ON THE RANKS OF CERTAIN SEMIGROUPS OF ORDER-PRESERVING PARTIAL ISOMETRIES OF A FINITE CHAIN

BY

MUHAMMAD JADA ALIYU

SPS/13/MMT/00016

BAYERO UNIVERSITY, KANO.

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MUHAMMAD JADA ALIYU

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Declaration

I Muhammad Jada Aliyu, hereby declare that this work is the product of my own research effort, undertaken under the supervision of Bashir Ali (Ph.D) and has not been presented to the best of my knowledge elsewhere for the award of a degree or certificate. All sources have been duly acknowledged.

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Muhammad Jada Aliyu   Date

SPS/13/MMT/00016.

Certification

This is to certify that Muhammad Jada Aliyu is a postgraduate student in the Department of Mathematical Sciences with registration number SPS/13/MMT/00016 has satisfactorily completed the requirements for research work for the degree of Masters of Science in Mathematics. The work embodied in this thesis is original and has not been submitted in part or full for any other certificate or degree of this or any other Institution.

Bashir Ali (Ph.D) Date

Supervisor.

Approval

This is to certify that this thesis titled "ON THE RANKS OF CERTAIN SEMI-GROUPS OF ORDER-PRESERVING PARTIAL ISOMETRIES OF A FINITE CHAIN" was carried out by Muhammad Jada Aliyu and has been approved by under signed as meeting the requirements for the award of Masters of Science degree in Mathematics, Faculty of Science Department of Mathematical Science Bayero University Kano.

Prof. Bashir Ali — Date
Supervisor.

Bashir Ali (Ph.D) — Date
Head of Department.

Amina M. Lawal (Ph.D) — Date
Internal Examiner.

Prof. Goje U. Garba — Date
External Examiner.

Nafiu Hussaini (Ph.D) — Date
P.G School Representative

Dedication

This research is dedicated to my lovely mum, Malama A'isha Abdulqadir, my dad, Mal. Aliyu Sa'ad Mbamba and my uncle Alh. Ibrahim Abdulqadir Jada.

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Abstract

Let $X_n = \{1, 2, ..., n\}$ be a finite chain and $\mathcal{ODP}_n$ be the semigroup of order-preserving partial isometries on X_n. In this work, we study the nilpotent elements in $\mathcal{ODP}_n$ and investigate the ranks of the nilpotent generated subsemigroup $\langle N \rangle$ and that of the the ideal $L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im(\alpha)| \leq r\}$ (where $1 \leq r \leq n$) of $\mathcal{ODP}_n$. We were able to show that, as an inverse semigroups the subsemigroup $\langle N \rangle$ has rank n while the ideal $L(n,r)$ has rank

$$\sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2} + \binom{n-2}{r-2}.$$

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION

This chapter gives some of the basic knowledge of semigroup theory which will help in understanding the work. The content of the chapter include the background of the research, the statement of the problem, the aim and objectives, as well as the justification of the study.

1.2 BACKGROUND OF THE RESEARCH

Let $X_n = \{1, 2, ..., n\}$ be a finite chain and S_n be a set consisting of all permutations on X_n more commonly known as *symmetric group*. It is well known that the set S_n plays a significant role in the study of group theory. In like manner, the set of all mappings of a finite set into itself (the semigroup of mappings) provides us with a corresponding objectives in semigroup theory. For one thing, such semigroups are rich source of example, as every semigroup can be embedded into some semigroups of mappings. Also, they are worth studying on their own right as natural occurring objects.

A lot of contributions concerning the study of the semigroup of mappings has been recorded over the years (see for example Howie [16, 17], Howie and Mc Fadden [18], Garba [10, 13], Ganyushkin and Mazorchuk [8], Umar [36, 37] and so many more). However, such contributions are by no means complete, the emergence of some new sets of semigroups of mappings in recent years, such as *the semigroup of contraction mappings* and *the semigroups of partial isometries* whose studies proves promising as seen in some recent significant journals and conference of proceedings (see for example Alkharousi [1, 2], A.D Adeshola [3], R. Kehinde [27]).

In this thesis, we are going to consider the *semigroup of order-preserving partial isometries.*

1.3 STATEMENT OF THE PROBLEM

Let $X_n = \{1, 2, ..., n\}$ be a finite chain and let $\mathcal{P}_n = \{\alpha : A \to B \quad \text{where } A, B \subseteq X_n\}$ be the *partial transformation semigroup*, $\mathcal{I}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is one-to-one mapping}\}$ be the *symmetric inverse semigroup*, $\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\forall x, y \in dom(\alpha))\, |x - y| = |x\alpha - y\alpha|\}$ and $\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x, y \in dom(\alpha)) \quad x \leq y \Longrightarrow x\alpha \leq y\alpha\}$. be the *semigroup of partial isometries* and *the semigroup of order preserving partial isometries* on the set X_n respectively.

A subset U of a semigroup S is called a generating set for S if every element of S can be expressed as product of some elements of U. If U generates S and any other generating set for S contains U, then we say that U is the minimal generating set for S.

What is the minimal generating set for $\mathcal{ODP}_n$?

If a semigroup S has a 0 (zero) element, a non zero element $a \in S$ is said to be nilpotent if there exist $m \in \mathbb{N}$ such that $a^m = 0$.

How do we characterize nilpotent in $\mathcal{ODP}_n$? Is $\mathcal{ODP}_n$ generated by nilpotents? If yes, what is the smallest subset consisting of nilpotents that generates $\mathcal{ODP}_n$? If no, how do we describe the subsemigroup generated by those nilpotents? And what is the minimal number of nilpotents that generates the subsemigroup?

1.4 MOTIVATION OF THE RESEARCH

Motivated by the works of Gomes an Howie 1987, G.U Garba 1994 and Alkharousi 2013, who worked on symmetric inverse semigroup, partial one-to-one order preserving semigroups and semigroup of partial isometries respectively.

1.5 AIM AND OBJECTIVES OF THE RESEARCH

Aim:

The aim of this research is to investigate some rank properties of some subsemigroups of $\mathcal{ODP}_n$.

Objectives:

The objectives include:

(i) To investigate some algebraic properties of $\mathcal{ODP}_n$.

(ii) To characterize elements of the nilpotent generated subsemigroup in $\mathcal{ODP}_n$.

(iii) To compute the rank of nilpotent generated subsemigroup of $\mathcal{ODP}_n$.

(iv) To compute the rank of the ideal

$L(n, r) = \{\alpha \in \mathcal{ODP}_n : |im(\alpha)| \leq r\}$ where $1 \leq r \leq n$.

1.6 JUSTIFICATION OF THE RESEARCH

Since the algebraic study of the semigroup $\mathcal{ODP}_n$ is not fully developed, it is our desire to investigate it's nilpotents and some rank properties of some of it's subsemigroups, with the hope that the result obtained in this research will form a part of the development of the theory of semigroup of mappings.

1.7 PRELIMINARIES

We give definition of some basic terms needed for the understanding of this thesis. All definitions and results given in this section are classical and can be found in any introductory text on semigroup theory. See Howie [19], J. A. Cain [21], Lallament G. [22] and T. Harju [35].

1.7.1 Semigroups

Definition 1.7.1 (Binary Operation). Let S be a non empty set, a binary operation $*$ on S is a mapping $* : S \times S \longrightarrow S$ such that $\forall a, b \in S$ $a * b$ is defined and $a * b$ is in S.

Example 1.7.2. If we consider $\mathbb{N} = \{1, 2, \ldots\}$ the set of all natural numbers, with the operation '+' (the usual addition), then '+' is a binary operation on $\mathbb{N}$. For, if

m, n are any two elements of $\mathbb{N}$ then $m + n$ is also in $\mathbb{N}$. On the other hand, if we consider the same $\mathbb{N}$ with the operation '-' (the usual substraction) then '-' is not a binary operation on S. Since 3 and 5 are elements of $\mathbb{N}$ but $3 - 5 = (-2) \notin \mathbb{N}$.

If $*$ is a binary operation on S, then $(S, *)$ is called an algebraic structure (groupoid to be specific). A binary operation $*$ on S is said to be associative if for any $a, b, c \in S$, $a * (b * c) = (a * b) * c$.

Definition 1.7.3 (Semigroup). A semigroup is a pair $(S, \cdot)$, where S is a non-empty set and "$\cdot$" is an associative binary operation on S. Obvious example is the set of natural numbers $\mathbb{N}$ with either addition or multiplication.

Throughout this thesis, S will denote a semigroup with operation '$\cdot$' instead of writing $(S, \cdot)$ unless we need to distinguish between different operations, for the same reason we will be writing xy instead of $x \cdot y$ (where $x, y \in S$) and we will call the operation multiplication and the element xy (i.e. the result of applying the operation to x and y) the product of the elements x and y.

In addition to associativity, if the operation "$\cdot$" on S is commutative, then we say that S is a commutative semigroup.

Definition 1.7.4 (Subsemigroup) A non-empty subset A of S is called a subsemigroup if it is closed with respect to multiplication, that is, if $a, b \in$ A then $ab \in A$. The associative condition of the operation follows immediately since it holds throughout S. Therefore A being subset of S need only to satisfy the closure property to become a subsemigroup with respect to the same operation.

Let A and B be two non-empty subsets of a semigroup S. We define the product of A and B as:

$$AB = \{ab : a \in A; b \in B\}$$

Therefore, in this sense, $A^2 = \{a_1 a_2 : a_1; a_2 \in A\}$ and $Ab = \{ab : a \in A\}$.

1.7.2 Ideals and Generating Sets

Definition 1.7.5 (Ideals). A non-empty subset A of a semigroup S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and (two-sided) ideal if it is both a left and a right ideal.

From this definition it is evident that every (right, left, or two-sided) ideal is a subsemigroup, but the converse is not the case. Among the ideals of S are S itself and (if S has a zero element) $\{0\}$. An ideal I such that $\{0\} \subset I \subset S$ (strictly) is called proper. A semigroup S is said to be *simple* if it has no proper ideal, and it is called a $0 -$ *simple* if $\{0\}$ is the only proper ideal it contains.

Let $\{T_i : i \in I\}$ be an indexed family of subsemigroups of S. The intersection $\cap T_i$, of all the subsemigroups T_i, is a subsemigroup of S. In particular, for any non-empty subset A of S, the intersection of all subsemigroup of S containing A is the smallest subsemigroup of S that contains A and we denote it by $\langle A \rangle$.

Definition 1.7.6 (Generating Set). Let A be a non empty subset of a semigroup S, the subsemigroup $\langle A \rangle$ generated by A is the smallest subsemigroup of S containing A which consists of all elements of S that can be expressed as finite products of elements in A. If $\langle A \rangle = S$, we say that A is a *generating set* for S.

In the above definition, if the subset A is finite, i.e. $A = \{a_1, a_2, \ldots, a_n\}$, then we shall write $\langle A \rangle = \langle a_1, a_2, \ldots a_n \rangle$. In the case $A = \{a\}$ a singleton, we write $\langle a \rangle = \{a, a^2, a^3, \ldots\}$ and refer to $\langle a \rangle$ as the *monogenic subsemigroup* of S generated by a. If $\langle a \rangle = S$, we say that S is a *monogenic* or *cyclic semigroup.* The order of an element $a \in S$ is defined as the order (the number of elements) of the subsemigroup

$\langle a \rangle$. A semigroup S is called periodic if all its elements are of finite order.

An element $e \in S$ is called an *idempotent* if $e^2 = e$. We will use $E(S)$ to denote the set of all idempotents in S. A semigroup S in which $E(S) = S$ is called a *band.* If $\langle E(S) \rangle = S$, then S is called a *semiband.*

1.7.3 Regular and Inverse Semigroups

An element a of a semigroup S is called regular if there exists $x \in S$ such that $axa = a$. The semigroup S is called regular if all its elements are regular. That is, if

$$(\forall a \in S)\, \exists x \in S : axa = a. \tag{1.7.1}$$

An element $a' \in S$ is an *inverse* of a if $aa'a = a$ and $a'aa' = a'$. A semigroup in which every element has a unique inverse is called an *inverse semigroup.* Note that this is entirely different from the notion of left/right inverses. In semigroups, the word 'inverse' (alone) is never used to mean left or right inverse. Although, every invertible element has an inverse element (in S) but not vise versa.

1.7.4 Equivalence Classes

Definition 1.7.6 (Binary Relation). Let X be a nonempty set. A (binary) relation on X is a subset δ of the cartesian product $X \times X$ of X such that $\forall x, y \in X$, the pair $(x, y) \in \delta$ if x is $\delta-$related to y. That is

$$\delta = \{(x, y) \in X \times X : x\delta y\}.$$

We shall frequently be writing $x\delta y$ if x is $\delta-$related to y instead of $(x, y) \in \delta$. Let $\mathcal{B}(X)$ be the set of all relations on X. Then, $\mathcal{B}(X)$ forms a semigroup under the operation of composition of relations. The operation is defined as: $\forall \rho, \delta \in \mathcal{B}(X)$,

$$\rho \circ \delta = \{(x, y) \in X \times X : (x, z) \in \rho,\ (z, y) \in \delta \quad \text{for some } \ z \in X\}.$$

The identity element of $\mathcal{B}(X)$ is the identity relation $\iota = \{(x, x) | \, x \in X\}$ (where every element is related to itself), which is the identity function on X. And the universal relation of $\mathcal{B}(X)$ is the relation $\omega = \omega_X = \{(x, y) | \, x, y \in X\}$ (in which everything is related to everything else).

Let $\delta \in \mathcal{B}(X)$. We adopt the following notations:

$$x\delta = \{y \,|\, (x, y) \in \delta\}$$

and

$$\delta^{-1} = \{(y, x) \,|\, (x, y) \in \delta\}.$$

A relation $\delta \in \mathcal{B}(X)$ is said to be *Reflexive* if $\iota_X \subseteq \delta$ (i.e $\forall x \in X,\ \ x\delta x$). It is said to be *Symmetric* if $\delta^{-1} = \delta$ (i.e $\forall x, y \in X,\ \ x\delta y \implies y\delta x$). And it is said to be *Transitive* if $\delta^2 \subseteq \delta$ (i.e $\forall x, y, z \in X$ if $\ x\delta y$ and $y\delta z \implies x\delta z$).

If δ satisfies all the three conditions above, then δ is called an *equivalence relation.* The set $x\delta$ defined above are called the *equivalence classes* of δ, and they formed partition of the set X. That is

$$X = \bigcup_{x \in X} x\delta \quad \text{and} \quad x\delta \cap y\delta \neq \emptyset \iff x\delta = y\delta.$$

1.7.5 Ordered Sets

Definition 1.7.7 (Partial-Order Relation). Let X be a non-empty set and $\leq$ be a binary relation on X. Then $\leq$ is said to be a partial order if it satisfies the following:

(O1) $\leq$ is *reflexive*; that is $\forall x \in X$, $x \leq x$.

(O2) $\leq$ is *antisymmetric*; that is $\forall x, y \in X$, if $x \leq y$ and $y \leq x$, the $x = y$.

(O3) $\leq$ is *transitive*; that is $\forall x, y, z \in X$, if $x \leq y$, and $y \leq z$, the $x \leq z$.

A set with a binary relation satisfying the above properties is called a *partial ordered set* or (*Poset* for short). We shall write $(X, \leq)$ or just simply X to refer to a partial ordered set. If a partial ordered set X satisfies an additional property that;

(O4) $(\forall x, y \in X)\, x \leq y$ or $y \leq x$,

then X is called a *total ordered set* (or *chain*).

1.7.6 Congruence Classes

An equivalence relation ρ on a semigroup S is called a *left congruence*, if

$$\forall x, y, z \in S \quad x\rho y \Longrightarrow (zx)\rho(zy),$$

and it is called a *Right congruence*, if

$$\forall x, y, z \in S \quad x\rho y \Longrightarrow (xz)\rho(yz)$$

and ρ is called a congruence, if it is both a left and a right congruence.

Recall that an equivalence relation ρ partitions the domain S into the equivalence classes $x\rho(x \in S)$. An equivalence class of a congruence is called a *congruence class.*

If ρ is a congruence, then it respects the product of S, that is, if the elements x_1, y_1 and x_2, y_2 are in the same equivalence class (i.e., $x_1\rho = y_1\rho$ and $x_2\rho = y_2\rho$), then x_1x_2 and y_1y_2 are in the same equivalence class (that is to say, you can also define an operation between the equivalence classes instead of the elements). A formal statement of this is given in the following lemma:

Lemma 1.7.1 [35, lemma 2.6] *An equivalence relation ρ on S is a congruence if and only if for all $x_1, x_2, y_1, y_2 \in S$*

$$\left.\begin{array}{l} x_1\rho y_1 \\ x_2\rho y_2 \end{array}\right\} \Longrightarrow (x_1x_2)\rho(y_1y_2).$$

1.7.7 Quotient Semigroups

Let ρ be a congruence on S, and let

$$S/\rho = \{x\rho \, : \, x \in S\}$$

be the set of all congruence classes of ρ. We define a new semigroup on the domain S/ρ by contracting each congruent class $x\rho$ into a single element. The operation is define as:

$$x\rho \cdot y\rho = (xy)\rho.$$

This operation is well defined by lemma 1.7.1 above. And for any $x, y, z \in S$,

$$x\rho \cdot (y\rho \cdot z\rho) = x\rho \cdot (yz)\rho = (x(yz))\rho = ((xy)z)\rho = (xy)\rho \cdot z\rho = (x\rho \cdot y\rho) \cdot z\rho,$$

therefore, the operation is associative as well. This semigroup is called the *quotient semigroup* (of S modulo ρ).

Let I be an ideal of a semigroup S. Define a relation ρ_I on S by $x\rho_I y \Longleftrightarrow$ either $x = y$ or both x and y are in I. This relation is an equivalence relation. One can also show that ρ_I is a congruent on S as follows: Let x, y be in S. Suppose that $x\rho_I y$, then either $x = y$ or $x, y \in I$. If $x = y$, then for any $z \in S$, $zx = zy$ and $xz = yz$, that is $(zx)\rho_I(zy)$ and $(xz)\rho_I(yz)$. And if both x and y are in I, an ideal of S, then clearly, for any z in S zx, zy, xz, yz are all in I. Hence the proof.

We can now conveniently define a quotient semigroup on S with respect to ρ_I as

$$S/\rho_I = \{I\} \cup \{\{x\} : x \in S \backslash I\},$$

which consist of elements of I together with the elements of $S \backslash I$.

In S/ρ_I, the product of two elements in $S \backslash I$ is the same as their product in S if this lies in $S \backslash I$; otherwise the product is I. Since the element I of S/ρ_I is the zero element of the semigroup, another useful way of thinking S/ρ_I is as $(S \backslash I) \cup \{0\}$, where all product not falling in $S \backslash I$ are zero. This semigroup S/ρ_I is called a *Rees quotient Semigroup.*

1.7.8 Green's Relations

Green's relations was introduced by J. A. Green [12] in 1951. The relations are $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{D}$. These five equivalence relations serve as fundamental tool in understanding the structure of a semigroup. They relate elements of a semigroup based on the ideals the elements generate, thereby giving a lot of information about the structure of that semigroup and how its elements interact.

Let a be an element of a semigroup S. The smallest left ideal of S containing an element a is $Sa \cup \{a\}$, denoted by $S^1 a$ and is called the principal left ideal generated by a. Similarly, the smallest respective right and (two sided) ideal of S containing an element a is denoted and defined as $aS^1 = aS \cup \{a\}$ and $S^1 aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

For any two elements $a, b \in S$, we define the equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{D}$ on S by:

$$a\mathcal{L}b \iff S^1 a = S^1 b \tag{1.7.2}$$

$$a\mathcal{R}b \iff aS^1 = bS^1 \tag{1.7.3}$$

$$a\mathcal{J}b \iff S^1 aS^1 = S^1 bS^1 \tag{1.7.4}$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \tag{1.7.5}$$

and

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} \tag{1.7.6}$$

In many cases the two equivalences $\mathcal{D}$ and $\mathcal{J}$ coincide. This happens when the semigroup is finite, and also when it is periodic.

1.8 TRANSFORMATION SEMIGROUPS

Let X be a non empty set, a transformation is a mapping $\alpha : X \longrightarrow X$. Various transformations of different sets appear frequently in mathematics and as the usual composition of transformations is associative, each set of transformations, which is closed with respect to composition, forms a semigroup called *Transformation* *semigroup.*

Let $X_n = \{1, 2, ..., n\}$ be a finite chain and α be a transformation of X_n. Then, for all $x \in X_n$, we shall write $x\alpha$ to mean the image of x under the transformation of α instead of $\alpha(x)$ so that the composition (product) of any two or more transformation will always start from left.

1.8.1 The Partial Transformation Semigroup

Let A and B are any two subsets of X_n, a mapping $\alpha : A \longrightarrow B$ is called a *partial transformation* of X_n. A and B are called domain and range of α respectively. The set of all partial transformations of X_n is denoted by $\mathcal{P}_n$ and as the composition of partial transformations will be a partial transformation, the set $\mathcal{P}_n$ is a semigroup under composition of mappings called *the partial transformation semigroup.*

The map $I : X_n \rightarrow X_n$ defined by $I(i) = i \, (\forall i \in X_n)$ acts as the identity of $\mathcal{P}_n$. The zero map $0 : \emptyset \rightarrow X_n$ is the zero element of $\mathcal{P}_n$. Thus, the partial transformation semigroup $\mathcal{P}_n$ is a monoid with zero.

If $\alpha \in \mathcal{P}_n$ is a partial transformation with domain $\{a_1, a_2, \ldots, a_m\} \subseteq X_n$, $(1 \leq m \leq n)$ α shall be written in array form as:

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ a_1\alpha & a_2\alpha & \ldots & a_m\alpha \end{pmatrix}. \tag{1.8.1}$$

Also, we define height of α to be the cardinality of the image set of α which is denoted by $h(\alpha)$. That is $h(\alpha) = |im(\alpha)|$.

Note that in the semigroup of partial transformation above, the subsets A and B defined there as the *domain* and *range* set are just arbitrary subsets of X_n, which could be up to X_n or proper subsets of X_n. The collection of all those mappings whose domain is exactly X_n gives a subsemigroup of $\mathcal{P}_n$ called the *Full transformation semigroup* denoted by T_n.

1.8.2 The Symmetric Inverse Semigroup.

A map $\alpha \in \mathcal{P}_n$ is called partial injection (one-to-one map) if for all $x, y \in dom(\alpha), x\alpha = y\alpha$ implies $x = y$. Obviously, the composition of two partial injections is a partial injection. Therefore the set of all partial injections, denoted by $\mathcal{I}_n$, is a subsemigroup of $\mathcal{P}_n$ which is an inverse semigroup ([19], Theorem 5.1.5), called the *symmetric inverse semigroup.* The semigroup $\mathcal{I}_n$ is the appropriate analogue in inverse semigroup theory of the symmetric group in group theory and the full transformation semigroup in semigroup theory. The following theorem known as *Vagner-Preston* theorem is the analogue of Cayley's Theorem.

Theorem 1.8.1 [19 theorem 5.1.7] *Every (finite) inverse semigroup is embeddable in a (finite) symmetric inverse semigroup.*

An element $\alpha \in \mathcal{I}_n$ is said to be $order-preserving$ if $(\forall x, y \in dom(\alpha)\, x \leq y \Longrightarrow x\alpha \leq y\alpha$. It is evident from this definition that, the composition of order preserving mappings is an order preserving, thus, the set $\mathcal{IO}_n = \{\alpha \in \mathcal{I}_n : \alpha$ is order-preserving$\}$ is subsemigroup of $\mathcal{I}_n$ called the *Semigroup of order preserving partial one-to-one transformation.*

As every inverse semigroup is automatically a regular semigroup, Garba in [10] deduced the following relations from [19, Proposition 2.4.2]:

$$\mathcal{L}(\mathcal{IO}_n) = \mathcal{L}(\mathcal{I}_n) \cap (\mathcal{IO}_n \times \mathcal{IO}_n)$$

$$\mathcal{R}(\mathcal{IO}_n) = \mathcal{R}(\mathcal{I}_n) \cap (\mathcal{IO}_n \times \mathcal{RO}_n).$$

$$\mathcal{J}(\mathcal{IO}_n) = \mathcal{J}(\mathcal{I}_n) \cap (\mathcal{IO}_n \times \mathcal{IO}_n).$$

The semigroup $\mathcal{IO}_n$ has $n+1$ $\mathcal{J}$ classes; $J_0, J_1, \ldots, J_n$ where $J_r\,(r = 1, 2, \ldots, n)$ consist of all element of *height* r. And each $\mathcal{J}$ class J_r consist of $\binom{n}{r}$ $\mathcal{L}-classes$ and $\binom{n}{r}$ $\mathcal{R}-classes$. And by the order-preserving nature of the elements, it is clear that each $\mathcal{H}-class$ contains only one element. Thus $|J_r| = \binom{n}{r}^2$ and

$$|\mathcal{IO}_n| = \sum_{r=0}^{n} \binom{n}{r}^2. \tag{1.8.2}$$

The following theorems describes the principal right (respectively left or two sided) ideals of transformation semigroups.

With each $\alpha \in \mathcal{P}_n$, we associate the binary relation π_α on X_n in the following way:

$$x\pi_\alpha y \Leftrightarrow \{x, y \in dom(\alpha) \text{ and } x\alpha = y\alpha \quad \text{or} \quad x, y \notin dom(\alpha)\} \tag{1.8.3}$$

In other words, $x\pi_\alpha y$ if either α is not defined on both x and y, or α is defined on both x and y and $\alpha(x) = \alpha(y)$.

Theorem 1.8.2 [8, theorem 4.2.4]. *Let S denote the semigroup $\mathcal{T}_n$, $\mathcal{P}_n$ or $\mathcal{I}_n$ and $\alpha \in S$. Then the left principal ideal generated by α has the following form*

$$S\alpha = \{\beta \in S : dom(\beta) \subset dom(\alpha) \text{ and } \pi_\alpha \subset \pi_\beta\}. \tag{1.8.4}$$

Theorem 1.8.3 [8, theorem 4.2.1]. *Let S denote the semigroup $\mathcal{T}_n$, $\mathcal{P}_n$ or $\mathcal{I}_n$ and $\alpha \in S$. Then the right principal ideal generated by α has the following form*

$$\alpha S = \{\beta \in S : im(\beta) \subset im(\alpha)\}. \tag{1.8.5}$$

Theorem 1.8.4 [8, theorem 4.2.8]. *Let S denote the semigroup $\mathcal{T}_n$, $\mathcal{P}_n$ or $\mathcal{I}_n$ and $\alpha \in S$. Then the principal ideal generated by α has the following form*

$$S\alpha S = \{\beta \in S : rank(\beta) \leq rank(\alpha)\}. \tag{1.8.6}$$

The next theorem, gives us the general characterization of an arbitrary ideal in transformation semigroup.

Theorem 1.8.5 [8, theorem 4.3.1]. *Let S denote the semigroup $\mathcal{T}_n$, $\mathcal{P}_n$ or $\mathcal{I}_n$. Then all two-sided ideals in S are principal and are generated by any element of the ideal, which has the maximal possible rank.*

Proof: Let $I \subseteq S$ be a two-sided ideal. Choose $\alpha \in I$ of maximal possible rank. As $SIS \subseteq I$, we have $S\alpha S \subseteq I$. On the other hand, by Theorem 1.8.4 the set $S\alpha S$ contains all elements whose rank does not exceed that of α. Hence $I \subseteq S\alpha S$ and thus $I = S\alpha S$.

Now, let S denote the semigroup $\mathcal{T}_n$, $\mathcal{P}_n$ or $\mathcal{I}_n$. For $k \leq n$, set

$$I_k = \{\alpha \in S : rank(\alpha) \leq k\} \tag{1.8.7}$$

Then by Theorem 1.8.4, each I_k is an ideal of S, and each ideal in S is of the form I_k for some k. In particular, the set of all ideals of S forms a chain with respect to inclusions. For $\mathcal{T}_n$ this chain has the form

$$I_1 \subset I_2 \subset \ldots \subset I_n = \mathcal{T}_n$$

While for $\mathcal{P}_n$ and $\mathcal{I}_n$ the chain has the form

$$\{0\} = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_n = \mathcal{P}_n(\text{ or } \mathcal{I}_n)$$

1.8.3 The Semigroup of Partial Isometries.

A transformation $\alpha \in \mathcal{P}_n$ is said to be *isometry* or (*distance preserving*) if $(\forall x, y \in dom(\alpha))\, |x-y| = |x\alpha - y\alpha|$. That is to say; if a transformation α is an isometry then the distance between any two points in the domain of α is the same as the distance between their images. It is clear from this definition that every transformation which is an isometries is also a one-to-one. Thus the set of all transformation isometries are denoted and define as

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\forall x, y \in dom(\alpha))\, |x-y| = |x\alpha - y\alpha|\} \tag{1.8.8}$$

is an inverse subsemigroup of $\mathcal{I}_n$ called the *semigroup of partial isometries.* Similarly we define

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x, y \in dom(\alpha))\, x \leq y \Longrightarrow x\alpha \leq y\alpha\} \tag{1.8.9}$$

to be a subsemigroup of $\mathcal{DP}_n$ consist of all order-preserving mappings, called the *Semigroup of order preserving partial isometries.* The following theorems gives us the cardinalities of the two semigroups:

Theorem 1.8.6 [1, theorem 2.15] *Let $\mathcal{DP}_n$ be the semigroup of partial isometries. Then $|\mathcal{DP}_n| = 3 \cdot 2^{n+1} - (n+1)^2 - 1$.*

Theorem 1.8.7 [1, theorem 2.6] *Let $\mathcal{ODP}_n$ be the semigroup of order preserving partial isometries. Then $|\mathcal{ODP}_n| = 3 \cdot 2^n - 2(n+1)$.*

Before moving into the literature review, we will give some definitions of some special types of numbers called the Stirling numbers. They are special type of numbers which arise in a variety of analytic and combinatorics problems. They are named after James Stirling, who introduced them in the 18th century. Two different sets of numbers bear this name: the Stirling numbers of the first kind and the Stirling numbers of the second kind. The most common notation used for the Stirling numbers for the first and second kind are $s(n, k)$ and $S(n, k)$ respectively.

Definition 1.8.8 (Stirling Numbers of the First Kind): *The Stirling numbers of the first kind are the coefficients in the expansion*

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

where $(x)_n$ denotes the falling factorial; $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$. Note that $(x)_0 = 1$ because it is an empty product.

A few of the Stirling numbers of the first kind are illustrated by the table below, starting with row 0, column 0:

1						
0	1					
0	−1	1				
0	2	−3	1			
0	−6	11	−6	1		
0	24	−50	35	−10	1	
0	−120	274	−225	85	−15	1

where

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k).$$

Definition 1.8.9 (Stirling Numbers of the Second Kind): *The Stirling numbers of the second kind count the number of ways to partition a set of n elements* *into k nonempty subsets. They are denoted by* $S(n,k)$ *or* $\left\{ n \atop k \right\}$ *which can be defined using the falling factorials as*

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, we review some of related literature on ranks and nilpotent ranks of some transformation semigroups. We begin by giving some historical development of semigroup theory.

2.2 HISTORICAL BACKGROUND OF SEMIGROUP THEORY

The first proper semigroup theory began in the 1920s with the work of the Russian mathematician Anthon Kazimirovich Suschkewitsch [28], who obtained a number of results. One of such is: "Every semigroup can be embedded in a full transformation monoid" - the result which is analogue to Cayleys theorem for groups. However, despite the publication of a textbook [29]; "The theory of generalised Groups" in 1937, Suschkewitschs work remained unknown for many years and his results were without being aware rediscovered by later researchers. For example, “The embedding of (finite) semigroup in a full transformation monoid” was reproduced by Stoll [30] in 1944.

During the 1930s, the study of semigroups began to take off, although at this early stage it was still influenced heavily by existing work on both groups and rings; Semigroups were approached either by dropping selected group axioms, or by discarding an entire operation, namely addition, from a ring. As the decade progressed, the theory gradually gained speed, resulting in the publication of three highly influential papers: Rees [27] in 1940, Clifford [6] and Dubreil [7] in 1941. Rees stated semigroup theory’s first major structure theorem, now known appropriately enough, as the Rees Theorem. This result completed a strand of research initiated by Suschkewitsch [28] and is analogous to the Wedderburn - Artin Theorem for rings. Clifford [6] stated the structure theorem which has no analogue in either group or ring theory and can therefore be taken to mark the beginning of an independent theory of semigroups.

The 1950s saw the introduction of three broad concepts which continue to be of use and interest in the modern theory of semigroups: Greens relations, regular semigroups and inverse semigroups. Inverse semigroups were introduced independently in Soviet Union and Great Britain by Wagner [31, 32] (1952-1953) and Preston [23, 24, 25] (1954) respectively. Green in 1951, first published Greens relations in a paper titled; On the structure of semigroup [14]; simply put, these are collections of equivalence relations which may be defined within a given semigroup(in terms of its principal ideals) to enable us to study its large scale structure as already explained in the previous chapter.

2.3 RANKS OF SOME TRANSFORMATION SEMI-GROUPS

As most of the semigroups we are going to discussed are inverse semigroup, it is important to clarify the notion of generating set in an inverse semigroup. Given a subset A of an inverse semigroup, by the inverse subsemigroup generated by A, we mean the smallest subsemigroup containing A, which consist of all finite product of elements of A and their inverses. This definition is similar to what we define earlier on generating sets for a given semigroup in the previous chapter, only in later one, we can see that the generated subsemigroup contain not only the finite product of elements of the set, but also their inverses. For instance, if we consider the set of integers $\mathbb{Z}$ with the operation '+' as our semigroup, then clearly the generating set of $\mathbb{Z}$ as a semigroup is $\{1, -1\}$. But as an inverse semigroup the generating set is just the singletone set $\{1\}$ or $\{-1\}$.

We have seen from definition of semigroup, if $a, b \in S$ then ab is also in S. Upon encountering any semigroup, perhaps one of the most natural question that one may have in mind is: Can some of this elements be expressed as product of other elements? If so what is the smallest subset of S (*i.e the minimal generating set*) that can generate the whole S. If such set exist (which always does as the set S itself is an obvious candidate), then the cardinality of such set is what we called the rank of the semigroup S.

Let $X_n = \{1, 2, \ldots n\}$ be a finite chain, and S_n be a symmetric group on X_n. It is well known that S_n is generated by two elements (the cyclic permutations $(1\ 2)$ and $(1\ 2\ \ldots\ n)$) while the symmetric inverse semigroup $\mathcal{I}_n$ has rank 3 being generated (as an inverse semigroup) by the two generators of S_n together with any transformation α for which $|dom(\alpha)| = n - 1$. Gomes and Howie [18], also examined the symmetric inverse semigroup $\mathcal{I}_n$ and showed that the rank (as an inverse semigroup) of the inverse semigroup

$$S\mathcal{I}_n = \{\alpha \in \mathcal{I}_n : |im\alpha| \leq n-1\}$$

consisting of all strict partial one-to-one transformation is n+1. Howie [15] generalized this by showing that for $r = 3, \ldots, n-1$ the rank of

$$L(n,r) = \{\alpha \in \mathcal{I}_n : |im\alpha| \leq r\} \text{ is } \binom{n}{r} + 1.$$

In [9] Garba studied the semigroup $\mathcal{P}_n$ of all partial transformations of X_n and showed that for the semigroup

$$K'(n,r) = \{\alpha \in \mathcal{P}_n : |im\alpha| \leq r\}$$

both the rank and the idempotent rank are equal to $S(n+1, r+1)$. Where $S(n+1, r+1)$ is the stirling number of the second kind.

In another paper [19], Gomes and Howie investigated the rank of the semigroups $\mathcal{O}_n$, $\mathcal{PO}_n$, and $S\mathcal{PO}_n$ (the semigroup of order-preserving full transformations, order-preserving partial transformations and order-preserving strictly partial transformations on X_n respectively). They showed that the rank of $\mathcal{O}_n$ is n, that of $\mathcal{PO}_n$ is $2n-1$ and $S\mathcal{PO}_n$ has rank $2n-2$. The idempotent rank of $\mathcal{O}_n$ is $2n-2$, $\mathcal{PO}_n$ is idempotent generated and its idempotent rank is $3n-2$. The semigroup $S\mathcal{PO}_n$ on the other hand is not idempotent-generated and so the question of its idempotent rank does not arise. These results have been generalised by Garba [12], (in line with Howie and McFadden [20]). He showed that For

$$L(n,r) = \{\alpha \in S : |im\alpha| \leq r \text{ and } r \leq n-2\}$$

the rank and the idempotent rank of $L(n,r)$ are both equal to $\binom{n}{r}$; if $S = O_n$. And if $S = \mathcal{PO}_n$, then the rank and the idempotent rank are both equal to

$$\sum_{k=r}^{n} \binom{n}{r} \binom{k-1}{r-1}$$

while for $S = S\mathcal{PO}_n$ the rank and the idempotent rank are both equal to

$$\sum_{k=r}^{n-1} \binom{n}{r} \binom{k-1}{r-1}.$$

2.3.1 Nilpotent Ranks

Let S be a semigroup S with 0 (zero) element, a non zero element $a \in S$ is said to be a nilpotent if there exist $m \in \mathbb{N}$ such that $a^m = 0$. Gomes and Howie [15] initiated the study of nilpotents by considering the subsemigroup $S\mathcal{I}_n$ of $\mathcal{P}_n$ consisting of all strictly partial one-to-one transformations. But before we present their results, we need the following definition

Let $\alpha \in \mathcal{P}_n$, then α is said to have *projection characteristic* (k,r) or to belong to $[k,r]$ if $|dom\alpha| = k$ and $|im\alpha| = r$. It is clear from this definition that given any element $\alpha \in [n-1, n-1]$ we must have $X_n \backslash dom(\alpha) = \{i\}$ and $X_n \backslash im(\alpha) = \{j\}$ for some $i, j \in X_n$. Hence we can always have a unique extension of the mapping α ; say α^* element in $[n,n]$ defined by:

$$i\alpha^* = j \quad \text{and} \quad x\alpha^* = x\alpha \, (\forall x \in dom\alpha).$$

Such α^* is called the completion of α .

Gomes and Howie proved that if n is even the subsemigroup $S\mathcal{I}_n$ of $\mathcal{P}_n$ is nilpotent generated. For n odd they showed that the nilpotents in $S\mathcal{I}_n$ generate $S\mathcal{I}_n \backslash W_{n-1}$

where W_{n-1} consists of all $\alpha \in [n-1, n-1]$ whose completions are odd permutations.

While Gomes and Howie were studying $\mathcal{P}_n$, simultaneously and independently Sullivan [34] considered the subsemigroup $S\mathcal{P}_n$ of $\mathcal{P}_n$ consisting all strict partial transformation. Surprisingly, their results turn out to be similar. For he showed that: If N is the set of nilpotents in $S\mathcal{P}_n$ then

$$\langle N \rangle = \begin{cases} S\mathcal{P}_n & \text{if } n \text{ is even} \\ S\mathcal{P}_n \backslash W_{n-1} & \text{if } n \text{ is odd} \end{cases}$$

where W_{n-1} as defined above, consists of all $\alpha \in [n-1, n-1]$ whose completions are odd permutations.

Although not all semigroups are nilpotent generated, but if a finite semigroup S contain zero element then certainly it contains some nilpotent elements. Therefore, one can always talk about the nilpotent generated subsemigroup of that semigroup. This problem was considered by Garba [13] for the semigroups $S\mathcal{I}_n$ and $S\mathcal{P}_n$. He showed that whether n is even or odd the nilpotent rank of the subsemigroups of $\mathcal{SI}_n$ and $S\mathcal{P}_n$ generated by nilpotents are equal to $n+1$ and $n+2$ respectively.

In [10], Garba studied the semigroup $\mathcal{I}O_n$ and show that for $r \leq n/2$, the rank and nilpotent rank of the two sided ideal

$$L(n,r) = \{\alpha \in \mathcal{I}O_n : |im\alpha| \leq r\}; \quad \text{is } \binom{n}{r} - 1.$$

While for $n/2 < r \leq n-2$; the rank and the nilpotent rank of the subsemigroup

$$M(n,r) = \{\alpha \in \mathcal{I}_n : |im\alpha| \leq r \,\text{and}\, \alpha \in \langle N \rangle\};$$

is

$$\binom{n}{r} - \binom{r-1}{n-r} - 1$$

The result which he extended [11] to the bigger semigroup $\mathcal{P}O_n$ and showed that the subsemigroup

$$\{\alpha \in \mathcal{P}O_n : |im\alpha| \leq n-2 \,\text{and}\; \alpha \in \langle N \rangle\}$$

has $6(n-2)$ and $7n-15$ as rank and nilpotent rank respectively.

2.4 THE SEMIGROUP OF PARTIAL ISOMETRIES

The semigroup $\mathcal{P}_n$ and some of its subsemigroups has been well studied and investigated over the years, the semigroup $\mathcal{DP}_n$ of all partial isometries, on the other hand receives less attention in comparison. Although, on more restrictive but richer mathematical structures, it has been studied by Wallen [33] and Bracci and Picasso [5], their corresponding study on chains was only initiated recently by Al-Kharousi *et al.* [1, 2]. In [1], some of the combinatorial properties of the semigroup $\mathcal{DP}_n$ and that of its subsemigroup $\mathcal{ODP}_n$ of all order-preserving partial isometries has been investigated, in particular, their cardinalities have been obtained. While in [2], they study some of the algebraic properties of the two semigroups namely Greens relations and ranks. For the ranks however, their results is being summarized in the following theorems:

Let

$$id_{X_n} = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 2 & \ldots & n-1 & n \end{pmatrix}$$

be the identity element of $\mathcal{DP}_n(\mathcal{ODP}_n)$. And let

$$h = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$$

be its reflection in $\mathcal{DP}_n$. Also, let

$$\eta = \begin{pmatrix} 2 & 3 & \dots & n \\ 1 & 2 & \dots & n-1 \end{pmatrix}$$

and $G = \{\eta, \eta^{-1}, id_{X_n\setminus\{i\}} \,:\, 2 \leq i \leq n-1\}$. Then,

Theorem 2.4.1 [2, theorem 3.1] *For $n \geq 2$*

(a) *rank($\mathcal{ODP}_n\backslash\{id_{X_n}\}$) $= n$. Moreover, G is the unique minimum generating set for $\mathcal{ODP}_n\backslash\{id_{X_n}\}$*

(b) *rank($\mathcal{ODP}_n$) $= n$ Moreover $G \cup \{id_{X_n}\}$ is the unique minimum generating set for $\mathcal{ODP}_n$*

Theorem 2.4.2 [2, theorem 3.2] *For $n \geq 2$*

(a) *as an inverse semigroup rank($\mathcal{ODP}_n\backslash\{id_{X_n}\}$) $= n-1$.*

(b) *as an inverse semigroup rank($\mathcal{ODP}_n$) $= n$.*

Before the next theorem, we give the following definition.

Let $x \in \mathbb{R}$, then the floor function of x (denoted by $\lfloor x \rfloor$) is the greatest integer that is less than or equals to x. While the Ceiling function of x (denoted by $\lceil x \rceil$) is the least integer that is greater than or equals to x. For example, if $x = 5.56$ then

$\lfloor x \rfloor = 5$ and $\lceil x \rceil = 6$, while for $x = 15$ both the ceiling and the floor of x equals to 15.

Theorem 2.4.3 [2, theorem 3.4] *For* $n \geq 2$

(a) *rank*($\mathcal{DP}_n \backslash \{id_{X_n}, h\}) = n$.

(b) as an inverse semigroup rank($\mathcal{DP}_n \backslash \{id_{X_n}, h\}) = \lfloor (n+3)/2 \rfloor$.

Where $\lfloor (n+3)/2 \rfloor$ is the floor function of $(n+3)/2$.

As we mentioned before, Green's relations have now become a standard tools for investigating the structure of semigroups. It is therefore become a customary that when one encounters a new class of semigroups, one of the questions that is often asked, concerns the characterization of Greens relations. Alkharousi [2] answered some of these questions in the following lemmas:

Lemma 2.4.4 [2, lemma 2.1] *Let* $\alpha \in \mathcal{DP}_n$ *or* $\mathcal{ODP}_n$ *:*
Then

(1) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* Dom $\alpha \subseteq$ D*om* β;

(2) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* im $\alpha \subseteq$ imβ;

(3) $\alpha \leq_{\mathcal{H}} \beta$ *if and only if* Dom $\alpha \subseteq$ D*om* β; *and* im $\alpha \subseteq$ *im* β;

To characterize Green's $\mathcal{D}$ relation, Alkharousi [2] introduced a terminology which they used in partitioning the sets $X_n = \{1, 2, ..., n\}$, called *gap* and *reverse gap* as follows:.

Definition 2.4.5 [Alkharousi, 2] *Let*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1\alpha & a_2\alpha & \dots & a_r\alpha \end{pmatrix}$$

be an element of $\mathcal{DP}_n$ *with* $|im\alpha| = r$*. A* gap *of the image set of* α *is an* $(r-1)$*-ordered tuples defined and denoted by:*

$$g(im\alpha) = (|a_2\alpha - a_1\alpha|, |a_3\alpha - a_2\alpha|, \dots, |a_r\alpha - a_{r-1}\alpha|)$$

while the reverse gap *of the image set is defined and denoted by:*

$$g^R(im\alpha) = (|a_r\alpha - a_{r-1}\alpha|, \dots, |a_3\alpha - a_2\alpha|, |a_2\alpha - a_1\alpha|).$$

Example 2.4.6 *If*

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 6 & 9 \\ 2 & 3 & 5 & 7 & 10 \end{pmatrix} \quad and \quad \beta = \begin{pmatrix} 2 & 3 & 5 & 10 \\ 9 & 4 & 2 & 1 \end{pmatrix},$$

then $g(im\alpha) = (1,2,2,3)$, $g(im\beta) = (5,2,1)$*. While* $g^R(im\alpha) = (3,2,2,1)$ *and* $g^R(im\beta) = (1,2,5)$.

As the elements of $\mathcal{DP}_n$ preserves distance, the following lemma can easily be deduced from the above definition.

Lemma 2.4.7 [2, lemma 2.3] *Let* $\alpha, \beta \in \mathcal{DP}_n$*. Then* $g(\mathrm{im}\alpha) = g(\mathrm{im}\beta)$ *or* $g^R(\mathrm{im}\alpha) = g(\mathrm{im}\beta)$ *if and only if there is an isometry between* $\mathrm{im}\alpha$ *and* $\mathrm{im}\beta$*.*

Next we have the theorems concerning the characterization of $\mathcal{D}$-classes of $\mathcal{DP}_n$ and $\mathcal{ODP}_n$.

Theorem 2.4.8 [2, theorem 2.4] *Let* $\alpha, \beta \in \mathcal{DP}_n$. *Then* $\alpha, \beta \in \mathcal{D}$ *if and only if* $g(\text{im}\alpha) = g(\text{im}\beta)$ *or* $g^R(\text{im}\alpha) = g(\text{im}\beta)$.

Theorem 2.4.9 [2, theorem 2.5] *In* $\mathcal{ODP}_n$ *we have:* $\alpha, \beta \in \mathcal{D}$ *if and only if* $g(\text{im}\alpha) = g(\text{im}\beta)$.

CHAPTER THREE

NILPOTENTS IN $\mathcal{ODP}_n$

3.1 INTRODUCTION

In this chapter, we study nilpotent elements in semigroup of order preserving partial isometries of finite chain ($\mathcal{ODP}_n$). We give the description of the subsemigroup generated by those nilpotents and obtain the rank of that subsemigroup.

3.2 NILPOTENTS IN $\mathcal{ODP}_n$

In this section, we define nilpotents in $\mathcal{ODP}_n$. We obtain the number of nilpotents in $\mathcal{ODP}_n$ and compute the order of the subsemigroup generated by those nilpotents. We begin with, let

$$\mathcal{PO}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is order preserving}\},$$

$$\mathcal{IO}_n = \{\alpha \in \mathcal{PO}_n : \alpha \text{ is one-to-one}\}$$

and

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{IO}_n : \alpha \text{ is an isometry}\}$$

which is the semigroup of *Order-preserving partial transformation,* the semigroup of Order-preserving partial one-to-one transformation and the semigroup of Order-preserving partial isometries respectively. Then the following relations should be observed:

$$\mathcal{IO}_n = \mathcal{I}_n \cap \mathcal{PO}_n \tag{3.2.1}$$

$$\mathcal{ODP}_n = \mathcal{DP}_n \cap \mathcal{IO}_n \tag{3.2.2}$$

It is clear from equation 3.2.2 that the semigroup $\mathcal{ODP}_n$ is a (proper) subsemigroup of $\mathcal{IO}_n$. We cite the following lemmas from [2].

Lemma 3.2.1 [2, lemma 1.1]. $\mathcal{ODP}_n$ *is an inverse subsemigroup of* $\mathcal{I}_n$ *.*

For a transformation α, if $x\alpha = x$ for some $x \in dom\alpha$ then x is called a fix point of α. The set of all fix points of α is denoted by $F(\alpha)$, that is $F(\alpha) = \{x \in dom\alpha : x\alpha = x\}$, while $f(\alpha)$ is define as the cardinality of the set $F(\alpha)$, that is $f(\alpha) = |F(\alpha)|$.

Lemma 3.2.2 [2, lemma 1.6]. *Let* $\alpha \in \mathcal{ODP}_n$*, if* $x\alpha = x$ *for any* x *in* $dom\alpha$*, then* $x\alpha = x$ *for all* $x \in dom\alpha$*, and thus* α *is an idempotent.*

Proof: Let x be a fixed point of α and suppose $y \in Dom\alpha$. If $x < y$ then by the order-preserving and isometry properties we see that $y - x = y\alpha - x\alpha = y\alpha - x \Longrightarrow y = y\alpha$. The case $y < x$ is similar. □

Next, we have the definition of nilpotents.

Definition 3.2.3 *Let S be a semigroup with 0 (zero) element. A non zero element $a \in S$ is said to be a nilpotent if there exist $m \in \mathbb{N}$ such that $a^m = 0$.*

Since the semigroup $\mathcal{ODP}_n$ is finite and contains a zero element (which is the empty map), it must therefore contain some nilpotent elements. We record the following lemma from [10].

Lemma 3.2.4 [10, lamma 2.1] *Let α be in $\mathcal{IO}_n$ such that $h(\alpha) < n$, then α is nilpotent if and only if $x\alpha \neq x$ for every $x \in dom\alpha$.*

As the semigroup $\mathcal{ODP}_n$ is a subsemigroup of $\mathcal{IO}_n$, every nilpotent in $\mathcal{ODP}_n$ is also a nilpotent in $\mathcal{IO}_n$. Therefore, the above lemma holds for $\mathcal{ODP}_n$ as well.

From lemma 3.2.2 and lemma 3.2.4, one can easily deduce the following lemma which we find indispensable in proving many results in this study.

Lemma 3.2.5 *Let α be in $\mathcal{ODP}_n$ such that $h(\alpha) \geq 1$, then α is either a nilpotent or an idempotent.*

Proof: Let $\alpha \in \mathcal{ODP}_n$ such that $h(\alpha) \geq 1$, if $f(\alpha) \geq 1$, then by lemma 3.2.2 α is an idempotent, otherwise is a nilpotent by lemma 3.2.4. $\square$

It is clear that if $h(\alpha) = 0$, then α is both nilpotent and idempotent.

Let N be the set of all nilpotent elements in $\mathcal{ODP}_n$, Alkharousi in [1, Proposition 2.8] obtained the cardinality of the set N from combinatorial point of view as the set of all elements of $\mathcal{ODP}_n$ with zero fixed points. Here we are going to restate the proposition using the word *nilpotents* instead of the fixed-points, which(to our own opinion) will fit better into the context of our work, and try to give an alternative proof using similar approach. But before then, we state the following Lemma which we will use in proving the Proposition.

Lemma 3.2.6 [8, corollary 2.7.3] *The element $\alpha \in \mathcal{I}_n$ is an idempotent if and only if α is the identity transformation of some $A \subseteq X_n$. In particular, $\mathcal{I}_n$ contains exactly 2^n idempotents.*

Proposition 3.2.7 *Let N be the set of all nilpotents in $\mathcal{ODP}_n$. Then*

$$|N| = 2^{n+1} - (2n + 1).$$

Proof: Let N be the set of all nilpotents in $\mathcal{ODP}_n$, and $E(\mathcal{ODP}_n)$ be the set of all idempotents in $\mathcal{ODP}_n$. From lemma 3.2.5 we have

$$|\mathcal{ODP}_n| = |E(\mathcal{ODP}_n)| + |N| + 1. \tag{3.2.3}$$

By lemma 3.2.6, $E(\mathcal{I}_n) \subset E(\mathcal{ODP}_n)$. But $\mathcal{ODP}_n \subset \mathcal{I}_n$, therefore, $E(\mathcal{I}_n) = E(\mathcal{ODP}_n)$ and $|E(\mathcal{ODP}_n)| = 2^n$. Now from theorem 1.8.7, we have that $|\mathcal{ODP}_n| = 3(2^n) - 2(n + 1)$. Therefore,

$$|N| = 3(2^n) - 2(n + 1) - 2^n = 2^{n+1} - 2n - 2 + 1.$$

Hence,

$$|N| = 2^{n+1} - (2n + 1).$$

One of the interesting question one may ask pertaining the semigroup that has nilpotents is that; can some of those non-nilpotent elements be expressed as the product of nilpotents? The collection of all those elements that can be expressed as product of nilpotents, together with the nilpotent elements themselves is what we call *nilpotent generated subsemigroup.* If every element of a semigroup S can be expressed as a product of some nilpotents, then S is called a semigroup generated by nilpotent elements. Otherwise, the nilpotent elements can only generate a proper subsemigroup.

Following [10] and considering the semigroup $\mathcal{ODP}_n$, let N be the set of all nilpotents in $\mathcal{ODP}_n$ and $\langle N \rangle$ be the subsemigroup generated by those nilpotents. Here, we give the description of elements of such subsemigroup. Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \tag{3.2.4}$$

be an element of $\mathcal{ODP}_n$ with height r, and suppose that α can be written as product of k-nilpotents for some $k \in \mathbb{N}$ i.e

$$\alpha = \theta_1\theta_2 \dots \theta_k \tag{3.2.5}$$

Then by [5] Ex. 2.1.4 each of this $\theta_i (i = 1, 2 \dots, k)$ must be of height atleast r. Now if any θ_i is of height greater than r, we can replace it by one of height exactly r by simply removing the redundant elements.

Example 3.2.8 *if*

$$\theta_1 = \begin{pmatrix} 1 & 2 & 4 & 6 & 8 \\ 2 & 3 & 5 & 7 & 9 \end{pmatrix} \quad \textit{and} \quad \theta_2 = \begin{pmatrix} 2 & 3 & 4 & 7 & 9 \\ 1 & 2 & 3 & 6 & 8 \end{pmatrix}$$

then

$$\theta_1\theta_2 = \begin{pmatrix} 1 & 2 & 6 & 8 \\ 1 & 2 & 6 & 8 \end{pmatrix}.$$

With respect to the product $\theta_1\theta_2$, the mapping $4 \mapsto 5$ in the transformation θ_1, is *redundant, because there is no 5 in the domain of* θ_2*. Also, in* θ_2*, the mapping* $4 \mapsto 3$ *is redundant, because there is no 4 in the image set of* θ_1*. Therefore, by removing the redundant elements,* θ_1 *and* θ_2 *can be rewritten as*

$$\theta_1' = \begin{pmatrix} 1 & 2 & 6 & 8 \\ 2 & 3 & 7 & 9 \end{pmatrix} \quad \textit{and} \quad \theta_2' = \begin{pmatrix} 2 & 3 & 7 & 9 \\ 1 & 2 & 6 & 8 \end{pmatrix}$$

respectively, and the product $\theta_1\theta_2$ *will be the same as the product* $\theta_1'\theta_2'$*.*

Considering α in equation 3.2.5, and assume that each θ_i has height exactly r. We have the following theorem.

Theorem 3.2.9 *For* $n \geq 2$*, let* α *be defined by* $a_i\alpha = b_i \, (1 \leq i \leq r)$ *be an element of* $\mathcal{ODP}_n$ *with* $h(\alpha) < n$*. Then* α *is a product of nilpotents if and only if it fails to satisfies the condition* $a_1 = 1$ *and* $a_r = n$*.*

Proof: Let α be in $\mathcal{ODP}_n$ with $h(\alpha) < n$. Suppose that α fail to satisfy the condition: $a_1 = 1$ and $a_r = n$. By Lemma 3.2.5, α in $\mathcal{ODP}_n$ implies that α is either a nilpotent or an idempotent. If α is a nilpotent then we are done. But if α is an idempotent we shall consider two cases:

Case 1: $a_1 = 1 \, (a_r \neq n)$. Then

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1 & a_2 & \dots & a_r \end{pmatrix}$$

can be written as a product of two nilpotents

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \text{and} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ a_1 & a_2 & \dots & a_r \end{pmatrix}$$

where $c_i = a_i + 1\,(1 \le i \le n)$.

case 2: If $a_1 \neq 1$, then we can define the $c_i's$ as $c_i = a_i - 1\,(1 \le i \le r)$, and express α as product of two elements as in case 1.

Conversely, suppose that α can be written as a product of k nilpotent as in equation 3.2.5, for some $k \in \mathbb{N}$ and suppose by contradiction that $a_1 = 1$ and $a_r = n$, by the assumption that each θ_i in 3.2.5 will be of height exactly r, hence we will have that $dom(\alpha) = dom(\theta_1)$. If we let

$$\theta_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ d_1 & d_2 & \dots & d_r \end{pmatrix}$$

then $d_i < d_{i+1}\,\forall i$ (since θ_1 is order preserving) and $a_i \neq d_i\,\forall i$ (since θ_1 is a nilpotent). Now

$$1 = a_1 \neq d_1 \Longrightarrow d_1 > 1 \tag{3.2.6}$$

And $|a_1 - a_r| = |d_1 - d_r|$ implies that $a_r - a_1 = d_r - d_1$ (by order preserving property of θ_1). Therefore, $n - 1 = d_r - d_1$ (since we assumed that $a_1 = 1$ and $a_r = n$), which implies that $n - 1 < d_r - 1$ (since $d_1 > 1$), and that implies that $n < d_r$ which is a contradiction; since $d_r \in X_n$ and α is a product of nilpotents.

Corollary 3.2.10 *Let α be defined as in equation 3.2.4, then α is a product of nilpotent if and only if it satisfy one of the following conditions:*

(I) $a_1 \neq 1$ *and* $b_1 \neq 1$;

(II) $a_1 \neq 1$, $b_1 = 1$ *and* $b_r \neq n$;

(III) $a_1 = 1$, $b_1 \neq 1$ *and* $a_r \neq n$;

(IV) $a_1 = 1$, $b_1 = 1$ *and* $b_r = a_r \neq n$;

It should be observe that by the corollary above, elements of the nilpotent generated subsemigroup $\langle N \rangle$ can be categorized into four different types. Henceforth, we shall say an element $\alpha \in \langle N \rangle$ is of type m (where m represents any of the conditions I, II, III, IV in the corollary), if α satisfies condition m.

Example 3.2.11 *consider the semigroup $\mathcal{ODP}_9$, then*

$$\alpha_1 = \begin{pmatrix} 2 & 6 & 8 \\ 3 & 7 & 9 \end{pmatrix} \quad and \quad \alpha_2 = \begin{pmatrix} 3 & 4 & 7 & 9 \\ 3 & 4 & 7 & 9 \end{pmatrix}$$

are both elements of type I, while

$$\begin{pmatrix} 2 & 3 & 7 & 9 \\ 1 & 2 & 6 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 6 & 8 \\ 2 & 3 & 7 & 9 \end{pmatrix} \quad and \quad \begin{pmatrix} 1 & 3 & 4 & 8 \\ 1 & 3 & 4 & 8 \end{pmatrix}$$

are elements of type II, III and IV respectively.

In the above example, we can see that, α_1 is a nilpotent element while α_2 is an idempotent. Therefore, by theorem 3.2.8 α_2 can be expressed as $\alpha_2 = \theta_1\theta_2$ product of two nilpotents, where θ_1 and θ_2 are respectively defined as:

$$\begin{pmatrix} 1 & 3 & 4 & 8 \\ 2 & 4 & 5 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 4 & 5 & 9 \\ 1 & 3 & 4 & 8 \end{pmatrix}.$$

Corollary 3.2.12 *Let N be the set of all nilpotent elements of $\mathcal{ODP}_n$. Then the nilpotent generated subsemigroup $\langle N \rangle$ is a proper subsemigroup of $\mathcal{ODP}_n$.*

Proof: Let $\alpha \in \mathcal{ODP}_n$. By theorem 3.2.7 above, $\alpha \in \langle N \rangle$ if $\{1, n\} \notin dom(\alpha)$. Therefore, all elements of $\mathcal{ODP}_n$ that are of the form:

$$\begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix}$$

are not in $\langle N \rangle$. Hence $\langle N \rangle$ is a proper subsemigroup of $\mathcal{ODP}_n$.

The next lemma gives the number of elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$.

Lemma 3.2.13 *For $1 < r \leq n$, the number of elements of $\mathcal{ODP}_n$ that are of the form $\begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix}$ is $\binom{n-2}{r-2}$.*

Theorem 3.2.14 *Let N be the set of all nilpotents in $\mathcal{ODP}_n$,and $\langle N \rangle$ be the subsemigroup generated by those nilpotents. Then $|\langle N \rangle| = 11 \cdot 2^{n-2} - 2(n+1)$.*

Proof: Let $\langle N \rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$. By theorem 1.8.3, lemma 3.2.12 and corollary 3.2.13 we have that;

$$\begin{aligned} |\langle N \rangle| &= 3 \cdot 2^n - 2(n+1) - \textstyle\sum_{r=2}^{n} \binom{n-2}{r-2} \\ &= 3 \cdot 2^n - 2(n+1) - 2^{n-2} \\ &= 11 \cdot 2^{n-2} - 2(n+1). \end{aligned}$$

3.2.1 Rank of Nilpotent Generated Subsemigroup

If a semigroup S is generated by nilpotent elements, then the cardinality of the smallest subset consisting of only nilpotent elements that generates S is called the nilpotent rank of S (written as $nilrank(S)$).

Let N be the set of all nilpotents in $\mathcal{ODP}_n$ and $\langle N \rangle$ be the subsemigroup generated by those nilpotents. We have seen from corollary 3.2.10 that, $\langle N \rangle$ generates only a proper subsemigroup of $\mathcal{ODP}_n$. Our aim in this section is to investigate the rank of $\langle N \rangle$.

As in 1.8.13, we define the two sided ideal of $\mathcal{ODP}_n$ as

$$L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \leq r\} \quad \text{where } 1 \leq r \leq n. \tag{3.2.7}$$

Now, let

$$K_r = L(n,r) \backslash L(n,r-1) \tag{3.2.8}$$

be the Rees quotient semigroup on $L(n,r)$. Then K_r is of the form $J_r \cup \{0\}$, where J_r is the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r. As already explained in chapter one, the product of any two elements in P_r say α and β will be of the form:

$$\alpha * \beta = \begin{cases} \alpha\beta, & \text{if } |h(\alpha\beta)| = r; \\ 0, & \text{if } |h(\alpha\beta)| < r \end{cases}$$

Also, let

$$M(n,r) = \{\alpha \in \langle N \rangle \; : \; |im\alpha| \leq r\} \tag{3.2.9}$$

be inverse subsemigroup of $\mathcal{ODP}_n$ generated by nilpotent elements of heights less than or equals to r. And let

$$W_r = M(n,r) \backslash M(n,r-1) \tag{3.2.10}$$

be Rees quotient semigroup on $M(n,r)$. Observe that $M(n,r) \subseteq L(n,r)$, therefore $W_r \subseteq K_r$.

Before stating the main theorem of the section, we would first of all try to partition the set X_n using some set theoretic terminology, so as to characterize the elements of $\mathcal{ODP}_n$ according their respective domain sets and image sets.

Let $X_n = \{1, 2, ..., n\}$ be a finite chain and $A = \{a_1, a_2, \ldots, a_r\}$ be a subset of X_n with r number of elements $(1 < r \leq n)$. Adopting the term used in [10], we shall say that A has *k-jumps* $(k \in \mathbb{N})$ between the elements a_i and a_{i+1} $(1 \leq i \leq r-1)$, if $a_{i+1} - a_i = k + 1$. And the sum of all jumps in A is called the *total jumps* in A. Observe that if a_i and a_{i+1} are consecutive numbers, then k is zero in that case, and subsets like $\{1, 2, \ldots, r-1, n\}$ are subset that has the maximum number of jumps. Therefore, the number k is in the interval $0 \leq k \leq n - r$.

Example 3.2.15 *consider the set* $X_9 = \{1, 2, \ldots, 9\}$ *and let* $r = 5$*, then the subset* $\{1, 3, 4, 5, 8\}$ *has 1-jump between it's first and second element and 3-jumps between it's forth and fifth element. While the subsets* $\{1, 2, 3, 4, 5\}$ *and* $\{3, 4, 5, 6, 7\}$ *has total jumps of zero each.*

Let $X_n = \{1, 2, \ldots, n\}$ and $A = \{a_1, a_2, \ldots, a_r\}$, $B = \{b_1, b_2, \ldots, b_r\}$ be any two ordered subsets of X_n with same cardinality say $r(1 \leq r \leq n)$. Define a relation $\sim$ on X_n as

$$A \sim B \quad \text{if } \forall i, j \in \{1, 2, \ldots, r\},\ |a_i - a_j| = |b_i - b_j|.$$

,

Remark 3.2.16 *It is clear from the way we define* $\sim$ *that, if* $A \sim B$ *then:*

(a) *A and B must have the same jumps in the same respective positions. And*

(b) *one can always define an order preserving partial isometry mapping between A and B.*

Lemma 3.2.17 *The relation* $\sim$ *is an equivalence relation on* X_n*.*

Proof: Let A, B, C be ordered subsets of X_n with $|A| = |B| = |C| = r$. Clearly from the definition of $\sim$, $\sim$ is reflexive. Therefore, to show the equivalence, we only need to show that $\sim$ is symmetric and transitive.

Now, suppose $A \sim B$, then by definition, $\forall i, j \in \{1, 2, \ldots, r\}$, $|a_i - a_j| = |b_i - b_j|$, this implies $|b_i - b_j| = |a_i - a_j|$ which implies that $B \sim A$. Also, if $A \sim B$ and $B \sim C$, then $\forall i, j \in \{1, 2, \ldots, r\}$, $|a_i - a_j| = |b_i - b_j|$ and $|b_i - b_j| = |c_i - c_j|$, which implies that $|a_i - a_j| = |c_i - c_j|$. Thus $A \sim C$. Hence the proof

We shall denote the equivalent class of a subset A of X_n by $[A]$, that is

$$[A] = \{B \subseteq X_n \,:\, B \sim A\}. \tag{3.2.11}$$

And we denote the set of all equivalent classes on X_n by $X_n/\sim$.

From the definition of $\sim$, equation (3.2.11) can be interpreted as

$$[A] = \{B \subseteq X_n \,:\, \forall b_i, b_j \in B \text{ and } a_i, a_j \in A \text{ (where } 1 \leq i, j \leq r),\ \ |b_i - b_j| = |a_i - a_j|\}.$$

But A and B are ordered subset of X_n, therefore, $|b_i - b_j| = |a_i - a_j| \Longrightarrow b_i - b_j = a_i - a_j$. Thus

$$[A] = \{B \subseteq X_n \,:\, \forall b_i, b_j \in B \text{ and } a_i, a_j \in A\ \ b_i = a_i + (b_j - a_j)\}. \tag{3.2.12}$$

Let $l_j = b_j - a_j$, for all $1 \leq j \leq r$, then, $b_i - b_j = a_i - a_j \Longrightarrow b_i - a_i = b_j - a_j$. Hence,

$$l_i = l_j \quad \forall i, j \in \{1, 2, \ldots r\}. \tag{3.2.13}$$

Now, $b_j \in B \subseteq X_n$ implies that $b_j \leq n$, which implies that $l_j = b_j - a_j \leq n - a_j$. Thus

$$l_j \leq n - a_j. \tag{3.2.14}$$

Also, $1 \leq b_1 \Longrightarrow 1 - a_1 \leq b_1 - a_1 = l_1$. This implies

$$1 - a_1 \leq l_1. \tag{3.2.15}$$

From equations (3.2.13), (3.2.14) and (3.2.12) we have

$$1 - a_1 \leq l_j \leq n - a_r. \quad \forall j \in \{1, 2, \ldots r\}. \tag{3.2.16}$$

Lemma 3.2.18 *Let X_n be a chain and $\sim$ be an equivalence relation on X_n. Then for any $A \subseteq X_n$,*

$$[A] = \{B \subseteq X_n \,:\, \forall b_i \in B;\, b_i = a_i + l, \quad 1 - a_1 \leq l \leq n - a_r\}$$

where $1 \leq i \leq r \leq n$.

Proof: It follows from equation (3.2.12) and (3.2.16).

We shall again adopt some terminology used in [10] and call a given subset of X_n, a *1-subset* if such subset contains the element 1. Obviously, if an ordered subset contains 1, then this “1” will always be the the first element of the set.

Lemma 3.2.19 *Let $\sim$ be an equivalence relation on X_n, then in each equivalent class of $\sim$, there exist a unique 1-subset.*

Proof: Let $A \subseteq X_n$ and $[A]$ be its equivalent class. If $a_1 = 1$ then A is a 1-subset. Now Suppose A is a not a 1-subset, then a_1 must be equal to $(1 + c)$ for some $1 \leq c \leq n - 1$.

Claim: The set $B = \{b_i \,:\, b_i = a_i - c\}$ is a 1-subset in $[A]$ for $1 \leq i \leq r$.

Proof of the claim: Observe that B is clearly a 1-subset by the way it is being defined, we only need to show that $B \sim A$. Now, $\forall 1 \leq i, j \leq r$; $|b_i - b_j| = |(a_i - c) - (a_j - c)| = |a_i - a_j|$, therefore $B \sim A$. Hence $B \in [A]$.

Next, we show that, this $1-$subset is unique in $[A]$. Suppose for contradiction, that there exist another set say $D = \{1, d_2, d_3, \ldots d_r\}$ which is also a 1-subset in $[A]$. Then, $D \sim B$, implies that $\forall j \in \{1, 2, \ldots, r\}$, $|d_j - 1| = |b_j - 1| \Longrightarrow d_j - 1 = a_j - 1 \Longrightarrow d_j = a_j \, \forall j$, which implies that $D = B$, hence a contradiction. Therefore B is unique in [A]. □

Remark 3.2.20 *For a given 1-subset, say $A = \{1, a_2, \ldots a_r\}$, if A has total jump of zero (that is all the elements of A are consecutive in X_n), then $a_r = r$, and if A has total jump of 1 (that is there exist two numbers in A with a missing digit between them), then $a_r = r + 1$. In general, if a 1-subset A has a total jump of m, then it's last element a_r would be equals to $r + m$, therefore A can be written as $A = \{1, a_2, \ldots r + m\}$.*

The next lemma tells us the number of subsets we will have in each equivalent class.

Lemma 3.2.21 *Let X_n be finite chain and $\sim$ be the equivalence relation on X_n. Then the number of subsets in each equivalent class is $n - (r + m) + 1$, where r is the cardinality of the subsets and m is the total jump in each class.*

Proof: Let $A \subseteq X_n$ and $[A]$ be its equivalent class. Suppose without lost of generality we consider A to be the 1-subset in that class, then by lemma 3.2.18 and remark 3.2.20,

$$[A] = \{B \subseteq X_n \, : \, \forall b_i \in B; \, b_i = a_i + l, \quad 0 \leq l \leq n - (r + m)\}.$$

That is to say all the other elements of $[A]$ will be obtained by adding a constant l, where l runs from 0 to $n - (r + m)$ (when $l = 0$, we have A itself). Therefore, we will have $n - (r + m) + 1$ number of them. $\square$

To see what we have been explaining clearly, consider the following example:

Example 3.2.22 *Let $X_9 = \{1, 2, \ldots, 9\}$, below is the set of subsets of X_9 with cardinality $r = 5$:*

$$\begin{array}{lllllllll} \{1,2,3,4,5\} & \{1,3,4,5,6\} & \cdots & \{1,2,3,4,6\} & \cdots & \{1,2,3,4,9\} & \cdots & \{1,6,7,8,9\} \\ \{2,3,4,5,6\} & \{2,4,5,6,7\} & \cdots & \{2,3,4,5,7\} & \vdots & & & \\ \{3,4,5,6,7\} & \{3,5,6,7,8\} & \cdots & \{3,4,5,6,8\} & & & & \\ \{4,5,6,7,8\} & \{4,6,7,8,9\} & \cdots & \{4,5,6,7,9\} & & & & \\ \{5,6,7,8,9\} & & & & & & & \end{array} \tag{3.2.17}$$

In the example above, each column of subsets represents a distinct equivalent class. The subsets in the first column has total jump of zero, while subsets in the second, (third, fourth and fifth) columns have the total jumps of 1 each (even though they are of different equivalent classes). Thus, two different equivalent classes of $\sim$ can have the same total jumps. The following lemma will give us the exact number of different classes with the same total jumps.

Lemma 3.2.23 *Let $X_n = \{1, 2, \ldots, n\}$. The number of different equivalent classes with total jump of m is $\binom{r+m-2}{r-2}$*

Proof: Let the 1$-$subset $\{1, \ldots, r + m\}$ represents each equivalent class with total jump of m. Now, between 1 and $r + m$ we have $r + m - 2$ elements in X_n and $r - 2$ elements in the 1-subset. So, to select a particular 1$-$subset we only need to select

$r-2$ elements from this $r+m-2$. Therefore, the total number of those 1−subsets is $\binom{r+m-2}{r-2}$. Hence the proof. $\square$

Remark 3.2.24

(a) Note that for $m = 0$, we have just 1 (class), while for $m = n - r$ (which is the maximum number m can reach) we have $\binom{n-2}{r-2}$ number of classes. And that is the number of subsets of the form $\{1, \ldots, n\}$ that characterized elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$.

(b) Observe also that, by lemma 3.2.21, those subsets $\{1, \ldots, n\}$ are the only elements in their own respective equivalent classes (i.e the order of their equivalent classes is just one).

Now suppose in each equivalent class we fixed the 1-subset to be a domain and and define an order-preserving partial isometry mapping with the remaining sets in that class, with the exception of the 1-subset itself (that is excluding the mapping of the 1-subset into itself). Then by lemma 3.2.21 we will have $n - (r + m)$ number of order-preserving partial isometries in each class. It should be observed that, subsets of the form $\{1, \ldots, n\}$ define no mapping in their own class (since they are the only elements in their own respective classes). While in the remaining classes, mappings defined there are all nilpotents.

Call G the set of all those mappings (having 1-subset as domain) then we have the following lemma which we need in proving the proposition that follows it.

Lemma 3.2.25 *Let $\alpha \in G$. Then for any $\beta, \gamma \in W_r$, $\alpha = \beta\gamma$ implies that either β is in G and γ is not, or γ is in G and β is not.*

Proof: Let Let $\alpha \in G$ and $\beta, \gamma \in W_r$ such that $\alpha = \beta\gamma$. For contradiction sake, suppose that neither $\beta \in G$ nor $\gamma \in G$. Then by our assumption that $\alpha = \beta\gamma$, and $\beta, \gamma \in W_r$ (having the same height), we must have that

$$\begin{aligned} dom\alpha &= dom\beta \\ im\beta &= dom\gamma \\ im\gamma &= im\alpha. \end{aligned} \tag{3.2.18}$$

Now $dom\alpha = dom\beta$ (which is a 1-subset) and β not in G, implies that $dom\beta = im\beta$ (i.e β must be partial identity). Also, $dom\gamma$ is a 1-subset (since from 3.2 18 above, $im\beta = dom\gamma$), and $\gamma \notin G$ implies that $dom\gamma = im\gamma$. And so, from equations in 3.2.18 above, we have: $dom\alpha = dom\beta = im\beta = dom\gamma = im\gamma = im\alpha$, which implies that $dom\alpha = im\alpha$, contradicting our assumption that α is in G. Therefore, if $\alpha = \beta\gamma$, then either β or γ must be in G. Next, we show that both β and γ cannot be in G at the same time.

It is clear that, if both β and γ are in G, then their domain sets must be 1-subset and their image sets are not. And so, $im\beta \neq dom\gamma$, which implies that $\beta\gamma \neq \alpha$. □

The next Proposition will give us the minimal generating set for the subsemigroup W_r, an inverse subsemigroup of $M(n, r)$.

Proposition 3.2.26 *For $n \geq 4$ and $1 < r \leq n-1$, The set G is the minimal generating set for W_r as an inverse semigroup.*

Proof: We first show that G is a generating set for W_r, we then show that it is indeed the minimal generating set. Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element of W_r. Then by corollary 3.2.8, α is either of type I ($a_1 \neq 1$ and $b_1 \neq 1$), type II ($a_1 \neq 1$, $b_1 = 1$ and $b_r \neq n$), type III ($a_1 = 1$, $b_1 \neq 1$ and $a_r \neq n$) or type IV ($a_1 = b_1 = 1$ and $a_r = b_r \neq n$).

Now, if α is of type I, then by lemma 3.2.19, there exist a 1-subset say $\{1, c_2, \ldots, c_r\}$ which will be of the same class with $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ and that

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ 1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} 1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} = \alpha$$

If α is of type II then α^{-1} is in G and if α is of type III then α is in G. Now if α is of type IV, that is

$$\alpha = \begin{pmatrix} 1 & a_2 & \ldots & a_r \\ 1 & a_2 & \ldots & a_r \end{pmatrix},$$

where $a_r \neq n$. Then, there exist a subset say $\{c_1, c_2, \ldots, c_r\}$ where $c_i = a_i + 1\,(i = 1, 2, \ldots, r)$, which is of the same class with $\{1, a_2, \ldots, a_r\}$ such that

$$\begin{pmatrix} 1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ 1 & a_2 & \ldots & a_r \end{pmatrix} = \alpha.$$

To show the minimality of G, we show that if G' is any generating set for W_r, then $|G'| \geq |G|$.

Now let $G' \subseteq W_r$ such that $\langle G' \rangle = W_r$ and suppose that there exist some $\alpha' \in G$ say

$$\alpha' = \begin{pmatrix} 1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

such that α' is not in G'. Then, since $\alpha' \in W_r$ and $\langle G' \rangle = W_r$ there exist some $\beta, \gamma \in G'$ such that $\beta\gamma = \alpha'$.

Claim: $\{\beta, \gamma\}$ generates no other non-zero element in W_r apart from α' and $(\alpha')^{-1}$.

Proof of the claim: Let $\beta, \gamma \in G'$ such that $\beta\gamma = \alpha'$, by lemma 3.2.23, either β in G or γ in G, but not both.

Case I: $\beta \in G$; then β and γ must be of the form:

$$\beta = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

Clearly $b_1 \neq 1$ (since α is in G), therefore, $\gamma\beta = \beta^{-1}\gamma = \beta^{-1}\gamma^{-1} = 0$ in W_r, and $\beta\gamma^{-1} = \alpha'$ if $\{c_1, c_2, \dots, c_r\} = \{b_1, b_2, \dots, b_r\}$, otherwise is 0, while $\gamma^{-1}\beta^{-1} = (\alpha')^{-1}$.

Case II: $\gamma \in G$; Then β and γ must be of the form:

$$\beta = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ 1 & a_2 & \dots & a_r \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

It follows that

$$\beta\gamma = \beta^{-1}\gamma = \alpha',$$

$$\beta\gamma^{-1} = \beta^{-1}\gamma^{-1} = 0 = \gamma\beta,$$

$$\gamma^{-1}\beta = \gamma^{-1}\beta^{-1} = (\alpha')^{-1}.$$

In both cases, $\{\beta, \gamma\}$ generate only α' and $(\alpha')^{-1}$. Therefore, $|G'| \geq |G|$. Hence, G is the minimal generating set for W_r. $\square$

To find the generating set for $M(n, r)$, we need the following proposition:

Proposition 3.2.27 *For $n \geq 4$, let $J_r = \{\alpha \in \mathcal{ODP}_n : |im\alpha| = r\}$ be the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r and N be the set of all nilpotents in $\mathcal{ODP}_n$. Then $\langle J_r \cap N \rangle \subseteq \langle J_{r+1} \cap N \rangle$ for all $1 < r \leq n-3$.*

Proof: Let $\alpha = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$ be an element of $\langle J_r \cap N \rangle$, essentially we consider two cases (that is when α is nilpotent and when α is idempotent).

Case (I): α is an idempotent, then α is either of type $I(a_1 = b_1 \neq 1)$ or type $IV(a_1 = b_1 = 1$ and $b_r = a_r \neq n)$. If α is of type I, let

$$t = \max\{x : x \in X \backslash dom\alpha\}$$

and

$$s = \max\{x : x \in X \backslash dom\alpha \,\text{and}\, x \neq t\}$$

then $t \neq 1$ and $s \neq 1$ (since $|X \backslash dom\alpha| \geq 3$).

Now suppose w.l.o.g that t are between a_i and a_{i+1} and s is between a_j and a_{j+1}. We define β and γ as

$$\beta = \begin{pmatrix} a_1 & a_2 & \ldots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \ldots & a_r \\ b_1 & b_2 & \ldots & b_j & b_{j+1} & \cdots & b_i & t & b_{i+1} & \ldots & b_r \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b_1 & b_2 & \ldots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \ldots & b_r \\ b_1 & b_2 & \ldots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \ldots & b_r \end{pmatrix}$$

It is clear that both β and γ are elements of $J_{r+1} \cap \langle N \rangle$ and $\alpha = \beta\gamma$.

If α is of type IV, we let $t = \min\{x : x \in X \backslash dom\alpha\}$ and $s = \min\{x : x \in X \backslash dom\alpha \;\text{ and }\; x \neq t\}$ (clearly $t \neq n$ and $s \neq n$) then β and γ are elements of $\langle J_{r+1} \cap N \rangle$ and $\alpha = \beta\gamma$.

Case (II) α is a nilpotent: Suppose $a_i > b_i \, \forall i$, then α is either of type $I(a_1 \neq 1$ and $b_1 \neq 1)$, or type $II(a_1 \neq 1$, $b_1 = 1$ and $b_r \neq n)$. If α is of type I, we must have that

$a_1 > b_1 > 1$. Define

$$\beta = \begin{pmatrix} a_1 - 1 & a_1 & a_2 & \dots & a_r \\ b_1 - 1 & b_1 & b_2 & \dots & b_r \end{pmatrix},$$

then $b_r < a_r \leq n$ and $1 \leq b_1 < a_1$. Therefore, β is in $\langle J_{r+1} \cap N \rangle$. Also, define γ as

$$\gamma = \begin{pmatrix} b_1 & b_2 & \dots & b_i & t & b_{i+1} & \dots & a_r \\ b_1 & b_2 & \dots & b_i & t & b_{i+1} & \dots & b_r \end{pmatrix}$$

where $t = \min\{x : x \in X \backslash im\beta\}$. It is clear that $t \neq n$ (since $|X \backslash im\beta| \geq 2$), therefore, γ is also in $\langle J_{r+1} \cap N \rangle$ and $\alpha = \beta\gamma$.

But if α is of type II (i.e $a_1 \neq 1$, $b_1 = 1$ and $b_r \neq n$), we first consider a case where all the skips in the domain set occur before a_1, and define

$$\beta = \begin{pmatrix} a_1 - 1 & a_1 & a_2 & \dots & a_r \\ 1 & 2 & 3 & \dots & r+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} 2 & 3 & \dots & r+1 & r+2 \\ 1 & 2 & \dots & r & r+1 \end{pmatrix}.$$

Otherwise we let $t = \max\{x : x \in X \backslash dom\alpha\}$ and define

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_i & t & a_{i+1} & \dots & a_r \\ 1 & b_2 & \dots & b_i & t\beta & b_{i+1} & \dots & b_r \end{pmatrix}$$

where $t\beta = b_i - a_i + t$. And let $s = \min\{x : x \in X \backslash im\beta\}$ and define

$$\gamma = \begin{pmatrix} 1 & b_2 & \cdots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \\ 1 & b_2 & \cdots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

Note that $t\beta = b_i - a_i + t < t \leq n$. Therefore, $\beta, \gamma \in \langle J_{r+1} \cap N \rangle$ and $\alpha = \beta\gamma$.

Now suppose $a_i < b_i \, \forall i$, then α is either of type I or type III. If α is of type I, then α^{-1} is also of type I with $a_i > b_i$, and if α is of type III, then α^{-1} is of type II with $a_i > b_i$. In both the cases $\alpha = \gamma^{-1}\beta^{-1}$ where β, γ are as defined in the respective previous cases.

Remark 3.2.28 *Observe that by the Proposition above, for $1 < r \leq n-2$ the set G is the minimal generating set for the whole $M(n,r)$ as well. So the task of finding the rank of $M(n,r)$ is reduced to just finding the cardinality of the set G. We have the following theorem:*

Theorem 3.2.29 *For $n \geq 4$ and $1 < r \leq n-2$, the rank of $M(n,r)$ as an inverse semigroup is*

$$\sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2}$$

where m stand for the total jumps in each class.

Proof: Let $M(n,r)$ be the inverse subsemigroup of $\mathcal{ODP}_n$ generated by nilpotents whose height is less than or equals to r. By remark 3.2.28, to compute the rank of $M(n,r)$ we only need to count the number of elements in G. Now, from lemma 3.2.23, the number of different equivalent classes with total jump of m is $\binom{r+m-2}{r-2}$. And in each class, the number of elements of G is $n-(r+m)$. Also, subsets that characterized elements in G have total jumps m ranging from 0 to $n-(r+1)$ (since subsets that have $m = n-r$ are subsets of the form $\{1, \ldots, n\}$ which is not in G). Therefore, for each $m \in (0, 1, \ldots, n-(r+1))$, we have $(n-(r+m))\binom{r+m-2}{r-2}$ elements of G. Hence,

$$|G| = \sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2}.$$

Corollary 3.2.30 *The rank of $M(n, n-2)$ is $n-1$.*

Proof: Follows directly by substituting for $r = n - 2$ in Theorem 3.2.25, and simplifying the summation.

Theorem 3.2.31 *Let $\langle N \rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$. Then, the rank of $\langle N \rangle$ as an inverse subsemigroup is n.*

Proof: Let $\langle N \rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$, from the above corollary, the rank of $M(n, n-2)$ is $n-1$, and elements of $\langle N \rangle$ that are not in $M(n, n-2)$ are those of $\langle N \rangle \cap J_{n-1}$ (since the only element of $\mathcal{ODP}_n$ with height n is the identity element which is clearly not in $\langle N \rangle$).

Let

$$\eta = \begin{pmatrix} 1 & 2 & \ldots & n-1 \\ 2 & 3 & \ldots & n \end{pmatrix}$$

Garba in [10] have shown that, the only elements of $\langle N \rangle \cap J_{n-1}$ are

$$\eta, \eta^{-1}, \eta^{-1}\eta, \text{ and } \eta\eta^{-1}.$$

which are generated by a single element η, therefore, the rank of $\langle N \rangle$ is $(n-1)+1$ which is equal to n. Hence the proof. □

CHAPTER FOUR

THE RANK OF IDEAL OF $\mathcal{ODP}_n$

4.1 INTRODUCTION

In this chapter we investigate the rank of the two sided ideal $L(n,r)$ of $\mathcal{ODP}_n$, and hence compute the rank of the semigroup $\mathcal{ODP}_n$.

4.2 RANK OF AN IDEAL OF $\mathcal{ODP}_n$

We try to extend the result obtained in chapter three to compute the rank of $L(n,r)$. Although, Alkharousi in [1] investigated the rank of $\mathcal{ODP}_n$ among other things and shown that; $\mathcal{ODP}_n$ as an inverse semigroup has rank n. We want to generalize this result to find the rank of the two sided ideal

$$L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \le r\} \quad \text{where } \ r \le n. \tag{4.2.1}$$

To do that, we first study elements of $\mathcal{ODP}_n$ that are neither nilpotents nor can be expressed as product of nilpotents (i.e elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$). We have seen from corollary 3.2.12 that, the elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$ are partial identities of the form

$$\begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix}.$$

It is clear that, multiplying any two element of this form will yield another element of the form, as such we have the following lemma:

Lemma 4.2.1 *Let $E_N = \{\alpha \in \mathcal{ODP}_n : \alpha$ is not in $\langle N \rangle\}$, then E_N is a subsemigroup of $\mathcal{ODP}_n$.*

Recall from equation 3.2.7 and 3.2.8, we define $L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \leq r\}$ where $1 \leq r \leq n$, as the two sided ideal of $\mathcal{ODP}_n$, and $K_r = L(n,r) \backslash L(n,r-1)$ as the Rees quotient semigroup on $L(n,r)$.

Now, for $1 < r \leq n$, let

$$D(n,r) = \{\alpha \in E_N : |im\alpha| \leq r\}$$

and

$$Q_r = D(n,r) \backslash D(n,r-1)$$

be the subsemigroup of K_r generated by the elements of E_N, then we have the following:

Lemma 4.2.2 *Let K_r be the Rees quotient semigroup on $L(n,r)$, and let W_r and Q_r be subsemigroups of K_r generated by nilpotent elements and elements of E_N respectively. Then K_r is a disjoint union of W_r and Q_r.*

Proof: It follows from Theorem 3.2.8 and the fact that W_r and Q_r are complement to each other. $\square$

Suppose we extend the set G constructed in chapter three (i.e the set of all nilpotent elements in W_r whose domain is a 1-subset) to another set Z by adding elements of Q_r (that is $Z = G \cup Q_r$). Then, we give a Lemma which is analogue to Lemma 3.2.25.

Lemma 4.2.3 *Let* α *be in* Q_r *then for any* $\beta, \gamma \in P_r$, $\alpha = \beta\gamma$ *if and only if* $\alpha = \beta = \gamma$

Proof: Let α be in Q_r such that $\alpha = \beta\gamma$ for some $\beta, \gamma \in K_r$, since $h(\alpha) = h(\beta\gamma)$ we have that

$$\begin{aligned} dom\alpha &= dom\beta \\ im\beta &= dom\gamma \\ im\gamma &= im\alpha. \end{aligned} \tag{4.2.2}$$

From the above equations, and the fact that $\alpha \in Q_r$, we have $\{1, n\} \subseteq dom(\beta)$, which implies that $dom(\beta) = im(\beta)$. Also, $\{1, n\} \subseteq im(\gamma)$ implies that $im(\gamma) = dom(\gamma)$. Therefore, $dom\alpha = dom\beta = im\beta = dom\gamma = im\gamma = im\alpha$, which implies that $\alpha = \beta = \gamma$.

Conversely suppose that $\alpha = \beta = \gamma$, then, since α is an idempotent; $\beta\gamma = \alpha\alpha = \alpha^2 = \alpha$.

The following Proposition is an extension of Proposition 3.2.26 in chapter three.

Proposition 4.2.4 *For* $n \geq 4$ *and* $1 < r \leq n - 1$, *let* $Z = G \cup Q_r$ *(i.e* Z *is the set* Q_r *union the set of all nilpotents in* K_r *whose domain is a 1-subset), then* Z *is a minimal generating set for* K_r *as an inverse semigroup.*

Proof: Just as we did in proving proposition 3.2.26, we begin by showing that the set Z is a generating set for K_r, before proving the minimality condition. Let $\alpha \in K_r$, by Lemma 4.2.2, α is either in W_r or in Q_r. If α is in W_r, then by Proposition 3.2.26, α is generated by elements of $G \subseteq Z$, and if α is in Q_r, then α is in Z

For the minimality of Z; we show that if Z' is any other generating set for K_r, then $|Z'| \geq |Z|$. Now let $Z' \subseteq K_r$ such that $\langle Z' \rangle = K_r$, suppose that there exist say $\delta \in Z$ such that δ is not in Z', then δ must come from W_r. For if δ is in Q_r, then by lemma 4.2.3 we cannot find any other elements in $K_r \supseteq Z'$ that can generate δ, contradicting our assumption that Z' generates K_r. Now δ not in Z' and $\langle Z' \rangle = K_r$ implies that $\delta = \eta\theta$ for some $\eta, \theta \in Z'$.

Claim: $\{\eta, \theta\}$ generates no other non-zero element in K_r apart from δ and δ^{-1}.

Proof of the claim: Let $\eta, \theta \in Z'$ such that $\eta\theta = \delta$. Since δ, η and θ are all in K_r, we must have $dom(\delta) = dom(\eta)$ and that implies that η cannot be in Q_r (since $\{1, n\} \nsubseteq dom(\eta)$). Also, (for the same reason) $im(\delta) = im(\theta)$ which implies that θ is not in Q_r (since $\{1, n\} \nsubseteq im(\theta)$). Therefore, η and θ must be elements of W_r. Thus δ, η and θ are all elements of W_r, hence, the result follows from proposition 3.2.26. □

Next, we give an extension of proposition 3.2.23 - where we will see that, the set Z constructed above generates not only K_r but also $L(n, r)$; for $r \leq n - 1$.

Proposition 4.2.5 *For $n \geq 4$ and $r \leq n - 2$, let $J_r = \{\alpha \in \mathcal{ODP}_n : |im\alpha| = r\}$ be the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r. Then $\langle J_r \cap \mathcal{ODP}_n \rangle \subseteq \langle J_{r+1} \cap \mathcal{ODP}_n \rangle$.*

Proof: Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element of $\langle J_{r-1} \cap \mathcal{ODP}_n \rangle$, then $|X \backslash dom\alpha| \geq 2$. Now let $t = \max\{x : x \in X \backslash dom\alpha\}$ and $s = \min\{x : x \in X \backslash dom\alpha\}$, suppose without lost of generality that t is between a_i and a_{i+1} and s is between a_j and a_{j+1}. Define β and γ as

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \dots & a_r \\ a_1 & a_2 & \dots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \dots & a_r \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} a_1 & a_2 & \dots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \dots & a_r \\ b_1 & b_2 & \dots & b_j & s\gamma & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

where

$$s\gamma = \begin{cases} b_j - a_j + s, & \text{if } s > a_1; \\ b_{j+1} - a_{j+1} + s, & \text{if } s < a_1 \end{cases}$$

then β and γ are elements of $\langle J_{r+1} \cap \mathcal{ODP}_n \rangle$ and $\beta\gamma = \alpha$. Hence α is in $\langle J_{r+1} \cap \mathcal{ODP}_n \rangle$.

Observe here also that, the task of finding the rank of $L(n, r)$ is reduced just to finding the cardinality of the set Z constructed in Proposition 4.2.4, since by Proposition 4.2.5 above, the set Z is a minimal generating set for $L(n, r)$ as well. As such we have the following theorem, which is an extension of theorem 3.2.29.

Theorem 4.2.6 *For $n \geq 4$ and $1 < r \leq n - 1$, the rank of $L(n, r)$ as an inverse semigroup is*

$$\sum_{m=0}^{n-(r+1)} (n - (r + m)) \binom{r + m - 2}{r - 2} + \binom{n - 2}{r - 2}$$

where m stand for the total jump in each class.

Proof: Let Z be a disjoint union of G and Q_r, then by theorem 3.2.29 and lemma 3.2.13, we have

$$|Z| = \sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2} + \binom{n-2}{r-2}.$$

Hence the proof.

Corollary 4.2.7 [2, theorem 3.2(a)] *Let 1_{X_n} be an identity element of $\mathcal{ODP}_n$. Then, the rank of $\mathcal{ODP}_n \backslash \{1_{X_n}\}$ as an inverse semigroup is $n-1$.*

Proof: Since the only element of $\mathcal{ODP}_n$ with height n is the identity element $\{1_{X_n}\}$, therefore, $\mathcal{ODP}_n \backslash \{1_{X_n}\} = L(n, n-1)$, and computing the rank of $L(n, n-1)$ from theorem 4.2.6 we obtain $n-1$.

Corollary 4.2.8 [2, theorem 3.2(b)] *The rank of $\mathcal{ODP}_n$ as an inverse semigroup is n.*

Proof: Follows from Corollary 4.2.7 above.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 INTRODUCTION

In this chapter, we give a summary of our findings along with some inferences made on them. We also give some directions for future research.

5.2 SUMMARY

We study nilpotent elements in the semigroup $\mathcal{ODP}_n$ of all order preserving partial isometries on a finite chain. We gave the description of the subsemigroup $\langle N \rangle$ generated by the set of all nilpotents. We also compute the cardinality of $\langle N \rangle$.

We defined an equivalence relation on the subsets of X_n and computed the order of each equivalence class. This equivalence relation lead us to finding the minimal generating set for the inverse subsemigroup $M(n,r)$ of $\mathcal{ODP}_n$ generated by all nilpotent elements whose height is less than or equal to $r(1 < r \leq n-2)$. Consequently the rank of the subsemigroup $M(n,r)$ and that of the nilpotent generated subsemigroup $\langle N \rangle$ was obtained.

The notion introduced above was extended to investigate the rank of the two sided ideal $L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \leq r\}$ (where $1 \leq r \leq n$) of the semigroup $\mathcal{ODP}_n$. This was done by first observing that the set $E_N = \{\alpha \in \mathcal{ODP}_n : \alpha \text{ is not in } \langle N \rangle\}$ consisting of all elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$ is a subsemigroup of $\mathcal{ODP}_n$. As a result, the minimal generating set for $L(n,r)$ was constructed and consequently the rank of the ideal $L(n,r)$ was computed.

5.3 CONCLUSION

The semigroup $\mathcal{ODP}_n$ is one of the new semigroups of transformation that are not yet being fully explored, and upon encountering any semigroup, one of the algebraic structure one may be interested to know about is the nilpotents of that semigroup. And going by the literature, the work on the nilpotents in $\mathcal{ODP}_n$ is not yet being done. We are therefore hoping that the results obtained in this research will form a part of the development of the theory of semigroup of transformations.

Also, Alkharousi in [1, 2] study the two semigroups $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ and shown in [2] among other things that the rank of $\mathcal{ODP}_n \backslash 1_{X_n}$ (where 1_{X_n} is the identity element) as an inverse semigroup is $n-1$. In our own study, we investigated the rank of the two sided ideal

$$L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \leq r\} \tag{5.3.1}$$

and shown it to be equals to

$$\sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2} + \binom{n-2}{r-2}. \tag{5.3.2}$$

It is clear from 5.2.1 that if $r = n-1$, then $L(n, r) = \mathcal{ODP}_n \backslash 1_{X_n}$. Also computing the rank of $L(n, n-1)$ from 5.2.2 give us $n-1$. Therefore, we conclude that our result is a generalisation of the result in Alkharousi [2].

5.4 RECOMMENDATION

This research can be extended to study the nilpotents in the semigroup of partial isometries $\mathcal{DP}_n$ and the semigroup of contraction mappings.

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