a particular element is present, which is r. But

$j \neq i$, then both the ith and jth rows have 1 in the same column iff both x_i and x_j belong to the same column. So every off diagonal entry of B is λ. Therefore;

$$HH^T = \begin{pmatrix} r & \lambda & \lambda & \lambda & . & . & \lambda \\ \lambda & r & \lambda & \lambda & . & . & \lambda \\ \lambda & \lambda & r & \lambda & . & . & \lambda \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & \lambda & \lambda & \lambda & . & . & r \end{pmatrix} = \lambda J_t + (r-\lambda) I_t$$

And $U_t H$ is a $1 \times b$ matrix whose ith entry counts the number of 1's in ith column, which is k. hence, $U_t H = k U_b$

Conversely, suppose that H is a $t \times b$ matrix s.t $HH^T = \lambda J_t + (r-\lambda) I_t$ and $U_t H = k U_b$. Let (Y, W) be a design whose incidence matrix is H. since H is a $t \times b$ matrix, $|Y| = t$ and $|W| = b$. From the second condition, it follows that every block of W contains K points. From the first condition, it follows that every point occurs in r blocks and every pair of points occurs in λ blocks. Hence, (Y, W) is a (t, b, r, k, λ)-BIBD.

3.3 Finite group in balanced incomplete block design (BIBD)

A finite group is a group of which the underlying set contains a finite number of elements. In other words, a group of finite number of elements is called a finite group. Order of a group G is denoted by O(G) which is the number of distinct elements in G. A group is a system <G,*>, where G is a non – void set and * is a binary operation on G satisfying the following postulates:

G1: Closure; $\forall a, b \in G$, then $a * b \in G$

G2: Associativity: $\forall a, b, c \in G$,; $a*(b*c) = (a*b)*c$

G3: Existence of identity: $\exists$ an element $e \in G$, called an identity s.t

$a * e = a = e * a \; \forall a \in G$,

G4: Existence of inverse: For each $a \in G$, $\exists$ an element $a^{-1} \in G$, called an inverse of a s.t $a * a^{-1} = e = a^{-1} * a$

A group is called a commutative or an abelian group if;

G5: Commutative law: $\forall a, b \in G$; $a * b = b * a$

A group <G,*> for which the postulate G5 does not hold is called a non – abelian group. If G is finite, then <G,*> is called a finite group, otherwise, it is called an infinite group.

A system <G,*> consisting of a non –void set G and a binary composition * on G is called a semigroup if it satisfies the postulate below;

SG1: $\forall a, b, c \in G$,; $a*(b*c) = (a*b)*c$

A group is always a semigroup while the converse is not true in general. The structural representations of group axioms are shown in figures (1a – 1d).

3.4 Ring algebra in balanced incomplete block design (BIBD)

A system $< R, +, \bullet >$, where R is a non – void set, + and * are two binary operations defined on the set R is called a ring if is satisfies the following postulate. For any $a, b, c \in R$;

R1: $a + b = b + a$

R2: $(a + b) + c = a + (b + c)$

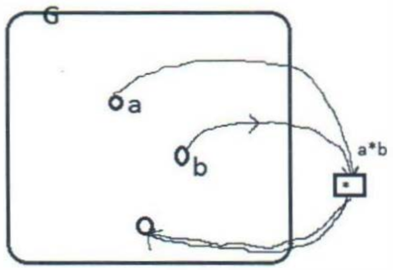
R3: $\exists \; 0 \in R$, $s.t \; a + 0 = a = 0 + a$

R4: For each $a \in R, \exists -a \in R \; s.t \; a + (-a) = 0$

R5: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$

R6: $a \bullet (b + c) = a \bullet b + a \bullet c$ (Left distributive law)

1 Closure: $\forall a, b \in G$, then $a * b \in G$



2 Associativity: $\forall a, b, c \in G$, ;

3 $(a*b)*c = a*(b*c)$

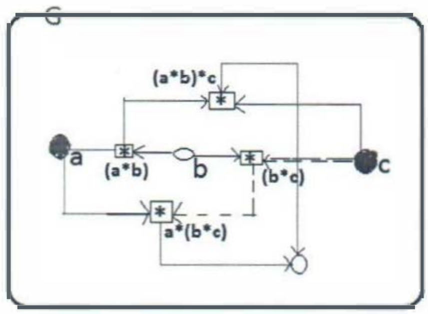


Fig 1b: Associativity structure

3 Identity: $e \in G$, $\exists\, e * a = a * e = a$

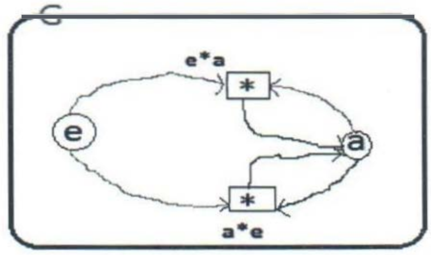


Fig 1c: Identity structure

4 Inverse: If $a \in G, a^{-1} \in G$ st

$a * a^{-1} = e = a^{-1} * a$

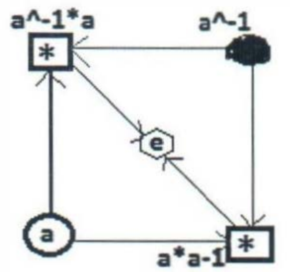


Fig 1d: Inverse structure

Fig 1: Group properties: (a) Closure property (b) associative property (c) Identity property (d) Inverse Property

$(b+c)\bullet a = b\bullet a + c\bullet a$ (Right distributive law)

The postulates R1 – R4 show that a ring R is an abelian additive group $<R,+>$. R5 show that a ring R is associative law under semigroup $<R,\bullet>$. A ring R in which $ab = ba$ for every $a,b \in R$; is a commutative ring. In other words, a ring $<R,+,\bullet>$ is commutative if $<R,\bullet>$ is a commutative semigroup. An element e of a ring R is called unity (or an identity) of R if $ae = ea = a$ for every $a \in R$. A non - empty subset S of a ring $<R,+,\bullet>$ is called a subring of $<R,+,\bullet>$ if $<S,+,\bullet>$ is also a ring. In general, a ring may or may not have a unity, however it can be easily shown that if a ring R has an element e $s.t$ $ae = ea\ a\ \forall\ a \in R$, then e is unique and this e is called the unity or the identity of R. the unity of a ring R is denoted by I. An element x of a ring is said to be idempotent if $x^2 = x$. A ring R in which every element is idempotent is known as boolean ring.

Proposition 3.7

1 The set Z is a ring under usual addition and multiplication with one as a unity.

2 For each set Q, $\Re$, C of all rational, real and complex numbers is a commutative ring with unity under addition and multiplication.

3.5 Field algebra in balanced incomplete block design (BIBD)

A commutative division ring is called a field. A ring with unity in which all non – zero elements form a group under multiplication is called a division ring. A field $<F,+,\bullet>$ is a set defined by two binary compositions + and *, respectively which satisfies the following postulates;

F1: $<F,+>$ ia an abelian (additive) group

F2: $< F/\{0\},\bullet >$ ia an abelian (multiplicative) group

F3: $\forall a,b,c \in F, a \bullet (b+c) = a \bullet b + a \bullet c$ (distributive law)

A non – void subset S of a field $< F >$ is said to be a sub – field of $< F >$ if;

(a) $a \in S, b \in S \Rightarrow a+b \in S, ab \in S$

(b) S is a field under the induced $< +,\bullet >$ operations

Any subset S of a field $< F >$, containing at least two elements is a subfield of $< F >$ iff;

(i) $a \in S, b \in S \Rightarrow a-b \in S$

(ii) $a \in S,\ b \in S,\ b \neq 0 \Rightarrow (ab)^{-1} \in S$

3.6 Isomorphism of balanced incomplete lock design (BIBD)

Suppose (X,C) and (Y,D) are two designs, then (X,C) and (Y,D) are isomorphic if there exist a bijection $\alpha : X \rightarrow Y$ $s.t$ $[\{\alpha(x) : x \in A\}, A \in C] = D$. If every point $x \in \alpha(x)$, then the collection of blocks C is transformed into D. Thus the bijection α is called an isomorphism. The isomorphic designs are related to the incidence matrices.

Proposition 3.7

Let $A = (a_{ij})$ and $B = (b_{ij})$ are both $t \times b$ incidence matrices of the designs, then two designs are isomorphic iff $\exists$ a permutation $\gamma\{1,2,\ldots,t\}$ and a permutation $\lambda\{1.2,\ldots,b\}$ $s.t$ $a_{i,j} = b_{\gamma(i)\lambda(j)}$ $\forall\, 1 \leq i \leq t, 1 \leq j \leq b$.

Proof

Assuming (X,C) and (Y,D) are designs having $t \times b$ incidence matrices A and B respectively, then, let $X = \{x_1, x_2, \ldots, x_t\}$, $Y = \{y_1, y_2, \ldots, y_b\}$, $C = \{c_1, c_2, \ldots, c_b\}$ and $D = \{d_1, d_2, \ldots, d_t\}$. Suppose (X,C) and (Y,D) are isomorphic then $\exists$ a bijective mapping $\alpha : X \to Y$ $s.t$ $[\{\alpha(x) : x \in A\}, A \in C] = D$ for $1 \leq i \leq t$ define;

$\gamma(i) = j$ iff $\alpha(x_i) = x_j$

Since α is a bijection of X and Y, γ is a permutation of $\gamma\{1,2, \ldots, t\}$. Let $\{\alpha(x) : x \in A\} = B_{\lambda(j)}$. Therefore $\exists$ a permutation $\lambda\{1.2, \ldots, b\}$ for $1 \leq j \leq b$. Since α is an isomorphism of (X,C) and (Y,D). Hence $a_{i,j} = b_{\gamma(i)\lambda(j)}$.

Conversely, suppose γ and λ are permutations s.t $a_{i,j} = b_{\gamma(i)\lambda(j)}$ $\forall\, i, j$ design $\alpha : X \to Y$ $s.t$ $\alpha(x_i) = x_j$ iff $\gamma(i) = j$. Then, $[\{\alpha(x) : x \in A\}\ A \in C] = D, \forall\, 1 \leq j \leq b$. Hence α defines an isomorphism of (X,C) and (Y,D).

Corollary 3.2

Suppose A and B are incidence matrices of two (t,b,r,k,λ) - BIBDs. Then the two BIBDs are isomorphic if there exist a $t \times t$ permutation matrix W, and $b \times b$ permutation matrix V $s.t$ $A = WBV$.

Proof

Let W and U be two matrices whose $(i,\gamma(i))th$ and $(j,\lambda(j))th$ entries are 1 and the rest entries are all 0. And let $U^T = V$. It is obvious that W and V are permutation matrices. WB is a restriction of rows B which correspond to the action of the onto mapping on points. Post multiplying by V gives a rearrangement of columns, but columns changed and the structure is preserved.

3.7 Automorphism of balanced incomplete block design (BIBD)

Suppose (X,C) be a design. An automorphism of (X,C) is an isomorphism of the design (X,C) to itself. In this design α is a permutation of X st $[\{\alpha(x) : x \in A\} : A \in C] = C$. A permutation α on a set X can be represented as disjoint cycle representation. Each cycle has the form $(x, \alpha(x), \alpha(\alpha(x)), ...)$ for some $x \in X$.

Algebraically, permutation is a bijection from a set S onto itself. That is, a function from S to S for which every element occurs exactly once as an image value. The collection of such permutations form a group called the symmetry group of S_n.

3.8 Construction of BIBD with specified automorphism

Let S_n denote symmetry group on a N – set. Let $\binom{X}{j}$ be the set of all $\binom{n}{j}$ j subset of X $\forall$ $j \leq n$ positive integer. For $X \subseteq Y$ and for a permutation $\sigma \in S_n$. Let $\sigma(X) = \{\sigma(y) : y \in X\}$. Suppose G is a semigroupof S_n. For positive integer $j \leq n$, for $U,V \in \binom{X}{j}$ define a relation $U \sim V$ if $\sigma(U) = V$ for some $\sigma \in G$. This relation ia an equivalent relation.

1. Reflexivity: since $\sigma(U) = A(U) = U, U \sim U$. Hence A is the identity permutation of σ.

2. Let $\sigma(U)=V$ for some $\sigma \in G$, then $\sigma^{-1}(U)=V$. Since $\sigma^{-1} \in G$. Hence $\sim$ is symmetric.

3. Transitive: Let $\sigma_1(U)=V$ and $\sigma_2(U)=W$ for some $\sigma_1\sigma_2 \in G$. Hence $\sigma_2 \circ \sigma_1(U)=\sigma_2(V)=W$. Since G is a subgroup $\sigma_2 \circ \sigma_1 \in G$.

This prove that "$\sim$" is an equivalent relation on $\binom{X}{j}$.the set of all automorphisms of a BIBD, (X,C) forms a group under the operation of composition of permutations. This group is called the automorphism group of the BIBD and is denoted by Aut (X,C). An $\mathrm{Aut}(X,C)$ is a group of the symmetric group $S_{|x|}$.

3.9 Galois field design

An algebraic structure satisfying all the axioms of the field but with F being a finite set of elements is known as a Galois field and it is denoted by $GF(s)$, where s is a finite number of elements in the set. The concept of a polynomial in ordinary algebra

$$a_i(a_j + a_k) = (a_j + a_k)a_i = a_i a_j + a_i a_k \quad (8)$$

This field exist for every finite number of elements which is the power of a prime. It is clear that every number of elements contained by a Galois field (a field with a finite number of elements) must be of the form p^n, where p is a prime integer and n any positive integer. Any two Galois field with the same number of elements are isomorphic. Thus every element of $GF(p^n)$ can be expressed in the standard form;

$$f(x) = a_0 + a_1 x_1 + a_2 x^2 + .. \quad . \quad . \quad . + a_{p-1} x^{n-1} \quad (9)$$

Where a_0, a_1, a_2 . . . are integers ranging from 0 to $p-1$. Any non – zero element β of $GF(p^n)$ satisfies $\beta^{p^n-1} = 1$. If t is the least positive integer such that $\beta^t = 1$, then t is the order of β. When $t = p^n - 1$, β is said to be a primitive element of $GF(p^n)$.

3.10 Finite geometry design

Combinatorial designs on a finite set of elements are known as a finite geometry. The elements of their domains are points of the geometry and their different subsets are lines of the geometry. The two standard axioms below are generally use in geometries:

A1: Two distinct points contained at most on a line.

A2: Two distinct lines intersect at most one point

The incidence matrix of a geometry $<X, L>$ with p points: $X = \{x_1, x_2, . . . , x_p\}$ and L lines: $L = \{l_1, l_2, ..., l_l\}$ is the $p \times l$ matrix

$$M_{<X,L>(i,j)} = \begin{cases} 1 & if \ \ x_i \in L_j \\ 0 & otherwise \end{cases} \tag{10}$$

Geometry is commonly specified by its incidence matrix. The dual of a geometry $<X,L>$ is the geometry $<X^*,L^*>$ with $X^*=L$ and $L^*=X$, whose incidence matrix is the transpose of the incidence matrix of $<X,L>$

3.11 Finite projective geometry design

A finite projective geometry of N – dimensions consist of ordered sets $(x_1,x_2,\ldots,x_N)$ called points, where $x_i's$ belong to Galois field on n – point $GF(p^n)$ and are not all simultaneously zero (i.e. not all $x_i's=0$). The space is denoted by $PG(N,p^n)$. The $x_i's$ are homogenous coordinates of the points that is for any $m \neq 0$, the point $(mx_0,mx_1,mx_2,\ldots,mx_N)$ is considered the same point as $(x_1,x_2,\ldots,x_N)$. It can be easily shown that the number of point in $PG(N,p^n)$ is exactly $S^N+S^{N+1}+\ldots+S+1=\frac{S^{N+1}-1}{S-1}$, where $s=p^n$ (11)

All the points which satisfy a set of N – m independent linear homogenous equation below may be said to form an m – dimensional subspace, or briefly an m – flat in $PG(N,s)$.

$$\left.\begin{array}{l} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = 0 \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = 0 \\ . \\ a_{N-m,0}x_0 + a_{N-m,1}x_1 + a_{N-m,2}x_2 + \ldots + a_{N-m,N}x_N = 0 \end{array}\right\} \tag{12}$$

Alternatively, a m – flat in a $PG(N,s)$ consists of all points with coordinates $a_0x_{00}+\ldots+a_Nx_{N,0},\ldots a_0x_{om}+\ldots+a_Nx_{Nm}\ldots$

Since a'_i s are not simultaneously zero, the matrix is;

$$\begin{pmatrix} x_{00} & x_{01} & . & . & x_{0m} \\ x_{10} & .x_{11} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ x_{N0} & . & . & . & x_{Nm} \end{pmatrix}$$

Which has the rank N + 1.

The equations (12) is the representation of the flat. Any set of N – m independent equations which can be obtained by linear combinations of the equations (12), will have the same set of solutions and will represent the same m – flat. The number of m – flats in $PG(N, s)$ can be shown to be;

$$\Omega(N,m,s) = \frac{(S^{N+1}-1)(S^{N}-1)\ldots(S^{N-m+1}-1)}{(S^{m+1}-1)(S^{m}-1)\ldots(S-1)} \qquad (13)$$

Where $s = p^n$

Therefore, $\Omega(N,m,s) = \Omega(N,N-1,s)$ (14)

It is convenient to set formally that, $\Omega(N,-1,s) = 1$

To every point of $PG(N,s)$ there correspond a variety (treatment) t and to every m – flat there correspond a block containing all those treatments whose corresponding points occur in m – flat. Every m – flat lie on $\Omega(m,0,s) = \frac{(S^{N+1}-1)}{S-1}$ points, through every point there pass $\Omega(N-1,m-1,s)$ m – flats, and through every pair of points there pass $\Omega(N-2,m-2,s)$ m – flats. Based on this flat, we have the following designs;

$$\left.\begin{array}{l} v=\Omega(m,0,s) \\ b=\Omega(N,m,s) \\ r=\Omega(N-1,m-1,s) \\ k=\Omega(m,o,s) \\ \lambda=\Omega(N-2,m-2,s) \end{array}\right\} \quad (15)$$

If N and m are taken as fixed, then all the designs in (15) for various values of p^n shall be said to belong to the series (P_N^m). A 0 – flat is identical with a point, 1 – flat is a line and 2 – flat is a plane.

3.12 Projective geometry design

A projective plane is a type of finite geometry in a combinatorial design. A projective plane p has a domain X, whose elements are called points and a collection of subsets of X is called lines such that the following properties hold:

P1: For each pair of distinct points, there is exactly one line containing them.

P2: Each pair of distinct line intersects in exactly one point

P3: There exist four points, no three of which lie on the same line

Proposition 3.8

In a projective plane P of order S, the number of points and lines is S^2+S+1. That is, a projective plane of order S is a balanced incomplete block design with parameters; $t=S^2+S+1$, $b=S^2+S+1$, $r=S$, $k=S$ and $\lambda=1$

An $(S^2+S+1, S+1, 1)$ - BIBD with $S\geq 2$ is called a projective plane of order S. Hence $(S^2+S)=(S+1)S$ shows that a projective plane is a symmetric balanced incomplete block design (BIBD)

3.13 Finite Euclidean geometry

Euclidean space of N – dimension denoted by $EG(N, p^n)$ is the set of all points of $PG(N, p^n)$ whose first coordinate x_0 is not zero. The points of an $EG(N, p^n)$ are given by the ordered set $(x_1, x_2, ..., x_N)$ where the $x_i^{'}s$ are the elements of $GF(p^n)$. The number of points in $EG(N, p^n)$ is S^N, where $S = p^n$. All the points which satisfy a set of consistent and independent linear equations may be said to form an m – flat of $EG(N, p^n)$ represented by the equations (16).

$$\left.\begin{array}{lllllllll} a_{10} & + & a_{11}x_1 & + & a_{12}x_1 & + & \dots & + & a_{1N}x_N = 0 \\ a_{20} & + & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_N = 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{N-m,0} & + & a_{N-m,1}x_1 & + & a_{N-m,2}x_2 & + & \dots & + & a_{N-m,N}x_N = 0 \end{array}\right\} \quad (16)$$

The set of N – m consistent and independent linear equations derived by linear combinations of (16), represent the same m – flat. The number of m – flat in $EG(N, p^n)$ is:

$$Q(N,m,s) - Q(N-1,m,s) = S^{N-m}Q(N-1,m-1,s) \quad (17)$$

$$Q(N,m,s) = S^{N-m}Q(N-1,m-1,s) + Q(N-1,m,s) \quad (18)$$

To every point of $EG(N, p^n)$ there correspond a treatment. To every m – flat there exist a correspond block containing all those treatments whose corresponding points form the m – flat. Then the design obtained from the flat is shown below:

$$v = Q(N,0,s) - Q(N-1,0,s) = S^N \quad (19)$$

$$b = Q(N,m,s) - Q(N-1,m,s) = S^{N-m}Q(N-1,m-1,s) \quad (20)$$

$$r = Q(N-1, m-1, s) \quad (21)$$

$$k = Q(N,0,s) - Q(m-1,0,s) = S^m \quad (22)$$

$$\lambda = Q(N-2, m-2, s) \quad (23)$$

If M and m are taken as fixed, then all the designs by (19 - 23) for various values of $s = p^n$ may be said to belong to the series $\left(E_N^m\right)$.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

In this section, different BIBD's are constructed in line with the techniques constructed in chapter three. Basic concepts of group algebra such as finite groups, rings and finite fields are tested using each of the constructed design. The Isomorphism and automorphism of the two constructed BIBDs are examined. Furthermore, some theorems are presented and proves to confirm the results.

4.1 Construction of BIBD with PG (3, 2)

To construct $PG(3,2)$, the coordinate of the points of the geometry are built up of the elements of $GF(2)$ which has only two elements 0 and 1. The blocks are obtained using the equations below:

$$x_i = 0 \quad (i=0,1,2,3)$$

$$x_i + x_j = 0 \quad (i,j=0,1,2,3: i \neq j)$$

$$x_i + x_j + x_k = 0 \quad (i,j,k=0,1,2,3: i \neq j, j \neq k, k \neq i)$$

$$x_0 + x_1 + x_2 + x_3 = 0$$

Here N = 3 and $S = p^n = 2$

The total number of points in $PG(N,s) \Rightarrow PG(3,2) = \dfrac{S^{N+1} - 1}{S - 1}$

$$b = \Omega(N,m,s) = \frac{S^{N+1} - 1}{S - 1} = 15 \text{points}$$

$$k = \Omega(m,0,s) = \frac{S^{m+1} - 1}{S - 1} = 7$$

$$r = \Omega(N-1,m-1,s) = \frac{S^{N} - 1}{S - 1} = 7$$

$$\lambda = \Omega(N-2, m-2, s) = \frac{S^{N-1}-1}{S-1} = 3$$

$$t = \Omega(N, 0, s) = \frac{S^{N+1}-1}{S-1} = 15$$

In this design, $b = t$

According to equation (11), the blocks are generated from the equation:

$8x_0 + 4x_1 + 2x_2 + x_3$

$x_0 =$ x_3, x_2, $x_2 + x_3$, x_1, $x_1 + x_3$, $x_1 + x_2$, $x_1 + x_2 + x_3$

0001 0010 0011 0100 0101 0110 0111

$x_1 =$ x_3, x_2, $x_2 + x_3$, x_0, $x_0 + x_3$, $x_0 + x_2$, $x_0 + x_2 + x_3$

0001 0010 0011 1000 1001 1010 1011

$x_2 =$ x_3, x_1, $x_1 + x_3$, x_0, $x_0 + x_3$, $x_0 + x_1$, $x_0 + x_1 + x_3$

0001 0100 0101 1000 1001 1100 1101

$x_3 =$ x_2, x_1, $x_1 + x_2$, x_0, $x_0 + x_2$, $x_0 + x_1$, $x_0 + x_1 + x_2$

0010 0100 0110 1000 1010 1100 1110

$x_0 + x_1 =$ x_3, x_2, $x_2 + x_3$, $x_0 + x_1$, $x_0 + x_1 + x_3$, $x_0 + x_1 + x_2$, $x_0 + x_1 + x_2 + x_3$

0001 0010 0011 1100 1101 1110 1111

$x_0 + x_2 =$ x_3, x_1, $x_1 + x_3$, $x_0 + x_2$, $x_0 + x_2 + x_3$, $x_0 + x_1 + x_2$, $x_0 + x_1 + x_2 + x_3$

0001 0100 0101 1010 1011 1110 1111

$x_0 + x_3 =$ x_2, x_1, $x_1 + x_2$, $x_0 + x_3$, $x_0 + x_2 + x_3$, $x_0 + x_1 + x_3$, $x_0 + x_1 + x_2 + x_3$

0010 0100 0110 1001 1011 1101 1111

$x_1 + x_2 =$ x_3, x_0, $x_0 + x_3$, $x_1 + x_2$, $x_1 + x_2 + x_3$, $x_0 + x_1 + x_2$, $x_0 + x_1 + x_2 + x_3$

0001 1000 1001 0110 0111 1100 1111

$x_1 + x_3 =$ x_2, x_0, $x_0 + x_2$, $x_1 + x_3$, $x_1 + x_2 + x_3$, $x_0 + x_1 + x_3$, $x_0 + x_1 + x_2 + x_3$

0010 1000 1010 0101 0111 1101 1111

$x_2 + x_3 = x_1, \quad x_0, \quad x_0 + x_1, \quad x_2 + x_3, \quad x_1 + x_2 + x_3, \quad x_0 + x_2 + x_3, \quad x_0 + x_1 + x_2 + x_3$

0100 1000 1100 0011 0111 1011 1111

$x_0 + x_1 + x_2 = x_2 + x_3, \quad x_1 + x_3, \quad x_1 + x_2, \quad x_0, \quad x_0 + x_2 + x_3, \quad x_0 + x_1 + x_3, \quad x_0 + x_1 + x_2$

0011 0101 0110 1000 1011 1101 1110

$x_0 + x_1 + x_3 = x_3 + x_2, \quad x_0 + x_3, \quad x_0 + x_2, \quad x_1, \quad x_1 + x_2 + x_3, \quad x_0 + x_1 + x_3, \quad x_0 + x_1 + x_2$

0011 1001 1010 0100 0111 1101 1110

$x_0 + x_2 + x_3 = x_1 + x_3, \quad x_0 + x_3, \quad x_0 + x_1, \quad x_2, \quad x_1 + x_2 + x_3, \quad x_0 + x_2 + x_3, \quad x_0 + x_1 + x_2$

0101 1001 1100 0010 0111 1011 1110

$x_1 + x_2 + x_3 = x_1 + x_2, \quad x_0 + x_2, \quad x_0 + x_1, \quad x_3, \quad x_1 + x_2 + x_3, \quad x_0 + x_2 + x_3, \quad x_0 + x_1 + x_3$

0110 1010 1100 0001 0111 1011 1101

To write this solution more compactly, recall for $PG(N, p^n)$, we have the coordinates $(x_0, x_1, \ldots, x_N)$. Then in $PG(3,2)$, the coordinates are correspond to the number x_0, x_1, x_2, x_3 excluded the point (0,0,0,0) written in the scale of numerator with mod 2. Transferring to the scale each point gets a unique number between 1 - 15. In this design, zero is closure under mod 2 and 1 is not. The number corresponds to x_0, x_1, x_2, x_3 be in consonant with the equation $8x_0 + 4x_1 + 2x_2 + x_3$. Therefore, the result is shown in Table 1.

4.2 Theorem 1

Let (X,B) be a BIBD where $(X = Z_{15})$ and B is a subset of X, then $\{X\}_1^{15}$ is a finite group, semigroup and abelian group under additive or multiplication operations

TABLE 1

Shows the number of treatments and blocks for BIBD using PG (3,2)

Points	Co – ordinates				Blocks
	x_0	x_1	x_2	x_3	x_0, x_1, x_2, x_3
1	0	0	0	1	$x_0 = 0$: 1, 2, 3, 4, 5, 6, 7
2	0	0	1	0	$x_1 = 0$: 1, 2, 3, 8, 9, 10, 11
3	0	0	1	1	$x_2 = 0$: 1, 4, 5, 8, 9, 12, 13
4	0	1	1	1	$x_3 = 0$: 2, 4, 6, 8, 10, 12, 14
5	0	1	0	1	$x_0 + x_1 = 0$: 1, 2, 3, 12, 13, 14, 15
6	0	1	1	0	$x_0 + x_2 = 0$: 1, 4, 5, 10, 11, 14, 15
7	0	1	1	1	$x_0 + x_3 = 0$: 2, 4, 6, 9, 11, 13, 15
8	1	0	0	0	$x_1 + x_2 = 0$: 1, 8, 9, 6, 7, 14, 15
9	1	0	0	1	$x_1 + x_3 = 0$: 2, 8, 10, 5, 7, 11, 15
10	1	0	1	0	$x_2 + x_3 = 0$: 4, 8, 12, 3, 7, 11, 15
11	1	0	1	1	$x_0 + x_1 + x_2 = 0$: 3, 5, 6, 8, 11, 13, 14
12	1	1	0	0	$x_0 + x_1 + x_3 = 0$: 3, 9, 10, 4, 7, 13, 14
13	1	1	0	1	$x_0 + x_2 + x_3 = 0$: 5, 9, 12, 2, 7, 11, 14
14	1	1	1	0	$x_1 + x_2 + x_3 = 0$: 6, 10, 12, 1, 7, 11, 13
15	1	1	1	1	$x_0 + x_1 + x_2 + x_3 = 0$: 3, 5, 6, 9, 10, 12, 15

Proof

(a) Let $B_1, B_2, . . ., B_{15} \in X$ and let $a_1, a_2, ..., a_{15} \in B$, selecting any three elements in X, say $a = 3$, $b = 6$ and $c = 7$. It follows that:

(i) $a * b \in X, \forall\ a, b \in Z_{15}$ (ii) $(a * b) * c = a * (b * c), \forall\ a, b, c \in Z_{15}$

$\Rightarrow 6 \bmod 15\ \therefore\ 6 \in X$ (iii) $b * c = c * b, \forall b, c \in Z_{15} \Rightarrow 9 \bmod 15\ \therefore\ 9 \in X$

$PG(3,2) = (x_0, x_1, x_2, x_3)$ $s.t$ $x's \in GF(2)$

Hence $PG(3,2)$ is a BIBD with $(t, b, r, k, \lambda) \Rightarrow (15, 15, 7, 7, 3)$ parameters. Thus $PG(3,2)$ is a symmetric BIBD.

Group table for $(Z_{15}, *)$ in appendix 1, the identity element does not exist in some rows and some integers occur more than once in a column. Therefore not all integers have an inverse. Hence $(X, *)$ is not a group. But from (ii) and (iii), X form a semigroup and abelian group.

(b) Choosing any block (B) from the design X $s.t$ $B \in X$. Let a_1, a_2, a_3 be any three integers in the block (B_2), say $a_1 = 2, a_2 = 8$, $a_3 = 11$. For any $a_1, a_2, a_3 \in B_2$

(i) $a_1 * (a_2 * a_3) = (a * b) * c$

$\Rightarrow 11 \bmod 15,\ \therefore\ 11 \in B_2$

(ii) $a_1 * a_2 = a_2 * a_1$

$\Rightarrow 1 \bmod 15,\ \therefore\ 1 \in B_2$

4.3 Theorem 2

Let a system $(X,+,*)$ be a design with two binary compositions defined on the varieties $(X = R_{15})$, then $(X, B's)$ is a ring algebra and a commutative ring.

Proof

To show that $(X, B's)$ is a ring algebra and a commutative ring, the design must be abelian additive group and semigroup defined on $(X,+,*)$. Also the left and right distributive law must hold.

For the design $(X,+,*)$, let a, b, c be any three real number in X, say $a = 4$, $b = 10$ and $c = 13$, the for any $a, b, c \in X$:

(i) $(a+b)+c = a+(b+c) \Rightarrow 12 \bmod 15$

$\therefore 12 \in X$

(ii) $a+b = b+a \Rightarrow 14 \bmod 15$

$\Rightarrow 14 \in X$

(iii) $(a*b)*c = a*(b*c) \Rightarrow 10 \bmod 15$

$\Rightarrow 10 \in X$

(iv) $a*(b+c) = a*b+a*c \Rightarrow 2 \bmod 15$

$\Rightarrow 2 \in X$

$(b+c)*a = b*a+c*a \Rightarrow 2 \bmod 15$

$\Rightarrow 2 \in X$

(v) $a*b = b*a \Rightarrow 10 \bmod 15$

$\Rightarrow 10 \in X$

Group table for $(X,+)$ in appendix 2 illustrates the existence of inverse and identity element. The table shows that X is closed under $\oplus$ and zero (0) is the identity of the elements are; -0 = 0, -1 = 14, -2 = 13, -3 = 12, -4 = 11, -5 = 10, -6 = 9, -7 = 8, -8 = 7, -9 = 6, -10 = 5, -11 = 4, -12 = 3, -13 = 2, -14 = 1. Since all the axioms of ring algebra are satisfied, therefore the design $(X,+,*)$ is a ring. The property (v) above having satisfied shows that the design is a commutative ring.

4.4 Theorem 3

Let $(X = Z_{15})$ and $B \in X$ be BIBD, then $(X,+,\bullet)$ and $(B,+,\bullet)$ is not a finite field

Proof

Based on the axioms of field and from table 3, $(Z_{15},+)$ is an abelian group. Consider $(Z_{15} - \{0\},\bullet)$, then $Z_{15} - \{0\}$ is closed under$(\bullet)$. Associativity, commutativity and distributive law follow from those of integers. One (1) is the multiplicative identity. But from Table 1, some integers have no inverse. Therefore, $Z_{15} - \{0\}$ does not satisfy the second properties of a field. Hence $(X,+,\bullet)$ is not a finite field. Also from table 2, the inverse does not exist for $(B_{15},+)$ and $(B_{15} - \{0\},\bullet)$, though associativity, commutativity and distributive law holds. Hence $(B,+,\bullet)$ is not a finite field.

4.5 Theorem 4

Any two BIBDs obtained from the same PG by permutation are isomorphic and automorphic to each order.

Proof

Let (X,B) be BIBD constructed from $PG(3,2)$ with the parameters $(t,b,r,k,\lambda) \Rightarrow (15,15,7,7,3)$ defined as follows:

$X = \{1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$

$$B=\left\{\begin{array}{l}(1234567),\ (12389AB),\ (14589CD),\ (2468ACE),(123CDEF),\\ (145ABEF),(2469BDF),(18967EF),(28A57DF),(48C37BF),\\ (3568BDE),(39A47DE),(59C27BE),(6AC17BD),(3659ACF)\end{array}\right\}$$

Suppose the permutation $(789ABCDEF)$ is applied to X and $(1,2,3,4,5,6,7)$ left unchanged, we have a new design Y $s.t$ C is obtain to form another (Y,C)-BIBD. By permutation; $\alpha(7)=8$, $\alpha(8)=9$, $\alpha(9)=A$, $\alpha(A)=B$, $\alpha(B)=C$, $\alpha(C)=D$, $\alpha(13)=E$, $\alpha(D)=F$, $\alpha(F)=7$ yields:

$Y=\{1,2,3,4,5,6,8,9,A,B,C,D,E,F,7\}$

$$C=\left\{\begin{array}{l}(1234568),\ (1239ABC),\ (1459ACDE),\ (2469ACF),(123DEF7),\\ (145BEF7),(246ABD7),(1967EF7),\ (29A58D7),(4,9D38C7),\\ (3569CDF),(3AB48EF),(5AD28CF),(6BD18CE),(365ABD7)\end{array}\right\}$$

This design is also $(15,15,7,7,3)$- BIBD. This result follows from the definition that (X,B) and (Y,C) is a mapping $\alpha:X\rightarrow Y$ $s.t$ $\alpha X=Y$ and $\alpha B=C$. Hence (X,B) and (Y,C) are isomorphic.

To show that $(15,15,7,7,3)$- BIBD is automorphic, let (X,B) be $(15,15,7,7,3)$- BIBD and let α be the permutation of X such that $[\{\alpha(x):x\in A\},A\in B]=B$. A permutation of x can be represented as a disjoint cycle so that each cycle has the form $(x,\alpha(x),\alpha(\alpha(x)),\ldots)$ for some $x\in X$. The permutation of X is defined by the rule $\alpha(x)=A$, where $A\in B$. This implies that; $\alpha(1)=2$, $\alpha(2)=3$, $\alpha(3)=4$, $\alpha(4)=5$, $\alpha(5)=6$, $\alpha(6)=7$, $\alpha(7)=8$, $\alpha(8)=9$, $\alpha(9)=10$, $\alpha(10)=11$, $\alpha(11)=12$, $\alpha(12)=13$, $\alpha(13)=14$, $\alpha(14)=15$, $\alpha(15)=1$

Based on the above permutation, the corresponding blocks are;

B_1 =2,3,4,5,6,7,8 $\quad$ B_2 =2,3,4,9,10,11,12 $\quad$ B_3 =2,5,6,9,10,13,14

B_4 =3,5,7,9,11,13,15 $\quad$ B_5 =2,3,4,13,14,15,1 $\quad$ B_6 =2,5,6,11,12,15,1

$B_7 = 3,5,7,10,12,14,1$ $B_8 = 2,9,10,7,8,15,1$ $B_9 = 3,9,11,6,8,13,1$

$B_{10} = 5,9,13,4,8,12,1$ $B_{11} = 4,6,7,9,12,14,15$ $B_{12} = 4,10,11,5,7,14,15$

$B_{13} = 6,9,13,3,7,12,15$ $B_{14} = 7,11,13,2,8,12,14$ $B_{15} = 4,5,7,10,11,13,1$

In a compact form, the blocks can be written as:

$$B = \begin{Bmatrix} (2,3,4,5,6,7,8),(2,3,4,9,10,11,12),(2,5,6,9,10,13,14), (3,5,7,9,11,13,15), \\ (2,3,4,13,14,15,1), (2,5,6,11,12,15,1),(3,5,7,10,12,14,1),(2,9,10,7,8,15,1), \\ (3,9,11,6,8,13,1),(5,9,13,4,8,12,1),(4,6,7,9,12,14,15),(4,10,11,5,7,14,15), \\ 6,9,13,3,7,12,15),(7,11,13,2,8,12,14),(4,5,7,10,11,13,1) \end{Bmatrix}$$

Hence the design is a mapping $\alpha : B \rightarrow B$. Therefore (X,B) is an automorphic BIBD.

4.6 Construction of BIBD with EG (3, 2)

To construct BIBD for 3 – D with two finite number of elements using EG(3,2), the number of points in EG(3,2)= S^N. The parameters (t,b,r,k,λ) - BIBD are obtained as;

$$t = \Omega(N,0,s) - \Omega(N-1,0,1) = \frac{S^{N+1} - S^N}{S-1} = 8$$

$$b = \Omega(N,m,s) - \Omega(N-1,m,s) = \frac{S^{N+1} - 1}{S-1} - \left(\frac{S^N - 1}{S^{m+1} - 1} \right) = 14$$

$$r = \Omega(N-1,m-1,s) = \frac{S^N - 1}{S-1} = 7$$

$$k = \Omega(m,0,s) - \Omega(m-1,0,s) = \frac{S^{m+1} - S^m}{S-1} = 4$$

$$\lambda = \Omega(N-2,m-2,\ s) = \frac{S^{N-1} - 1}{S^{m-1} - 1} = 3$$

In EG (3,2), every point has coordinate of the form x_1, x_2, x_3. The blocks are obtained from the equations below;

$x_i = 0 or 1 \ (i = 1,2,3)$

$x_i + x_j = 0 or 1 \ (i, j = 1,2,3; i \neq j)$

$x_1 + x_2 + x_3 = 0 or 1$

The blocks are generated from the equation $4x_1 + 2x_2 + x_3$

$x_1 = 0$: $x_1 = x_2 = x_3 = 0$, x_3, x_2, $x_2 + x_3$
000 001 010 011

$x_2 = 0$: $x_1 = x_2 = x_3 = 0$, x_3, x_1, $x_1 + x_3$
000 001 100 101

$x_3 = 0$: $x_1 = x_2 = x_3 = 0$, x_2, x_1, $x_1 + x_2$
000 010 100 110

$x_1 = 1$: x_1, $x_1 + x_3$, $x_1 + x_2$, $x_1 + x_2 + x_3$
100 101 110 111

$x_2 = 1$: x_2, $x_2 + x_3$, $x_1 + x_2$, $x_1 + x_2 + x_3$
010 011 110 111

$x_3 = 1$: x_3, $x_2 + x_3$, $x_1 + x_3$, $x_1 + x_2 + x_3$
001 011 101 111

$x_1 + x_2 = 0$: $x_1 = x_2 = x_3 = 0$, x_3, $x_1 + x_2$, $x_1 + x_2 + x_3$
000 001 110 111

$x_1 + x_3 = 0$: $x_1 = x_2 = x_3 = 0$, x_2, $x_1 + x_3$, $x_1 + x_2 + x_3$
000 010 101 111

$x_2 + x_3 = 0$: $x_1 = x_2 = x_3 = 0$, x_1, $x_2 + x_3$, $x_1 + x_2 + x_3$
000 100 011 111

$x_1 + x_2 = 1$: x_2, $x_2 + x_3$, x_1, $x_1 + x_3$
010 011 100 101

$x_1 + x_3 = 1$: x_3, $x_2 + x_3$, x_1, $x_1 + x_2$
001 011 100 110

$x_2 + x_3 = 1$: x_3, $x_1 + x_3$, x_2, $x_1 + x_2$
001 101 010 110

$x_1 + x_2 + x_3 = 0$: $x_1 = x_2 = x_3 = 0$, $x_2 + x_3$, $x_1 + x_3$, $x_1 + x_2$

000 011 101 110

$x_1 + x_2 + x_3 = 1$: x_3, x_2, x_1, $x_1 + x_2 + x_3$

001 010 100 111

The co-ordinate of EG(N,s)is $(x_1, x_2, \ldots, x_N)$. Then for EG(3,2), the co-ordinates are correspond to the number x_1, x_2, x_3. Transferring the co-ordinates, each point has a unique number between 1 and 8. The number corresponds to x_1, x_2, x_3 is in line with the equation $4x_1 + 2x_2 + x_3$. The solution can be written in a tabular form as shown in Table 2.

Therefore EG (3,2) form a design below;

$X = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$$B = \begin{Bmatrix} (0123), (0145), (0246), (0167), (0257), (0437), (0356), (4567), (2367), \\ (1367), (2345), (1346), (1526), (1247) \end{Bmatrix}$$

Hence $EG(3,2)$ is $(8, 14, 7, 4, 3)$ - BIBD corresponds to the parameters (t, b, r, k, λ).

4.7 Theorem 5

Let $(X, B, +, *)$ be a system of the designs with two binary operations defined on the varieties $(X = Z_8)$, then (i) $(X, B, +)$ is a group (ii) $(X, B, *)$ is a semigroup

Proof

Let b_1, b_2, b_3 be any elements in (X, B) design, then $b_1, b_2, b_3 \in X$. For $b_1 = 2$, $b_2 = 5$, $b_3 = 7 \in X$, for any $b_1, b_2, b_3 \in Z$,

(i) $b_1 * (b_2 * b_3) = (b_1 * b_2) * b_3$

TABLE 2

Shows the number of treatments and blocks for BIBD using EG(3,2)

Points	Co – ordinates			Blocks	
0	x_1	x_2	x_3	$x_1 = 0$	$x_1 = 1$
1	0	0	1	$x_1 = 0: 0\ 1\ 2\ 3$	$x_1 = 1: 4\ 5\ 6\ 7$
2	0	1	0	$x_2 = 0: 0\ 1\ 4\ 5$	$x_2 = 1: 2\ 3\ 6\ 7$
3	0	1	1	$x_1 = 0: 0\ 2\ 4\ 6$	$x_3 = 1: 1\ 3\ 5\ 7$
4	1	0	0	$x_1 + x_2 = 0: 0\ 1\ 6\ 7$	$x_1 + x_2 = 1: 2\ 3\ 4\ 5$
5	1	0	1	$x_1 + x_3 = 0: 0\ 2\ 5\ 7$	$x_1 + x_3 = 1: 1\ 3\ 4\ 6$
6	1	1	0	$x_2 + x_3 = 0: 0\ 4\ 3\ 7$	$x_2 + x_3 = 1: 1\ 5\ 2\ 6$
7	1	1	1	$x_1 + x_2 + x_3 = 0: 0\ 3\ 5\ 6$	$x_1 + x_2 + x_3 = 1: 1\ 2\ 4\ 7$

$\Rightarrow 6 \in Z_8$. Therefore the operation (*) is associative

(ii) $b_1 * b_2 = b_2 * b_1$

$\Rightarrow 2 \in Z_8$. Therefore the operation (*) is commutative

Group table for $(Z_8,+,*)$ in appendix 3 are used to identify the existence of an identity element and inverse. One (1) is an identity element of the binary operation (*) but it is observed that some integers repeat more than once in rows and columns. Therefore, some integers have no inverse, hence $(Z_8,*)$ is not a group.

For $(Z_8,+)$ system of design, we have

(i) $b_1 + b_2 \in Z,\ b_1, b_2 \in Z_8$

$\Rightarrow 7 \in Z_8$, therefore Z_8 is closed under (+)

(ii) for any $b_1, b_2, b_3 \in Z,\ b_1 + (b_2 + b_3) = (b_1 + b_2) + b_3$

$\Rightarrow 6 \in Z_8$, therefore Z_8 is associative under (+)

(iii) for any $b_1, b_2 \in Z_8,\ b_1 + b_2 = b_2 + b_1$

$\Rightarrow 7 \in Z_8$, therefore Z_8 is commutative under (+)

(iv) 0 is an identity element for the binary operation (+), since

$b_1 + 0 = b,\ \forall\ b \in Z_8$

(v) The inverse elements are; -0 = 0, -1 = 7, -2 = 6, -3 = 5, -4 = 4, -5 = 3, -6 = 2, -7 = 1

Therefore $(Z_8,+)$ is a group. Hence the design $(X,+,*)$ is a group under the binary operation (+) and it is a semigroup under the binary operation (*).

4.8 Theorem 6

Suppose a system $< X,+,* >$ is a non – void set of BIBD, then $< X,+,* >$ is a ring algebra.

Proof

For $< X,+,* >$ to be a ring, the design must be abelian additive group, semigroup and satisfy right and left distributive law. Following the above theorem 5, and using appendix 3 for $< Z_8 + >$ shows that the design $< X,+,* >$ is abelian additive group and semigroup. For any $a,b,c \in X$, then

$a*(b+c) = a*b+a*c \in X$

$(b+c)*a = b*a+c*a \in X$

Therefore the right and left distributive law is satisfied. Hence $< X,+,* >$ is a ring.

4.9 Theorem 7

Any two $EG(N, p^n)$ - BIBDs are isomorphic and automorphic to each order.

Proof

Let (V_1, B) be BIBD constructed from $EG(3,2)$ with the parameters (t,b,r,k,λ) defined as follows;

$V_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$$B = \begin{cases} (0123), (0145), (0246), (0167), (0257), (0437), (0356), (4567), (2367), \\ (1367), (2345), (1346), (1526), (1247) \end{cases}$$

Suppose the permutation of (4567) is apply to B and left (0123) unchanged, we have a new design (V_2, C) such that C is gotten from B. The Permutation of (4567) means; $\alpha(4)=5$, $\alpha(5)=6$, $\alpha(6)=7$, $\alpha(7)=4$. The design (V_2, C) is shown below:

$V_2 = \{0, 1, 2, 3, 5, 6, 7, 4\}$

$$B = \begin{cases} (0123), (0156), (0257), (0174), (0264), (0534), (0367), (5674), (2374), \\ (1364), (2356), (1357), (1627), (1254) \end{cases}$$

From the definition, (V_1, B) and (V_2, C) is a bijection $\alpha : V_1 \to V_2$ st $\alpha V_1 = V_2$ and $\alpha B = C$. Hence the two $(8,14,7,4,3)$ - BIBDs are isomorphic.

To show that $(8,14,7,4,3)$ - BIBD is automorphic, let (V_1, B) be $(8,14,7,4,3)$ - BIBD and let the design Φ be the permutation of V such that $[\{\Phi(v) : v \in k\}, k \in B] = B$. The disjoint cycle for permutation V for some $v \in V$ defined as $\Phi(v) = k$ where $k \in B$. This implies that $\Phi(0) = 1$, $\Phi(1) = 2$, $\Phi(2) = 3$, $\Phi(3) = 4$, $\Phi(4) = 5$, $\Phi(5) = 6$, $\Phi(6) = 7$, $\Phi(7) = 0$. The rule of permutation gives the corresponding blocks below:

B_1 = 1,2,3,4	B_2 = 1,2,5,6	B_3 = 1,3,5,7	B_4 = 1,2,7,0
B_5 = 1,3,6,0	B_6 = 1,5,4,0	B_7 = 1,4,6,7	B_8 = 5,6,7,0
B_9 = 3,4,7,0	B_{10} = 2,4,6,0	B_{11} = 3,4,5,6	B_{12} = 2,3,5,7
B_{13} = 2,6,3,7	B_{14} = 2,3,5,0		

In a compact form, the blocks can be written as;

$B = \{B_1, B_2, B_3, B_4, B_5, \ldots \ldots, B_{14}\}$

$$B = \begin{Bmatrix} (1234),(1256),(1357),(1270),(1360),(1540),(1467),(5670),(3470), \\ (2460),(3456),(2357),(2637),(2350) \end{Bmatrix}$$

This design is a mapping $\Phi : B \rightarrow B$, hence the design (V,B) with $(8,14,7,4,3)$ is automorphic balanced incomplete block design (ABIBD).

4.10 Theorem 8

Suppose B_1, B_2, B_3 be any three blocks in (t,b,r,k,λ) - BIBD are equivalent classes, then;

(i) $B_1 \cong B_1$

(ii) $B_1 \cong B_2 \Rightarrow B_2 \cong B_1$

(iii) $B_1 \cong B_2$, $B_2 \cong B_3 \Rightarrow B_3 \cong B_1$

Proof

(i) Let B_1 defined as $\omega : B_1 \Rightarrow B_1$ $s.t$ $\omega(a) = a$, $\forall a \in B_1$. Then ω is clearly injective as well as subjective mapping on B_1 . Fornany $a,b \in B_1$, $\omega(a,b) = ab = \omega(a)\omega(b)$. Then ω is an automorphismof B_1 . Hence $B_1 \cong B_1$

(ii) For $B_1 \cong B_2$ $\exists$ an automorphism of X of B_1 onto B_2 . Since X is injective as well as onto, the mapping $X^{-1} : B_2 \rightarrow B_1$ given by $X^{-1}(b) = a$ if $X(a) = b$ for any $b \in B_2$ is 1 – 1 as well as onto. Let $b_1, b_2 \in B_2$ since X is onto, then $a_1, a_2 \in B_1$ $s.t$ $X(a_1) = b_1$ and $X(a_2) = b_2$. This implies that $X^{-1}(b_1) = a_1$ and $X^{-1}(b_2) = a_2$.

However, $X(a_1a_2) = X(a_1)X(a_2) = b_1b_2$ gives $X^{-1}(b_1b_2) = a_1a_2$

$= X^{-1}(b_1)X^{-1}(b_2)$. Hence X^{-1} is an isomorphism of B_2 onto B_1 and $B_2 \cong B_1$

(iii) Let $B_1 \cong B_2$, $B_2 \cong B_3$ $\exists$ an isomorphism X of B_1 onto B_2 and an isomorphism Y of B_2 onto B_3. Since X and Y are both injective as well as onto mapping. For any $a,b \in B_1$, $(Y \circ X)(ab) = Y(X(ab)) = Y(X(a)X(b))$ $\Rightarrow Y(X(a))Y(X(b))$ since $X(a), X(b) \in B_2$, then $[(Y \circ X)(a)][(Y \circ X)(b)]$. Hence, $Y \circ X$ is an isomorphism of B_1 onto B_2 and $B_1 \cong B_3$ as required.

4.11 Theorem 9

Suppose $B = \{v_1, v_2, \ldots, v_t\}$ be a semigroup and X be a non – void design

$s.t\ (X,B) = \left\{ \sum_i^n a_i v_i \,/\, a_i \in X, v_i \in B \right\}$ then $(X, B, +, *)$ is a ring.

Proof

For any $a,b,c \in X$ and $v_i \in B$, then $(+,*)$ defined on (X,B) as;

(i) $\sum_i^t a_i v_i + \sum_i^t b_i v_i = \sum_i^t (a_i + b_i) v_i$

(ii) $\left(\sum_i^t a_i v_i\right)\left(\sum_i^t b_i v_i\right) = \sum_i^t c_k v_k$ where $\sum_i^t a_i b_i = c_k$ and $v_i v_j = v_k$

$\sum_i^t a_i v_i$ is just a formal symbol. Then

(a) $$\left(\sum_i^t a_i v_i + \sum_i^t b_i v_i\right) + \sum_i^t c_i v_i = \sum_i^t (a_i + b_i) v_i + \sum_i^t c_i v_i = \sum_i^t [(a_i + b_i) + c_i] v_i$$

$$= \sum_i^t [a_i (b_i + c_i)] v_i = \sum_i^t a_i v_i + \sum_i^t (b_i + c_i) v_i = \sum_i^t a_i v_i + \left(\sum_i^t b_i v_i + \sum c_i v_i\right)$$

$$\Rightarrow \left(\sum_i^t a_i v_i + \sum_i^t b_i v_i\right) + \sum_i^t c_i v_i = \sum_i^t a_i v_i + \left(\sum_i^t b_i v_i + \sum c_i v_i\right)$$

(b) $$\sum_i^t a_i v_i + \sum_i^t b_i v_i = \sum_i^t (a_i + b_i) v_i = \sum_i^t (b_i + a_i) v_i$$

$$= \sum_i^t b_i v_i + \sum_i^t a_i v_i$$

(c) Let $\sum_i^t f_i v_i = 0$ where $f_i = 0, \forall\, i$, then

$$\sum_i^t a_i v_i + \sum_i^t f_i v_i = \sum_i^t (a_i + f_i) v_i = \sum_i^t a_i v_i \text{ as } f_i = 0, \forall i$$

(d) Also $\sum (-a_i) v_i + \sum a_i v_i = \sum (-a_i + a_i) v_i = 0, \ \forall i$

Hence $(X, B, +)$ is an abelian group.

Similarly, let $\sum_{(v_i v_j) v_k = v_u} a_i b_j c_k$ and $\sum_{(v_i v_j) v_k = v_p} a_i b_j c_k$ represent the elements $\alpha_u = \sum_{(v_i v_j) v_k = v_u} a_i b_j c_k$

and $\beta_v = \sum_{(v_i v_j) v_k = v_u} a_i b_j c_k$ of the ring X, where the two summations run over triples

a_i, b_j, c_k s.t

$(v_i v_j) v_k = v_u$ and $v_i (v_j v_k) = v_p$ respectively. Now

(e) $[(\sum a_i v_i)(\sum b_j v_j)](\sum c_l v_l) = \left(\sum_k \alpha_k v_k\right)\left(\sum_l c_l v_l\right)$, Where

$$\alpha_k = \sum_{v_i v_j = v_k} a_i b_j = \sum_u \beta_u v_u$$

$$\text{But } \beta_u = \sum_{v_k v_l = v_u} \alpha_k c_l = \sum_{v_k v_l = v_u} \left(\sum_{v_i v_j = v_k} a_i b_j\right) c_l = \sum_{(v_i v_j) v_l = v_u} a_i b_j c_l$$

Again $(\sum a_i v_i)[(\sum b_j v_j)(\sum c_l v_l)] = \left(\sum_i \alpha_i v_i\right) \sum_k \beta_k v_k$ where

$$\beta_k = \sum_{v_j v_l = v_k} b_j c_l = \sum_p \gamma_v v_p$$

$$\text{But } \gamma_v = \sum_{v_i v_k = v_p} \alpha_i \beta_k = \sum_{v_i v_k = v_p} a_i \sum_{v_j v_l = v_k} b_j c_l = \sum_{v_i (v_j v_k) = v_p} a_i b_j c_l$$

Since β is a semi – group under multiplication, $v_u = v_p, \forall u, p = 1, 2, \ldots, t$.

Hence

$[(\sum a_i v_i)(\sum b_j v_j)](\sum c_l v_l) = (\sum a_i v_i)[(\sum b_j v_j)(\sum c_l v_l)]$. Thus $(X, B, *)$ is a semi – group.

(f) $\sum_i^t a_i v_i \left(\sum_j^t b_j v_j + \sum_l^t c_l v_l\right) = \sum_i^t a_i v_i \left(\sum_j (b_j + c_l) v_j\right)$

$\Rightarrow \sum_i^t a_i v_i \left(\sum_j m_j v_j\right)$ where $m_j = b_j + c_l, \ \forall \ j = 1, 2, \ldots, t$

$\Rightarrow \sum_k^t \alpha_k v_k$, where $\alpha_k = \sum_{v_i v_j = v_k} a_i m_j = \sum_{v_i v_j = v_k} a_i (b_j + c_l)$

$\sum_{v_i v_j = v_k} a_i b_j + \sum_{v_i v_j = v_k} a_i c_l = \beta_k + \gamma_k$ where $\beta_k = \sum_{v_i v_j = v_k} a_i b_j$ and $\gamma_k = \sum_{v_i v_j = v_k} a_i c_l$

Also $\left(\sum_i a_i v_i\right)\left(\sum_j b_j v_i\right)+\left(\sum_i a_i v_i\right)\left(\sum_l c_l v_l\right) = \sum_k \beta_k v_k + \sum_k \gamma_k v_k$

$$= \sum_k (\beta_k + \gamma_k) v_k = \sum_k w_k v_k \text{ where } w_k = \beta_k + \gamma_k$$

Hence $\sum_i^t a_i v_i \left(\sum_j^t b_j v_j + \sum_l^t c_j v_j\right) = \left(\sum_i a_i v_i\right)\left(\sum_j b_j v_j\right)+\left(\sum_i a_i v_i\right)\left(\sum_l c_j v_j\right)$

Similarly

$$\left(\sum_j^t b_j v_j + \sum_l^t c_j v_j\right)\sum_i^t a_i v_i = \left(\sum_j^t (b_j + c_j) v_j\right)\sum_i a_i v_i$$

$\left(\sum_j n_j v_j\right)\sum_i a_i v_i$ where $n_j = b_j + c_j, \ \forall \ j = 1,2,3,\ldots,t$

$$= \sum_k z_k v_k \quad \text{where } z_k = \sum_{v_i v_j = v_k} n_j a_l = \sum_{v_i v_{j=v_k}} (b_j + c_j) a_i$$

$\sum_{v_i v_j = v_k} b_j a_l + \sum_{v_i v_{j=v_k}} c_j a_i = T_k + S_k$ where $T_k = \sum_{v_i v_j = v_k} b_j a_l$ and $S_k = \sum_{v_i v_{j=v_k}} c_j a_i$

Also $\left(\sum_l b_j v_j\right)\left(\sum_i a_i v_i\right)+\left(\sum_l c_j v_j\right)\left(\sum_i a_i v_i\right) = \sum_k T_k v_k + \sum_k S_k v_k$

$$= \sum_k (T_k + S_k) v_k = \sum_k Z_k v_k$$

Hence $\left(\sum_l b_j v_j + \sum_l c_j v_j\right)\sum_i a_i v_i = \left(\sum_l b_j v_j\right)\left(\sum_i a_i v_i\right)+\left(\sum_l c_j v_j\right)\left(\sum_i a_i v_i\right)$

Therefore $< X, B, +, * >$ is a ring.

4.12 Theorem 10

Suppose a design (X,B) be (t,b,r,k,λ) - BIBD. Let $B = \{b_1,b_2,b_3,\ldots,b_v\}$ be multiplicative abelian group $(B-\{0\},\bullet)$, then $(X,B,+,\bullet)$ is a field *iff* $(X,+)$ is an abelian group.

Proof

For any $x,y,z \in X$, then the binary operation (+) define on X as;

$\phi(x)+\phi(y)=\phi(x+y)$, where ϕ denote the number of each element in X. Given $x,y,z \in X$, then ;

(i) $(\phi(x)+\phi(y))+\phi(z) = \phi(x+y)+\phi(z) = \phi((x+y)+z)$

$$= \phi(x+(y+z)) = \phi(x)+\phi(x+z)$$

$$= \phi(x)+(\phi(y)+\phi(z))$$

Hence $(\phi(x)+\phi(y))+\phi(z) = \phi(x)+(\phi(y)+\phi(z))$. X is a associative under (+)

(ii) $\phi(x)+\phi(y) = \phi(x+y) = \phi(y+x) = \phi(y)+\phi(x)$

Therefore X is commutative under (+)

(iii) $\forall\ x \in X\ \exists\ \phi(-x) \in X\ s.t\ \phi(-x)+\phi(x) = \phi(-x+x) = 0$

Hence the inverse exist in X

(iv) Let $\phi(t)=0$, where $t = 0$, $0 \in X$, $s.t\ \phi(x)+\phi(t) = \phi(x+t) = \phi(x)\ as\ t = 0$

Then $x \in X$, hence $\phi(x)$ is an identity element

The results (i) – (iv) shows that the design $(X,+)$ under the additive binary operation is an abelian group. And since B is an abelian multiplicative group $(B-\{0\},\bullet)$, then the design $(X,B,+,\bullet)$ is a field.

4.14 Discussion of results

The projective geometry of three dimensional space with two finite elements PG(3,2) was used to generate (15,15,7,7,3) – BIBD where the co – ordinate of the points are built up of the elements of Galios field GF(2) which has only two elements 0 and 1. The finite PG(3,2) gives the equation $8x_0 + 4x_1 + 2x_2 + x_3$ correspond to the number x_0, x_1, x_2, x_3 without the point (0,0,0,0). The new equation obtained from PG(3,2) is used to generate each block of the 15 blocks. In this design, X is used as a variety and B as blocks which is represented as(X,B)

Furthermore, the algebraic concept of group theory, finite fields and rings were used to explain the designs obtained from PG(3,2). The result shows that the design (X,B) is a group additive binary operation. But under the multiplicative binary operation, it is a semigroup and not a group. In PG(3,2), X satisfied all the postulates of ring algebra, hence X is a ring under the binary operations of (+) and (*). But the block vector (B) is not a ring since some blocks does not satisfy some properties of a ring, hence B under the binary operation (+), that is, $(X,+)$ is a commutative ring. Based on the axioms of a field, the variety(X)form an abelian group $(X,+)$ but $(X-\{0\},\bullet)$ does not satisfy the second properties of a field as an abelian multiplicative group. Hence the variety (X) with two binary operations $(X,+,\bullet)$ is not a field. The prove of two balanced Incomplete Block Design (BIBD) are isomorphic has been shown by a mapping $\alpha : X \rightarrow Y$ *s.t* $\alpha X = Y$ and $\alpha B = C$, where

(X,B) and (Y,C) are two BIBDs. From the design (X,B), (15,15,7,7,3) - BIBD is automorphic by permutation (X) defined as $\alpha(x) = A$ where $A \in B$. The automorphism of (X) is a design which map B into itself. That is, $X : B \rightarrow B$, hence (X,B) is an automorphic balanced incomplete block design.

The design (E_3^2) series generate (8, 14, 7, 4, 3) – BIBD. In geometry $EG(3,2)$, every point has the co – ordinate of the form $x_1x_2x_3$. Each co – ordinate, we consider two elements either 0 or 1 to obtain the required number of blocks. The number of points in $EG(3,2)$ is obtained using the expression S^N. The blocks are generated in each co – ordinate from the equation $4x_1 + 2x_2 + x_3$ whose first co – ordinate is not zero (0). In accordance with the axioms of group, ring and field algebra, a design (X,B), where X is a variety and B is blocks vector were tested. The result shows that X is not a group under multiplicative binary operation (*) but under additive (+) binary operation, X is a group, semigroup and abelian group. It is observed that the variety (X) from the Euclidean geometry $EG(3,2)$ form a ring algebra. But some blocks obtained from X does not satisfy some properties of a ring. The results also show that $(X,+,\bullet)$ from $EG(N,p^n)$ is not field algebra. From the design X, we have B which is the block vector. The block $(B,+,\bullet)$ with two binary operations does not satisfy some axioms of field hence B is not field algebra. From the $EG(3,2)$, any two BIBDs (V_1,B) and (V_2,C) has been shown to be isomorphic under the defined rule $\alpha : V_1 \rightarrow V_2$ such that $\alpha V_1 = V_2$ and $\alpha B = C$. The automorphism of the design (V_1,B) was confirmed by letting ϕ to be the permutation of V such that $\phi(v)=k$ where $k \in B$. Many theorems have been established and the basic three are; to proof that any three blocks in (t,b,r,k,λ) - BIBD form an equivalence relationship with respect to isomorphism. Secondly, a finite semigroup $\{B\}_{v_1}^{v_r}$ and the design X is said to be $(X, B, +, \bullet)$ from ring algebra. Also, the abelian multiplicative group $\{B\}_{b_1}^{b_v}$ of $(B/\{0\}, \bullet)$ and the variety (X) is $(X, B, +, \bullet)$ form a field iff $(X, +)$ is an abelian group.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this research work, construction of balanced incomplete block design (BIBD) using geometrical approach was established. Block design originated from the statistical theory of experimental design and finite geometry. In experimental design view, balanced incomplete block design (BIBD) has a direct correlation with combinatorial design where the domain X is the variety and the subsets of X are blocks (B). Finite geometry is a combinatorial design on a finite set of elements, where the elements of their domains are known as points of the geometry called varieties and their distinguished subsets are known as lines of the geometry called blocks. A geometry design is commonly specified by its incidence matrix.

In harmony with the concept of this research work, block design is an ordered pair (V,B), such that V is a finite non – empty set whose elements are points and B is a finite non – empty multiset of non – void subsets of V called blocks. A balanced design is complete if $t = k$ so that each block contains all the design (X). if $k > t$, then the design X is incomplete. Based on these ideas, balanced incomplete block design (BIBD) is a design with treatments (t) in blocks (b) each of size (k), $(k < t)$ such that each treatment occurs at most once in a block, each treatment occurs in exactly r blocks and each pair of treatments occurs together in exactly λ blocks. Balanced Incomplete Block Design (BIBD) is related to algebra due to the concept of geometry. In line with this research, finite projective geometry denoted as $PG(N, p^n)$ and finite Euclidean geometry represented as $EG(N, p^n)$ was employed to construct BIBD which is used to test various aspects of algebra such as group, ring, field, isomorphism and automorphism of a design. In this context, group, ring and field is defined as follows:

A group is a system $< X,* >$, where X is a non – empty set called a design in BIBD with a binary operation (*) that satisfied the following axioms. For any $a,b,c \in X$

(i) $a*(b*c) = (a*b)*c$

(ii) $\exists\, I \in X \; s.t \; a*I = I*a = a, \forall\, a \in X$

(iii) for each $a \in X \; \exists\, a^{-1} \in X \; s.t \; a*a^{-1} = a^{-1}*a = I$

The design X is commutative or abelian group if, $\forall\, a,b \in X,\; a*b = b*a$. A BIBD defined as $< X,+,* >$ is a ring iff the following properties hold. For any $a,b,c \in X$

(i) $a+b = b+a$

(ii) $(a+b)+c = a+(b+c)$

(iii) $\exists\, 0 \in X \; s.t \; a+0 = 0+a = a, \forall\, a \in X$

(iv) For each $a \in X \; \exists -a \in X \; s.t \; a+(-a) = (-a)+a = 0$

(v) $(a*b)*c = a*(b*c)$

(vi) $a*(b+c) = a*b + a*c$

$(b*c)*a = b*a + c*a$

A design X in which $ab = ba$ for every $a,b,c \in X$ is a commutative ring. A design X from BIBD of the form $(X,+,*)$ is a finite field algebra with two binary operations which satisfies the following postulates:

$X_1 : (X,+)$ is an abelian (additive) group. The axioms of a ring (i) – (iv) is an abelian additive group.

$X_2 : (X-\{0\},\bullet)$ is an abelian (multiplicative) group defined as:

(i) $ab = b * a$, for every $a,b,c \in X$

(ii) $(a*b)*c = a*(b*c)$

(iii) $\exists\ I \in X\ s.t\ a*I = I*a = a,\ \forall\ a \in X$

(iv) $\forall\ a \neq 0 \in X\ \exists\ a^{-1} \in X\ s.t\ a*a^{-1} = a^{-1}*a = I,$

$X_3 : \forall\ a,b,c \in X,\ a*(b*c) = a*b + a*c$

The isomorphic of two BIBDs (X,B) and (Y,C) were evaluated by the rule $\alpha X = Y$ and $\alpha B = C$ such that a mapping $\alpha : X \to Y$ is a bijection. Also, an isomorphism of a design (X,B) to itself is automorphic were demonstrated. The constructed BIBD is from a finite projective geometry and finite Euclidean geometry of N – dimensions of ordered sets $(x_1, x_2, ..., x_N)$, where $x's$ are the elements of Galois field $GF(s)$.

$PG(3,2)$ was used to construct (15,15,7,7,3) – BIBD which is also a symmetry (t,k,λ) - BIBD of (15,7,3) – BIBD. This design was tested to verify whether the properties of group theory, ring and field algebra are satisfied. Two designs was formed from (15,7,3) – BIBD and used to examine whether there are isomorphic or not. The result of the isomorphism of the two BIBDs give rises to check for automorphic balanced incomplete block design. $EG(3,2)$ was also used to construct (8,14,7,4,3) – BIBD. The result does not concise with the symmetry properties of BIBD. The variety and the blocks of $EG(3,2)$ design was tested to ascertain whether the constructed BIBD from $EG(3,2)$ is a group, ring and a field. The results of the three algebra vary based on the binary operations defined on them. The isomorphism and automorphism of (8,14,7,4,3) – BIBD was evaluated. Hence, many theorems were established and proof accordance with the concept of an algebra.

5.2 Conclusion

Based on the results, it is observed that geometrical approach is the best technique to construct balanced incomplete block design (BIBD) irrespective of the block size and number of varieties. In construction of BIBDs using Projective geometry and Euclidean geometry with the same dimensions and finites number of elements, we observed that the number of blocks and varieties are not the same. In this design (X, B), finite group under multiplicative binary operation does not form, but it does under additive binary operation. That is; the design (X,B) is an abelian additive group. The design (X, B) form ring algebra for the two methods used in this research. Also the design (X, B) for the two methods is not field algebra. The BIBDs of the form (X,B) formed a semigroup, commutative group, semiring, commutative ring and subfield. An isomorphism of two BIBDs and automorphism of BIBD was shown by the rule of disjoint permutation. Hence, several theorems with proves have been established in harmony with the algebraic structure mentioned above.

5.3 Recommendations

Based on the findings, the following recommendations are made for further study:

1. To established a relationship between $PG(N, p^n)$ and $EG(N, p^n)$, the nature of the two orthogonal series and construct a design belonging to the series (P_3^3), (E_3^3), (P_3^4) and (E_3^4) respectively for the case $s = 3$.

2. To construct a higher dimensional geometrical designs involve an increasing number of replicates.

3. To find an isomorphism for the Fano plane to check the uniqueness of (3, 7, 1) up to automorphism and determine the order of its full automorphism group.

5.4 Contribution to knowledge

1. This research proposed the construction of balanced incomplete block design (BIBD) using the concept of Projective Geometry (PG) and Euclidean Geometry (EG).

2. The algebraic properties of the designs are explored

3. This research also showed that balanced incomplete block design (BIBD) constructed from the same Projective Geometry (PG) and Euclidean Geometry (EG) are isomorphic and automorphic to each other

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Appendix 1

Group table for $(Z_{15},*)$ design

*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13
3	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
4	0	4	8	12	1	5	9	13	2	6	10	14	3	7	11
5	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10
6	0	6	12	3	9	0	6	12	3	9	0	6	12	3	9
7	0	7	14	6	13	5	7	4	11	3	10	2	9	1	8
8	0	8	1	9	2	10	3	11	4	12	5	13	6	14	7
9	0	9	3	12	6	0	9	3	12	6	0	9	3	12	6
10	0	10	5	0	10	5	0	10	5	0	10	5	0	10	5
11	0	11	7	3	14	10	6	2	13	9	5	1	12	8	4
12	0	12	9	6	3	0	12	9	6	3	0	12	9	6	3
13	0	13	11	9	7	5	3	1	14	12	10	8	6	4	2
14	0	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Appendix 2

Group table for $(X,+)$ design

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Appendix 3

Group operations table for $(Z_8,+,*)$ design

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(a)

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(b)