Numerical Solution of Ordinary Differential Equcation Using Taylor's Series and Modified Euler's Method

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Federal Universitg Gusau

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Yahaya Kabiru

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DECLARATION

I hereby declare that this project is written by me and it has not been presented before in any application for a Bachelor Degree except for quotations and summaries which have been duly acknowledged.

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Yahaya Kabiru

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Date

CERTIFICATION

This project entiled "Numerical solution of ordinary differencial equation using Taylor's series and modifies Euler's method" meets regulation governing the award of Bachelor of sciences of the federal university Gusau and approved for its contribution to knowledge and literacy presentation.

Dr. A. U Moyi
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11/12/2019
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12.12.19
Date

DEDICATION

This project is dedicated to my beloved Father Alh. Kabiru Umar and my Mother Haj. Karima Muhammad who work tireless to see me through my studies.

ACKNOWLEDGEMENTS

I have to thank indefinitely to Allah (SWT) who bestowed on me his mercy and health which enable me accomplished this project. I wish to thank my project supervisor and constructive interpretations made it possible for this project to be done properly owing to his other official commitments that do not stop him from reading and correcting the scripts.

In a special way, I wish not to forget to convey my in-depth appreciation to our H.O.D of Mathematical Science Dr. Aliyu Usman Moyi. Furthermore, much of adoration goes to our project coordinator Dr. Jibril Lawal, I am sincerely grateful. And I wish to thank my lecturers in the Department of Mathematical Science which are Dr. Emmanuel Omokhuale, Dr. Jibril Lawal, Mal. Salisu Muhammad, Mr. Joshua Ibidoja, Mal. Kabiru Bello Gamagiwa, Mr. Thiophilus Danjuma and others who taught me and contributed immensely during the period of my studies and as well in this research work.

I wish not to forget to convey my in-depth appreciation to the family of Alh. Kabiru Umar Asha-Fura. I wish not to specially say thanks to my mother Malama Karima Muhammad and my siblings which are Shafi'u Kabiru, Saminu Kabiru, Sabitu Sanusi, Ibrahim Kabiru, Kasimu Kabiru and others and adoration goes to my lovely wife Hauwa'u Jibril, I will like to express my gratitude to my friends Yahuza Abbas, Ibrahim Muhammad, Abubakar Al-Majir, Buhari Salisu, Yasinu Umar.

I am sincerely grateful to my course mates which are Ighorie Dawhosa, Yusuf Shamsu Bakura, Ayuba Lawal, Hassan Bako and others for their encouragement and friendship during the course of my study.

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ABSTRACT

The aim of the study is to compare the Taylor's series method and modified Euler's method for the computation of the numerical solution of initial values problems of ordinary differential equation. Examples were pulled and both methods were used to arrive at the solutions. These solutions were compared with respect to the exact solutions. Comparisons were based on the quality of approximation and value of errors found for both Taylor's series method and modified Euler's method in comparison to the exact solution which formed the bases for conclusion from results gotten.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solution of mathematical problem and for extracting useful information from available solution which are not expressed in traceable forms such problem may each be formulated.

This formulation may correspond exactly to the situation which it is intended to describe. more often, it will not. Analytical solutions, when available, may be precise in themselves, but may be of unacceptable form because of the fact that they are not amendable to direct interpretation in numerical terms, in which numerical analysis attempted to derive method of interpreting them into numerical terms. Hildebrand (1974).

With the coming of the technological age problems have required solutions which have not yet been solved by great mathematician yet which technology has demanded solution. For example the weather on the earth surface is governed by complicated mathematical equations which have not been solved to date analytically. The answer of this seemingly impossible situation is to accept an approximation of the required solution rather the exact answer. The accuracy depends on the method of approximation proposed. This leads us to the definition of numerical analysis i.e the study of behavior of numerical method. Morris(1983).

1.2 Aims and Objectives of the Study

Arm of this project is to solve and compare Taylor's series method and modified Euler's method.

The objective of this project is to know which one of them has a lesser error and gives a better approximation.

1.2 Scope and Limitations of the Study

In this project we shall use some computational technique that will minimize errors we may encounter during numerical computations, we shall therefore restrict ourselves on those technique for solving ordinary differential equation, since this area is very wide.

Some methods are better than others, that is some methods are more efficient than others (in a less number of iteration, it will converge to the actual solution)

1.3 Definition of Some Basic Terms

1.1.1 **DEFINITION:** A differential equation is an equation which involves differential co-efficient and also differential equation are subdivided into two types namely

i. ordinary differential equation and

ii. partial differential equation

Ordinary differential equation involves only one independent variable (and only ordinary differential co-efficient). Hence, any function of x,y and the derivatives of y up to any order such that.

$$(x, y, \frac{dy}{dx}, \frac{dx}{dx^2} \ldots\ldots\ldots\ldots\ldots) = 0$$

This defines ordinary differential equation for y.

Example: $\frac{dy}{dx} = 3$ (b) $\frac{d^2y}{dx^2} + y = \sin x$ (c) $\left(\frac{dy}{dx}\right)^3 + y^2 x$ are all ordinary differential equation.

Dass (2006)

(ii) Partial differential equation: involves two or more independent variable

Example:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

1.4.2 Definition

A dependent variable of an ordinary differential equation is the variable being differential.

Example: if $\frac{dy}{dx} = 2x$ then y is the dependent variable that is a function of X.

1.4.3 Definition

An independent variable is a variable with respect to which the differentiation is performed.

Example: $\frac{dy}{dx} = 2x$, x is the independent variable

1.4.4 Definition

The order of an ordinary differentiation is the order of the highest differential co-efficient present in the equation consider.

$$\frac{dy^2}{dt^2} + R\,\frac{dq}{dt} + \frac{q}{c} = \text{esimwt} \qquad (1)$$

$$\text{Cos } x\frac{d^2y}{dx^2} + \sin x\left(\frac{d^2y}{dx}\right)^2 + 8y = \tan x \qquad (2)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx}\right)^2 \qquad (3)$$

The order of the above equation is 2.Dass (43)

The degree of an ordinary differential equation is the degree of the highest derivation after removing the radical sign and function.

The degree of eqn (1) and (2) is 1.And the degree of eqn (3) is 2.

1.4.5 Definition

A differential equation is said to be linear if it is of the first degree in the dependent variable and all its derivatives.

Example: $\frac{dy}{dx} + py = q$

Where P and Q are functions of x (but not of y) or constants.

Example:

1. $\frac{dy}{dx} = \sin x$
2. $\frac{dy}{dx} - 2x = x$

1.4.6 Definition

A function f(x,y) is said to be homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x.y)$ or $(M(x,y)dy + N(x,y)dx = 0)$ otherwise it is called non-homogeneous.

Example of homogeneous equation are:

$3\frac{dy}{dx} = 17y = 0$

$\frac{dy}{dx} 2y = 0$

Example of non-homogeneous equations are:-

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} + 3x - 17y = 2x$$

Error = true value – Approximate value

For instance, the true value of π = 3.141592653 while the approximate value is 22/7 = 3.142857143.

:. ERROR = 3.14159654-3.142857143

Error = -0.00126449

Relative error: the relative error of a numerical measurement or calculation is the numerical difference between the true values of the calculation and the approximate value divided by the true value.

Let Vt be the true value, let Xn be the approximate value.

Then the error = $= \frac{Vt-Xn}{Vt}$

:. To find the relative error of # we will have 3.141592654-3.1412857143/3.141592654

Relative error = 0.00126448967/3.141592654

= 0.0004025

= 0.0004

Percentage error: this is simply define as the relative error multiplied by 100 using the example on the π the percentage error

i.e percentage error = relative error x 100

$$= -0.0004025 \times 100$$

$$= 0.04\%$$

Absolute error: this is the absolute value of the error.

I.e. if error = -0.00126449,

Then absolute error = |-0.00126449| = 0.00126449 Ibrahim (2006).

Having seen the types of error we have in numerical analysis. We will proceed to look at the different sources of error in numerical analysis which we are likely to encounter.

1.4.7 Definition

If the initial condition are not known exactly (or must expressed in exactly as a terminated decimal number). The solution will be affected to a greater or lesser degree depending on the sensibility of the equation. Highly sensitive equations are said to be subject to inherit instability Jain (1984).

1.4.8 Definition

Since we can carry only a finite number of decimal places, our computations are subject to inaccuracy from this source no matter whether we round or whether we chop off.

Carrying more decimal places in the intermediate calculations than we required in the final answer is the answer is the normal practice to minimize this, but in length calculations this is a source of error that is most difficult to analyzed and control. Jain (1984).

1.4.9 Definition

It is convenient to define a truncation error by exclusion as any error which is either a grass error or a round off error. Thus a truncation error is one which would be present even in the hypothetical situation in which no "mistakes" were made, all given data are exact, and infinitely many digits were retained in all calculations. Frequently a truncation error corresponds to the fact that, where as an exact result would be affordable (in the limit) by an infinite sequence of steps, the process is truncated after a certain finite number of steps. Hildebrand (1974).

Having seen the definition and the example of ordinary differential equation we can say that ordinary differential equation can be classified into linear and non-linear. Their degree could also be of first degree or any other degree higher than one.

CHAPTER TWO

LITERATURE VIEW

2.1 Introduction

Numerical solution of ordinary differential equation is the most important technique ever developed in continuous time dynamics. Since most ordinary differential equation are not soluble analytically, numerical integration is the only way to obtain information about the trajectory. Many different methods have been proposed and used in an attempt to solve accurately, various types of ordinary differential equation. All these discredited the differential system to produce a difference equation or map Ochoche (2007).

The methods, obtain different maps from the same equation, but they have the same aim: that the dynamics of the maps, should correspond closely to the dynamics of the differential equation Julyan and Pirol (1992) and Ochoche (2007).

With the advent of computer, numerical method is now an increasingly attractive and efficient way to obtain the approximate solution to differential equation that had to prove difficultly even impossible to solve analytically Ochoche (2007).

Abhulimenand Otunta (2006) talked on the stiff ordinary differential equation: most conventional numerical integrations solvers cannot effectively cope with stiff problems.

$$Y^1 = f(x,y)y\,(x_0) = y_0 \;\; xe(a,b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

As they lack adequate stability characteristics, for this reason, there has been research attention focus on this class of stiff problem. Several authors including Jain (1972), Enright (1974), Jackson and kenue (19740, Fatokum (2000, 2001, 2002) Xiao et al (2001), Fatunla

(1976,1980), etc., have developed A-stable algorithms for solving stiffs initial value problem in ordinary differential equations.

Liniger and Willoughby (1970) introduced the concept of exponentially fitting and suggested three new A-stable schemes with k = 1. However, cash (1981) derived an exponentially fitted multi-derivative, multi-step method of order up to 5 with the step number k=1 and 2. A numerical investigation of those methods shows that all are A-stable for all close fitting parameters.

Okunga (1994,1999) developed second derivative multi-step method order 2,4,5 and 6 for stiff lupe in ODES, these methods were found to be A-stable efficient for stiff problems in which the method applicable.

Abhulimananand Otunba (2006) developed a sixth order multi derivative method for stiff system of differential equation which compete favorable with the other existing methods. They also observed that for exponentially fitted problems, the methods need not to use a small step length as it may be required by many multi step method before a good accuracy is obtained.

Adeyemiluyi and Babatola (2006) viewed the existing one step algorithms developed for stiff iv problems for ordinary differential equation which include:

i. Generalized rumge – kutta scheme by Lawson (1966)

ii. Implicit runge – kutta scheme by butcher (1965)

iii. Explicit one and two stage inverse runge – kutta method by Adeyemiluyi (2005)

They concluded that the most recent and efficient scheme for stiff ODES is the (iii) scheme and they also discussed its disadvantage and suggested that the implicit discretization scheme can take care of those disadvantages. They also went ahead to discuss its consistency, convergence and stability. They finally concluded that its accuracy is quite better than the accuracy of the corresponding 2-stage classical Runge – kutta method of the same order and its interval of A-stability is $(-\infty,0)$. It was finally recommended to be suitable for solution of real life problems arising in physics, chemistry and economics.

Odetunde and Egbetade (2006) describe and applied a power series to the solution of non-linear two point BvPs. The result obtained in example (2) shows that as the value of x increases, the accuracy of the method improves the just as the resulting error decreases. Over all, the method has been shown to be computationally efficient because yield solution that are reasonable and easy to express.

In (1982), Hong yuafu introduced a rationalized Runge-kutta scheme which was based on the quotient of weighted average of several estimates of the function f(x,y) generated by the conventional Runge-kutta scheme to obtain a more accurate approximation.

Babatola et al (2006), in their scheme of a new one step implicit inverse Runge-kutta method for solution of system of ODE. Concluded that the scheme is accurate, they also observed that the new method demonstrated better accuracy in solving system of ODEs. They also discovered that the new scheme can solve system of ODEs accurately for problems arising in electrical transmission network, meter transfer, control theory and action kinetic.

Kayoed (2006) reported in his new work on an improved Numerous method than that the method of reducing higher order ODEs to a system of first order equation, suffers some setbacks which include increased dimension of the resulting system to be solved wastage in computer time, computational burdens and cost implication are discussed in Awoyemi (1991, 2001,2002).

A number of articles had been written on numerous method for a direct numerical solution of special second order initial value problem of ODES of the form.

$Y = f(x,y)y(x_o) = y_2 \; y^{1}(x_0) = T$...................... (2)

In which the first derivative y1 is absent (see lambe (1973) jain (1984) Awoyemi (1992), Gonzalez and Thompson (1997) among others. This method because in effectives or incapable when the problem to be solve involve the first derivative y1 explicitly. This is the reason why kayoed (2006) developed a numerical method capable of handing general second order IvPs of the form.

$Y = f(x,y,y^{1})y(x_0) = y_2 \quad y(x_0) = T$...................(3)

In which first derivative y1 is present. Kayoed (2006) concluded that the new method is applied to solve linear and non-linear test problems. The result obtained for problem (i) and (ii) of the special type (1.1) are compared with Adee et al (2005). These results show better accuracy of the new method over Adee et al.

Ochoche (2007) proposed an improved technique for the computation of the numerical solution of IvPs. The method they improved is the modified Euler's method in which when effected a much better performance was gotten and the improved Euler is also of order two.

Richards (2005) also used different numerical method to compete ordinary differential equation. In his result, it is observed that Adams Bashforthmoulton method is more accurate followed by miles method. Then runge-kutta, then modified Euler's method, miles, method Euler and Euler's method area all predictor character methods while Runge-tutta is not.

Ibrahim (2006) also in his lecture note did the comparative of Euler's method modified Euler method and Runge-kutta method in which he found that the Runge-kutta method has a better approximation than modified Euler's method then Euler's method. He also sued the Taylor's series method and Picard's method to solve some problems of ordinary differential equation.

Uba (2008) in his lecture note also used Picard's method to solve problems in ordinary differential equation it is observed that the Picard's method is very tedious and confusing. It takes time and also that the accuracy increases as the order of the ordinary differential equation increases. Theeconomical effect of the environment on the population of field mine. He also discuss on other numerical method such as Taylor's series method and its short coming. Euler's method modified Euler's method. Runge-kutta method of order 4 and their short comings. He also discuss on the multi steps method which also include Adams moultou method and the mine's method finally he summarized their comparison in the table below:

TABLE 2.1: COMPARISONS AMONG MODIFIED EULER, FORTH ORDER RUNGE-KUTTA, MILNE AND ADAMS MOUTON METHODS TABLE.

Method	Types	Local errors	Global error	Function eval/step	Stability	Ease of changing step size	Recommendation
Modified Euler	Singles step	$^{0}(h^{3})$	$^{0}(h^{2})$	2	Good	Good	No
Forth order RuneaKutta	Single step	$^{0}(h^{5})$	$^{0}(h^{1})$	4	Good	Good	Yes
Milne	Multi step	$^{0}(h^{5})$	$^{0}(h^{4})$	2	Poor	Poor	No
Adams moulton	Mult step	$^{0}(h^{5})$	$^{0}(h^{4})$	2	Good	Poor	Yes

2.2 Taylor's Series Method

The Taylor's series method has long been regarded as an efficient procedure for solving system of ordinary differential equation. Frequently it is necessary to algebraically manipulate the differential system into an equivalent system. The Taylor's coefficient for this modified system may be simply written. However the required modification is a tedious and error prone task for all bet the simplest system for this reason. The Taylor's series method has often been excluded by numerical analysis from consideration as a general purpose integrator.

Jain (1984) discussed on Taylor's series method and its short coming on ordinary differential equation. Ibrahim (2006) in this lectures note used the Taylor's series method to solve problems of ordinary differential equation.

2.3 Modified Euler's Method

The Euler's method is a first order method, which means that the local error (error per step) is proportional to the square of the step size and the global error (error at a given time) is proportional to the step size. The Euler's method often serves as basis to construct more complicated method.

Jain (1984) discussed on modified Euler's method of order 4 and its short coming on ordinary differential equation.

Ochoche (2007) modified Euler's method as an improved technique for the computation of the numerical solution of IvPs. In which when effected a much better performance was gotten and the improved Euler is also of order two. Richard (2005) Euler's method are all predictor characters method while Runge-kutta is not.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

The purpose of this project is to compare the Taylor's series method to modified Euler's method for the computation of the numerical solution of initial values problems of ordinary differential equations.

Differential equations are one of the most important mathematical tools used in modeling problems in physical science. Historically, differential equations have originated in chemistry, physics and engineering. More recently, they have also arise in medicine, biology, anthropology and the like. Ordinary differential equation arise frequently in the study of physical system, unfortunately, many cannot be solve exactly. This is why the ability to numerically appropriate these methods is so important Rattenbury (2005), and Ocheche (2007).

In this chapter we shall use the Taylor's series method and the modified Euler's method to solve problems of ordinary differential equation and then analyzed the error so as to see which of these methods has less prone to error.

The equation we'll use for this purpose to errors.

1. $Y = y - x$ $\quad y(0) = 2 \quad [0,1] \quad h = 0.1$
2. $Y = y^2 + 1$ $\quad y(0) = 0 \quad [0,1] \quad h = 0.1$
3. $Y = 2x$ $\quad y(0) = 1 \quad [0,1] \quad h = 0.2$
4. $Y = y$ $\quad y(0) = 3 \quad [0.1] \quad h = 0.1$

3.2 Taylor's Series' Method

Before applying the Taylor's series method well first of all find the exact equation using the analytic method which will later be use for comparison the two methods above.

Q1. Taylor's series

$y = y - x$

$\frac{dy}{dx} = y - x$

Let $m = y - x$ $\frac{dy}{dx} = m$

$\frac{dm}{dx} = \frac{dy}{dx} - 1$

$\frac{dm}{dx} = m - 1$

$\frac{dm}{m-1} = dx$

$\ln(m-1) = x + c$

$m - 1 = e^{x+c} = e^{x}.e$

$m = ke^{x} + 1$

$y - x = ke^{x} + 0 + 1$

$y = ke^{x} + x + 1$

$y(0) = 2$

$2 = k.1 + 1$

2-1=k

1=k

$Y = e^x + x + 1$

Having the exact solution we shall now apply Taylor's series method and the modified Euler method. It is very important to note that the higher the order of differential equation the better the chances of convergence to the exact solution (using Taylor's series method) here we shall use order 4.

Taylor's series y (0) = 2

$y' = y - x \Rightarrow y' = 2$

$y'' = y - 1 \Rightarrow y'' = 1$

$y''' = y'' \Rightarrow y''' = 1$

$y''' = y''' \Rightarrow y^{iv} = 1$

Putting all these values into Taylors expansion formula we'll have

$$y = y_0 + hy + \frac{h2}{2} y + \frac{h3}{6} y + \frac{h4}{24} y$$

$$= 2 + 0.1(2) + \frac{(0.1)2}{2} + \frac{(0.1)3}{6} + \frac{(0.1)4}{24}$$

$$= 2 + 0.2 + 0.005 + 0.0001666666 + 0.000000000416666$$

y(0.1)= 2.2055171

when y=2.205171 x= 0.1

y' = y – x => 2.105171

y'' = y' => 1.105171

y''' = y'' => 1.105171

y=2.205171 + 0.1(2.105171) + $\frac{(0.1)2}{2}$ (1.105171) + $\frac{(0.1)2}{6}$ (1.105171) + $\frac{(0.1)4}{26}$ (1.105171)

= 2.205171 +0.2.105171+0.0005525855+0.000046087

y(0.2) = 2.421403

y = y – x => 2.221403

y =y -1 => 1.221403

y = y => 1.221403

y = y => 1.221403

y = 2.41403 + 0.1(2.221403) + $\frac{(0.1)2}{2}$ (1.221403)+ $\frac{(0.1)2}{6}$ (1.221403)+ $\frac{(0.1)4}{26}$ (1.221403)

= 2.421403 + 0.2221404 + 0.006107015 +00020356716+0.00000508917

y= (0.3) =2.649859

when y = 2.649859   x = 0.3

y' = y- x =>2.349859

y'' = y' -1 =>1.349859

y = 2.2649695 + 0.1(2.349859) + $\frac{(0.1)^2}{2}$ (1.349859) + $\frac{(0.1)^2}{6}$ (1.349859) + $\frac{(0.1)^4}{24}$ (1.349859)

$=2.2649859 + 0.2349859 + 0.006749295 + 0.0002249765 + 0.0000056244I$

$y(0.4) = 2.891825$

when $y = 2.891825$ $x = 0.4$

$y' = y - x \Rightarrow 2.891825$

$y'' = y' - 1 \Rightarrow 1.491825$

$y''' = y'' \Rightarrow 1.491825$

$y''' = y'' \Rightarrow 1.491825$

$y^{iv} = y''' \Rightarrow 1.491825$

$y = 2.89182.5 + 0.1(2.49125) + \frac{(0.1)^2}{2}(1.49125) + \frac{(0.1)^2}{6}(1.49125) + \frac{(0.1)^4}{24}(1.49125)$

$y(0.5) = 3.148721$

when $y = 3.148721$ $x = 0.5$

$y' = y - x \Rightarrow 2.64872$

$y'' = y' - 1 \Rightarrow 1.648721$

$y''' = y'' \Rightarrow 1.\ 648721$

$y''' = y'' \Rightarrow 1.\ 648721$

$y^{iv} = y''' \Rightarrow 1.\ 648721$

$y = 3.148721 + 0.1(2.648721) + \frac{(0.1)^2}{2}(1.648721) + \frac{(0.1)^2}{6}(1.648721) + \frac{(0.1)^4}{24}(1.648721)$

$y(0.6) = 3.422118$

when $y = 3.422118$ $x = 0.6$

$y' = y - x \Rightarrow 2.422118$

$y'' = y' - 1 \Rightarrow 1.422118$

$y''' = y'' \Rightarrow 1.422118$

$y''' = y'' \Rightarrow 1.422118$

$y^{iv} = y''' \Rightarrow 1.422118$

$y = 3.422118 + 0.1(2.822118) + \frac{(0.1)^2}{2}(1.822118) + \frac{(0.1)^2}{6}(1.822118) + \frac{(0.1)^4}{24}(1.822118)$

$y(0.6) = 3.713752$

when $y = 3.713752$ $x = 0.7$

$y' = y - x \Rightarrow 3.013752$

$y'' = y' - 1 \Rightarrow 1.013752$

$y''' = y'' \Rightarrow 1.013752$

$y''' = y'' \Rightarrow 1.013752$

$y^{iv} = y''' \Rightarrow 1.013752$

$y = 3.713752 + 0.1(2.013752) + \frac{(0.1)^2}{2}(1.013752) + \frac{(0.1)^2}{6}(1.013752) + \frac{(0.1)^4}{24}(1.013752)$

$y(0.6) = 4.025540$

when $y = 4.025540$ $x = 0.8$

$y' = y - x \Rightarrow 3.22554$

$y'' = y' - 1 \Rightarrow 1.22554$

$y''' = y'' \Rightarrow 1.22554$

$y''' = y'' \Rightarrow 1.22554$

$y^{iv} = y''' \Rightarrow 1.22554$

$y = 4.02554 + 0.1(2.22554) + \frac{(0.1)^2}{2}(1.22554) + \frac{(0.1)^3}{6}(1.22554) + \frac{(0.1)^4}{24}(1.22554)$

$y(0.6) = 3.025540$

when $y = 3.025540$ $x = 0.9$

$y' = y - x \Rightarrow 3.459602$

$y'' = y' - 1 \Rightarrow 1.459602$

$y''' = y'' \Rightarrow 1.459602$

$y''' = y'' \Rightarrow 1.459602$

$y^{iv} = y''' \Rightarrow 1.459602$

$y = 3.359602 + 0.1(2.459602) + \frac{(0.1)^2}{2}(1.459602) + \frac{(0.1)^2}{6}(1.459602) + \frac{(0.1)^4}{24}(1.459602)$

$y(0.6) = 3.359602$

TABLE 3.1: THE TAYLOR'S SERIES METHODS

N	X_n	y_n	y_n Taylor	Exact y_n	Error
1.	0.0	2	2	2	0
2.	0.1	2	2205171.	2.205171	0
3.	0.2	2.205171	2.421403	2.421403	0
4.	0.3	2.421403	2.649859	2.649859	0
5.	0.4	2.649859	2.891825	2.891825	0
6.	0.5	2.891825	2.148721	2.148721	0
7.	0.6	2.148721	3.422118	3.422119	0
8.	0.7	3.422118	3.713752	3.713753	$1x10^{-6}$
9.	0.8	3.713752	3.025540	3.025541	$1x10^{-6}$
10.	0.9	4.025540	4.359602	4.359603	$1x10^{-6}$
11.	1.0	4.359602	4.718280	4.718282	$2x10^{-6}$

3.3 Modified Euler's Method

1. $y' = y - x$ $y(0)=2$ [0,1] $h = 0.1$

Applying

$y_{n+1} = y_n = hf(x_n + y_n)$

$y_{n+1} = y_n = \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

$f(x,y) = y - x$

$y_{n+1} = 2 + 0.1(2 - 0)$

$= 2 + 0.2$

Euler $y_{n+1} = 2.2$

$y_{n+1} = 2 + (.0.1/2)([(2)+(2.2 - 0.1)]$

$= 2 + 0.005(4.1)$

Euler $y_{n+1}(0.1) = 2.205000$

$y_n = 2.205 \quad x\ 0.1$

$y_{n+1} = 2.205 + (0.1)(2.205 - 0.1)$

$= 2.205 + 0.2105$

M. Euler $y(0.2) = 2.421025$

When $y_n = 2.421025 \quad x_n = 0.2\ x_{n+1} = 0.3$

Euler $y_{n+1} = 2.421025 + (0.1)(2.421025 - 0.2)$

$= 2.6431275$

$y_{n+1} = 2.421025 + 0.05((2.221025) + (2.6431275 - 0.3))$

$= 2.421025 + 0.05(2.221025 + 2.3431275)$

M. Euler $y(0.3) = 2.649233$

When $y_n = 2.649233$ $x_n = 0.3$ $x_{n+1} = 0.4$

Euler $y_{n+1} = 2.649233 + (0.1)(2.649233 - 0.3)$

$= 2.649233 + 0.1(2.349233)$

$y_{n+1} = 2.421025 + 0.05((2.221025) + (2.6431275 - 0.3))$

$= 2.421025 + 0.05(2.221025 + 2.3431275)$

Euler $y_{n+1} = 2.8841563$

$y_{n+1} = 2.649233 + 0.05((2.349233) + (2.8841563 - 0.3))$

$= 2.649233 + 0.5(2.349233 + 2.4841563)$

M. Euler $y(0.4) = 2.890902$

When $y_n = 2.890902$ $x_n = 0.4$ $x_{n+1} = 0.5$

Euler $y_{n+1} = 2.890902 + (0.1)(2.890902 - 0.3)$

$= 2.890902 + 0.1(2.890902)$

Euler $y_{n+1} = 3.1399922$

$y_{n+1} = 2.890902 + 0.05((2.490902) + (3.1399922 - 0.5))$

$= 2.890902 + 0.5(2.490902 + 2.6399922)$

Euler $y(0.5) = 3.147447$

When $y_n = 3.147447$ $x_n = 0.5$ $x_{n+1} = 0.6$

Euler $y_{n+1} = 3.147447 + (0.1)(3.147447 - 0.5)$

$= 3.147447 + 0.1(2.647447)$

Euler $y_{n+1} = 3.4121916$

$y_{n+1} = 3.147447 + 0.05((2.647447) + (3.4121916 - 0.6))$

$= 3.147447 + 0.5(2.647447 + 2.81219167)$

Euler $y(0.6) = 3.411780$

When $y_n = 3.411780$ $x_n = 0.6$ $x_{n+1} = 0.7$

$y_{n+1} = 3.411780 + 0.05(3.411780 - 0.6)$

$= 3.411780 + 0.5(2.811780)$

Euler $y_{n+1} = 3.692958$

$y_{n+1} = 3.411780 + 0.05((2.811780)+(3.692958 - 0.7))$

$= 3.411780 + 0.5(2.811780 + 2.992958)$

Euler $y(0.7) = 3.702017$

When $y_n = 3.702017$ $x_n = 0.7$ $x_{n+1} = 0.8$

$y_{n+1} = 3.702017 + 0.1(3.702017 - 0.7)$

$= 3.702017 + 0.1(3.002017)$

Euler $y_{n+1} = 4.012229$

$y_{n+1} = 3.702017 + 0.05((3.002017)+4.0022187 - 0.8)$

$= 3.702017 + 0.5(3.002017 + 3.20220187)$

Euler $y(0.8) = 4.012229$

When $y_n = 4.012229$ $x_n = 0.8$ $x_{n+1} = 0.9$

$y_{n+1} = 4.012229 + 0.1(4.012229 - 0.8)$

$= 4.012229 + 0.1(3.212229)$

Euler $y_{n+1} = 4.333419$

$y_{n+1} = 4.012229 + 0.05((3.212229) + (4.3334519 - 0.9)$

$= 4.012229 + 0.5(3.212229 + 3.433451)$

Euler $y(0.9) = 4.344513$

When $y_n = 4.344513$ $x_n = 0.9$ $x_{n+1} = 1$

$y_{n+1} = 4.344513 + (0.1)(4.344513 - 0.9)$

$= 4.344513 + 0.1(3.444513)$

Euler $y_{n+1} = 4.6889643$

$y_{n+1} = 4.344513 + 0.05((3.444513) + (4.6889643 - 1)$

$= 4.344513 + 0.5(3.444513 + 3.6889643)$

Euler y(1) = 4.701187

TABLE 3.2: THE MODIFIED EULER'S METHOD

N	X_n	Y_n	$X_n - Y_n$	Euler's Fomula	X_{n+}	X_{n+1^-} Y_{n+1}	Modified Euler	Exact Y_{n+1}	Error
1.	0.0	2.000000	2000000	2.200000	0.1	2.100000	2.205000	2.205171	$1.7\text{x}10^{-4}$
2.	0.1	2.205000	2.105000	2.155000	0.2	2.215500	2.421025	2.421025	$3.78\text{x}10^{-4}$
3.	0.2	2.421025	2.221025	20643128	0.3	2.343128	2.649233	2.649859	$6.26\text{x}10^{-4}$
4.	0.3	2.649233	2.349233	2.884156	0.4	2.481563	2.890902	2.891825	$9.23\text{x}10^{-4}$
5.	0.4	2.890902	2.490902	3.13992	0.5	2.63662	3.147447	3.148721	$1.27\text{x}10^{-3}$
6.	0.5	3.147447	2.647447	3.412196	0.6	2.812192	3.411780	3.422119	0.010339
7.	0.6	3.411780	2.811780	3.692958	0.7	2.992958	3.702017	3.713753	0.011736
8.	0.7	3.702017	3.002017	4.002218	0.8	3.202202	4.012229	4.025541	013312
9.	0.8	4.012229	3.212229	4.333452	0.9	3.433458	4.344513	4.359603	0.01509
10.	0.9	4.344513	3.444513	4.688964	1.0	3.688964	4.701187	4.718282	0.01709
11.	1.1	4.701187							

Q2.

y = y2 + 1 y(0) = 0[0.1] h = 0.1

To find the exact solution

$\frac{dy}{dx} = y^2 + 1$

$\int \frac{dy}{y^2+1} = \int dx$

tan-1 y = x + c

y = tan (x + c)

y = tan x + k where k = tan c

But $y = 0$ $x = 0$

$0 = 0 + k \Longrightarrow k = 0$

$y' = y^2 + 1$

$y'' = 2yy'$

$y''' = 2(y')^2 + 2yy'' \Rightarrow y'' = 2$

$y^{iv} = 6y'y'' = 2yy'' \Rightarrow y^{iv} = 0$

$y = 0 + (0.1) + 0 + \frac{(0.1)^3}{6}(2) + 0$

$y(0.1) = 0.100333$

When $y = 0.100333$ $\quad x = 0.1$

$y' = y^2 + 1 \quad \Rightarrow \quad y' = 1.010066711$

$y'' = 2yy' \quad \Rightarrow \quad y'' = 0.202680466$

$y''' = 2(y')^2 + 2yy'' \quad \Rightarrow \quad y'' = 2.081141719$

$y^{iv} = 6y'y'' = 2yy'' \quad \Rightarrow \quad y^{iv} = 1.645972955$

$y = 0.10033 + 0.1(0.010066711) + \frac{(0.1)^2}{2}(0.202680466) + \frac{(0.1)^3}{6}(2.081141719) +$

$\frac{(0.1)^4}{24}(1.645972955)$

$y(0.2) = 0.202707$

When y = 0.202707 $\qquad$ x = 0.2

$y' = y^2 + 1 \quad \Rightarrow \quad y' = 1.041090128$

$y'' = 2yy' \quad \Rightarrow \quad y'' = 0.4220725131$

$y''' = 2(y')^2 + 2yy'' \quad \Rightarrow \quad y'' = 2.3388511414$

$y^{iv} = 6y'y'' = 2yy'' \quad \Rightarrow \quad y^{iv} = 3.584696267$

$y = 0.202707+(0.1)(1.041090128)+\frac{(0.1)^2}{2}(0.422072513) + \frac{(0.1)^3}{6}(2.338851414) +$

$\frac{(0.1)^4}{24}(3.58696267)$

y(0.3) = 0.309331

$y=0.422785+(0.1)(1.78747156)+\frac{(0.1)^2}{2}(0.9967132329) + \frac{(0.1)^3}{6}(3.621680525) +$

$\frac{(0.1)^4}{24}(10.11162173)$

y(0.5) = 0.546289

When y = 0.546289 $\qquad$ x = 0.5

$y' = y^2 + 1 \quad \Rightarrow \quad y' = 1.298431672$

$y'' = 2yy' \quad \Rightarrow \quad y'' = 1.418637879$

$y''' = 2(y')^2 + 2yy'' \quad \Rightarrow \quad y'' = 4.821822148$

$y^{iv} = 6y'y'' = 2yy'' \quad \Rightarrow \quad y^{iv} = 16.42950071$

$y = 0.546289 + (0.1)(1.298431672) + \frac{(0.1)^2}{2}(1.418637879) + \frac{(0.1)^3}{6}(4.821822148) + \frac{(0.1)^4}{24}(16.42950071)$

$y(0.6) = 0.684114$

When $y = 0.684114$ $\quad x = 0.6$

$y' = y^2 + 1 \quad \Rightarrow \quad y' = 1.468011965$

$y'' = 2yy' \quad \Rightarrow \quad y'' = 2.008575075$

$y''' = 2(y')^2 + 2yy'' \quad \Rightarrow \quad y'' = 7.058306916$

$y^{iv} = 6y'y'' = 2yy'' \quad \Rightarrow \quad y^{IV} = 27.34904661$

$y = 0.684114 + (0.1)(1.468011965) + \frac{(0.1)^2}{2}(2.008575075) + \frac{(0.1)^3}{6}(7.058306916) + \frac{(0.1)^4}{24}(27.34904661)$

$y(0.7) = 0.842248$

When $y = 0.842248$ $\quad x = 0.7$

$y' = y^2 + 1 \quad \Rightarrow \quad y' = 1.709382385$

$y'' = 2yy' \quad \Rightarrow \quad y'' = 2.879449195$

$y''' = 2(y')^2 + 2yy'' \quad \Rightarrow \quad y'' = 10.69439929$

$y^{iv} = 6y'y'' = 2yy''$ ⇒ $y^{iv} = 47.54716002$

$y = 0.842248 + (0.1)(1.709382385) + \frac{(0.1)^2}{2}(2.879449195) + \frac{(0.1)^3}{6}(10.69439929) + \frac{(0.1)^4}{24}(47.54716002)$

$y(0.8) = 1.029564$

When $y = 1.029564$ $x = 0.8$

$y' = y^2 + 1$ ⇒ $y' = 2.060002871$

$y'' = 2yy'$ ⇒ $y'' = 4.241811273$

$y''' = 2(y')^2 + 2yy''$ ⇒ $y'' = 17.22156948$

$y^{iv} = 6y'y'' = 2yy''$ ⇒ $y^{iv} = 87.89047571$

$y = 1.029564 + (0.1)(0.060002871) + \frac{(0.1)^2}{2}(4.241811273) + \frac{(0.1)^3}{6}(17.22156948) + \frac{(0.1)^4}{24}(87.89047571)$

$y(0.9) = 1.260010$

When $y = 1.260010$ $x = 0.9$

$y' = y^2 + 1$ ⇒ $y' = 2.587625801$

$y'' = 2yy'$ ⇒ $y'' = 6.520870007$

$y''' = 2(y')^2 + 2yy''$ $\Rightarrow$ $y'' = 29.82434052$

$y^{iv} = 6y'y'' = 2yy''$ $\Rightarrow$ $y^{iv} = 176.39777$

$$y = 1.260010+(0.1)(2.587625801)+\frac{(0.1)^2}{2}(6.520870007) + \frac{(0.1)^3}{6}(29.82434052) + \frac{(0.1)^4}{24}(176.39777)$$

y(1.;0) = 1.557083

TABLE 3.3: THE TAYLOR'S SERIES METHOD

N	X_n	Y_n	Y_n Taylor	Exact Y_n	Error
1.	0.0	0.00	0.00	0.00	0
2.	0.1	0.00	0.100333	0.100335	$2x10^{-6}$
3.	0.2	0.100333	0.202707	0.202710	$3x10^{-6}$
4.	0.3	0.202707	0.309331	0.309336	$5x10^{-6}$
5.	0.4	0.309331	0.422785	0.422793	$8x10^{-6}$
6.	0.5	0.422785	0.546289	0.546302	$1.3x10^{-5}$
7.	0.6	0.546289	0.684114	0.684132	$2.3x10^{-5}$
8.	0.7	0.684114	0.842248	0.842288	$4x10^{-5}$
9.	0.8	0.842248	1.029564	1.0295639	$7.5x10^{-5}$
10.	0.9	1.029564	1.260010	1.2600158	$1.4x10^{-4}$
11.	1.0	1.260010	1.557073	1.557408	$3.25x10^{-4}$

2.

2. $y' = y^2 + 1$ y(0) [0,1] h = 0.1

Applying

$y_{n+1} = y_n = hf(x_n, y_n)$

$y_{n+1} = y_n = \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

$y = 0 \quad x_n = 0 \quad h = 0.1 \quad x_{n+1} = 0.1$

Euler $y_{n+1} = 0.1$

$y_{n+1} = 0 + 0.05[1+(0.1)^2 + 1]$

$= 0.005(1 + 1.01$

M. Euler $y(0.1) = 0.1005$

When $y_n = 0.1005 \quad x_n = 0.1 \quad x_{n+1} = 0.2$

$y_{n+1} = 0.1005 + 0.1((0.1005)2 + 1)$

$= 0.1005 + 0.1(1.01010025)$

Euler $y_{n+1} = 0.201510025$

$y_{n+1} = 0.1005 + 0.5\ [(1.01010025) + (0.201510025)^2 + 1]$

$= 0.1005 + 0.5\ (1.01010025) + 1.04060629)$

M. Euler $y_n\ (0.2) = 0.203035$

When $y_n = 0.203035 \quad x_n = 0.1 \quad x_{n+1} = 0.2$

$y_{n+1} = 0.203035 + 0.1\ ((0.203035)^2 + 1)$

$= 0.203035 + 0.1\ (1.041223211)$

M. Euler y_n (0.2) = 0.3071573211

y_{n+1} = 0.203035 + 0.05 [(0.041223211) + $(0.3071573211)^2$ + 1]

= 0.203035 + 0.1 (1.041223211 + 1.09434562)

M. Euler y_n (0.3) = 0.309813

When y_n = 0.309813 x_n =0.3 x_{n+1} = 0.4

y_{n+1} = 0.309813 + 0.1 (($0.309813)^2$ + 1)

= 0.309813 + 0.1 (1.095984095)

Euler y_{n+1} = 0.4194114095

y_{n+1} = 0.309813 + 0.05 [(0.095984095) + $(0.4194114095)^2$ + 1]

= 0.309813 + 0.1 (1.095984095 + 1.7590593)

M. Euler y_n (0.4) = 0.423041

When y_n = 0.423041 x_n =0.3 x_{n+1} = 0.5

y_{n+1} = 0.423041 + 0.1 ($(0.423041)^2$ + 1)

= 0.423041 + 0.1 (1.178963688)

Euler y_{n+1} = 0.5309373688

y_{n+1} = 0.423041 + 0.05 [(0.5309373688) + $(0.4194114095)^2$ + 1]

$= 0.423041 + 0.1\ (1.178963966 + 1.292613237)$

M. Euler $y_n\ (0.5) = 0.546620$

When $y_n = 0.546620$ $x_n = 0.5$ $x_{n+1} = 0.6$

$y_{n+1} = 0.546620 + 0.1\ ((0.54662)^2 + 1)$

$= 0.546620 + 0.1\ (1.298793424)$

Euler $y_{n+1} = 0.6764993424$

$y_{n+1} = 0.546620 + 0.05\ [(0.298793424) + (0.6764993424)^2 + 1]$

$= 0.546620 + 0.5\ (1.298793424 + 1.45765136)$

M. Euler $y_n\ (0.6) = 0.684442$

When $y_n = 0.684442$ $x_n = 0.6$ $x_{n+1} = 0.7$

$y_{n+1} = 0.684442 + 0.1\ ((0.684442)^2 + 1)$

$= 0.684442 + 0.1\ (1.46846085)$

Euler $y_{n+1} = 0.8312880851$

$y_{n+1} = 0.684442 + 0.05\ [(1.468460851) + (0.8312880851)^2 + 1]$

$= 0.684442 + 0.5\ (1.468460851 + 1.69103988)$

M. Euler y_n (0.7) = 0.842417

When y_n = 0.842417 x_n =0.7 x_{n+1} = 0.8

y_{n+1} = 0.842417 + 0.1 ((0.842417)^2 + 1)

= 0.842417 + 0.1 (1.709666402)

Euler y_{n+1} = 1.01338364

y_{n+1} = 0.842417 + 0.05[(1.709666402) + (1.01338364)^2 + 1]

= 0.842417 + 0.5(1.709666402 + 2.026946402)

M. Euler y_n (0.8) =

When y_n = 0.842417 x_n =0.7 x_{n+1} = 0.8

y_{n+1} = 0.842417 + 0.1 ((0.842417)^2 + 1)

= 0.842417 + 0.1 (1.709666402)

Euler y_{n+1} = 1.01338364

y_{n+1} = 0.842417 + 0.05[1.709666402) + (1.01338364)^2 + 1]

= 0.842417 + 0.05(1.709666402 + 2.026946402)

M. Euler y_n (0.8) = 1.029248

When y_n = 1.029248 x_n =0.8 x_{n+1} = 0.9

$y_{n+1} = 1.029248 + 0.1\ ((1.029248)^2 + 1)$

$= 1.029248 + 0.1\ (2.059351446)$

Euler $y_{n+1} = 1.235183145$

$y_{n+1} = 0.029248 + 0.05[(2.059351446) + (1.235183145)^2 + 1]$

$= 0.029248 + 0.05(2.059351446 + 2.026946402)$

M. Euler $y_n\ (0.8) = 1.258499$

When $y_n = 1.258499$ $x_n = 0.9$ $x_{n+1} = 1.0$

$y_{n+1} = 1.258499 + 0.1\ ((1.258499)^2 + 1)$

$= 1.258499 + 0.1\ (2.583819733)$

Euler $y_{n+1} = 1.516880973$

$y_{n+1} = 1.258499 + 0.05[(2.583819733) + (1.516880973)^2 + 1]$

$= 1.258499 + 0.05(1.583819733 + 3.300927887)$

M. Euler $y_n(0.8) = 1.552736$

TABLE 3.4: THE MODIFIED EULER'S METHOD

N	X_n	Y_n	$X_n - Y_n$	Euler's Fomula	X_{n+}	$X_{n+1}-Y_{n+1}$	Modified Euler	Exact Y_{n+1}	Error
1.	0.0	0.000000	1.000000	1.100000	0.1	1.01	1.100500	1.100335	-1.65×10^{-4}
2.	0.1	1.1005	1.010100	0.201510	0.2	1.040606	0.203035	0.203710	-3.25×10^{-4}
3.	0.2	0.203035	1.041223	0.307157	0.3	1.094346	0.209813	0.309336	-4.79×10^{-4}
4.	0.3	0.309813	1.095984	0.419411	0.4	1.195906	0.423041	0.422793	-2.48×10^{-4}
5.	0.4	0.423041	1.178964	0.540937	0.5	1.292613	0.546620	0.546302	-3.18×10^{-4}
6.	0.5	0.546620	1.298793	0.676499	0.6	1.457651	0.684442	0.684137	-3.05×10^{-4}
7.	0.6	0.684442	1.465461	0.831288	0.7	1.691040	0.842417	0.842288	-1.29×10^{-4}
8.	0.7	0.842417	1.709666	1.013384	0.8	2.026946	1.029248	1.029639	3.9×10^{-4}
9.	0.8	1.029248	2.059351	1.235184	0.9	2.525677	1.258499	1.258158	1.659×10^{-3}
10.	0.9	1.258499	2.583820	1.516881	1.0	3.300928	1.552736	1.552408	4.672×10^{-3}
11.	1.0	1.552736							

3. $y' = 2x \; y(1) = 1 \quad [0,1] h = 0.2$

Solution

$$\frac{dy}{dx} = 2x$$

$$\int dy = \int 2x dx$$

$$y = \frac{2x^2}{2} + c$$

$$y = x^2 + c$$

When $x = 1 y = 1 c = 7$

$$1 = 1 = c \Rightarrow c = 0$$

$$Y = x^2$$

Taylor's series

$y' = 2x$ when $x = 1 \Rightarrow y' = 2$

$y' = 2 \Rightarrow y'' = 2$

$y = 1 + (0.2)(2) + (0.2)^2$

$=1 + 0.4 + 0.04$

$y(1.2) = 1.44$

When $y = 1.44$ $x = 1.2$

$y' = 2x \Rightarrow 2.4$

$y'' = 2$

$y = 1.44 + (0.2)(2.4) + (0.2)^2$

$=1 + 44 + 0.48 + 0.04$

$y(1.4) = 1.96$

When $y = 1.96$ $x = 1.4$

$y' = 2x \Rightarrow 2.8$

$y'' = 2 \Rightarrow 2$

$y = 1.96 + (0.2)(2.8) + (0.2)^2$

$=1.\ 96 + 0.56 + 0.04$

$y(1.6) = 2.56$

When $y = 2.56$ $x = 1.6$

$y' = 2x \Rightarrow 3.2$

$y'' = 2 \Rightarrow 2$

$y = 2.56 + (0.2)(3.2) + (0.2)^2$

$= 2.56 + 0.64 + 0.04$

$y(1.8) = 3.24$

When $y = 3.24$ $x = 1.8$

$y' = 2x \Rightarrow 3.6$

$y'' = 2 \Rightarrow 2$

$y = 3.24 + (0.2)(3.6) + (0.2)^2$

$= 3.24 + 0.72 + 0.04$

$y(2.0) = 4$

TABLE 3.5: THE TAYLOR'S SERIES METHOD

N	X_n	Y_n	Y_n Taylor	Exact Y_n	Error
1.	1.0	1.0000	1.0000	1.0000	0
2.	1.2	1.0000	1.4400	1.4400	0
3.	1.4	1.4400	1.9600	1.9600	0
4.	1.6	1.9600	2.5600	2.5600	0
5.	1.8	2.5600	3.2400	3.2400	0
6.	2.0	3.2400	4.0000	4.0000	0

3. $y' = 2x \quad y(1) = 1 \; [0.1] \quad h = 0.2$

Applying

$y_{n+1} = y_n = hf(y_n + x_n)$

$y_{n+1} = y_n = \frac{h}{2}[f(y_n + x_n) + f(y_{n+1} \,.\, x_{n+1})]$

$f(y_n + x_n) = 2x$

$x_n = 1, y_n = 1 \quad x_{n+1} = 1.2$

Euler $y_{n+1} = 1 + 0.2(2)$

$= 1.4$

$y_{n+1} = 1 + 0.1[2 + (2(1.2))]$

$= 1.0.1 [2 + 2.4]$

M. Euler $y(1.2) = 1.44$

When $y = 1.44 \quad x_n = 1.2 \quad x_{n+1} = 1.4$

$y_{n+1} = 1.44 + 0.2[2(1.2)]$

$= 1.44 + 0.2(2.4)$

Euler $y_{n+1} = 1.92$

$y_{n+1} = (1.4) + 0.1[2.4 + (2(4))]$

$= 1.44 + 0.1[2.4 + 2.8]$

M. Euler $y(1.4) = 1.96$

When $y = 1.96 \quad x_n = 1.4 \quad x_{n+1} = 1.3$

$y_{n+1} = 1.96 + 0.2(2.8)$

Euler $y_{n+1} = 2.52$

$y_{n+1} = (1.96) + 0.1[2.8 + (2(1.6))]$

$= 1.96 + 0.1[2.8 + 3.2]$

M. Euler $y(1.6) = 2.56$

When $y = 2.56 \quad x_n = 1.6 \quad x_{n+1} = 1.8$

$y_{n+1} = 2.56 + 0.2(2(1.6))$

$= 2.56 + 0.2(3.2)$

Euler $y_{n+1} = 3.2$

$y_{n+1} = 2.56 + 0.1[3.2 + 2(1.8)]$

$= 2.56 + 0.1[3.2 + 3.6]$

M. Euler y(1.8)3.24

When y = 3.24 $x_n = 1.8$ $x_{n+1} = 2.0$

$y_{n+1} = 3.24 + 0.2(2(1.8))$

$= 3.24 + 0.2(3.6)$

Euler $y_{n+1} = 3.96$

$y_{n+1} = 3.4 + 0.1[3.6 + (2(1.8))]$

$= 3.24 + 0.1[3.6 + 4]$

M. Euler y(2) = 4

TABLE 3.6: THE MODIFIED EULER'S METHOD

N	X_n	Y_n	Euler's Formula	x_{n+1}	Modified Euler	Exact y_{n+1}	Error
1.	0.0	0.000000	1.100000	0.1	1.100500	1.100335	-1.65×10^{-4}
2.	0.1	1.1005	0.201510	0.2	0.03035	0.203710	-3.25×10^{-4}
3.	0.2	0.203035	0.307157	0.3	0.209813	0.309336	-4.79×10^{-4}
4.	0.3	0.309813	0.419411	0.4	0.423041	0.422793	-2.48×10^{-4}
5.	0.4	0.423041	0.540937	0.5	0.546620	0.546302	-3.18×10^{-4}

4. $y' = y$ $y(0)$ $[0.1]h = 0.1$

Solution

$\frac{dy}{dx} = y$

$\int\frac{dy}{y} = \int dx$

$\ln y = x + e$

$y = ke^x$

When $y = 1$ $x = 0$ $k = ?$

$\frac{1}{e^2} = k$

$\frac{1}{1} = k \Rightarrow k = 1$

$y = e^x$

Taylor's series

$y' = y \Rightarrow 1$

$y'' = y' \Rightarrow 1$

$y''' = y'' \Rightarrow 1$

$y^{iv} = y''' \Rightarrow 1$

$y = 1\ (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24}$

$= 1 + 0.1 + 0.005 + 0.0001666666 + 0.00000416666$

$y(0.1) = 1.1051710$

When $y = 1.1051710 \quad x = 0.1$

$y' = y \Rightarrow 1.1051710$

$y'' = y' \Rightarrow 1.1051710$

$y''' = y'' \Rightarrow 1.1051710$

$y^{iv} = y''' \Rightarrow 1.1051710$

$$y = 1.105171 + (0.1)(1.105171) + \frac{(0.1)^2}{2}(1.105171) + \frac{(0.1)^3}{6}(1.105171) + \frac{(0.1)^4}{24}(1.105171)$$

$$= 0.105171 + 0.1105171 + 0.005525855 + 0.00018419516 + 0.00000460487$$

$y(0.2) = 1.221403$

When $y = 1.221403 \quad x = 0.1$

$y' = y \Rightarrow 1.221403$

$y'' = y' \Rightarrow 1.221403$

$y''' = y'' \Rightarrow 1.221403$

$y^{iv} = y''' \Rightarrow 1.221403$

$$y = 1.221403 + (0.1)(1.221403) + \frac{(0.1)^2}{2}(1.221403) + \frac{(0.1)^3}{6}(1.221403) + \frac{(0.1)^4}{24}(1.221403)$$

$y(0.3) = 1.349859$

When $y = 1.349859 \quad x = 0.3$

$y' = y \Rightarrow 1.349859$

$y'' = y' \Rightarrow 1.349859$

$y''' = y'' \Rightarrow 1.349859$

$y^{iv} = y''' \Rightarrow 1.349859$

$$y = 1.349859 + (0.1)(1.349859) + \frac{(0.1)^2}{2}(1.349859) + \frac{(0.1)^3}{6}(1.349859) + \frac{(0.1)^4}{24}(1.349859)$$

$y(0.5) = 1.648721$

When $y = 1.648721$ $x = 0.5$

$y' = y \Rightarrow 1.648721$

$y'' = y' \Rightarrow 1.648721$

$y''' = y'' \Rightarrow 1.648721$

$y^{iv} = y''' \Rightarrow 1.648721$

$$y = 1.648721 + (0.1)(1.648721) + \frac{(0.1)^2}{2}(1.648721) + \frac{(0.1)^3}{6}(1.648721) + \frac{(0.1)^4}{24}(1.648721)$$

$y(0.6) = 1.822119$

When $y = 1.822119$ $x = 0.6$

$y' = y \Rightarrow 1.822119$

$y'' = y' \Rightarrow 1.822119$

$y''' = y'' \Rightarrow 1.822119$

$y^{iv} = y''' \Rightarrow 1.822119$

$$y = 1.822119 + (0.1)(1.822119) + \frac{(0.1)^2}{2}(1.822119) + \frac{(0.1)^3}{6}(1.822119) + \frac{(0.1)^4}{24}(1.822119)$$

$y(0.7) = 2.013753$

When $y = 2.013753 \quad x = 0.7$

$y' = y \Rightarrow 2.013753$

$y'' = y' \Rightarrow 2.013753$

$y''' = y'' \Rightarrow 2.013753$

$y^{iv} = y''' \Rightarrow 2.013753$

$$y = 2.013753 + (0.1)(2.013753) + \frac{(0.1)^2}{2}(2.013753) + \frac{(0.1)^3}{6}(2.013753) + \frac{(0.1)^4}{24}(2.013753)$$

$y(0.8) = 2.225541$

When $y = 2.225541 \quad x = 0.8$

$y' = y \Rightarrow 2.225541$

$y'' = y' \Rightarrow 2.225541$

$y''' = y'' \Rightarrow 2.225541$

$y^{iv} = y''' \quad \Rightarrow 2.225541$

$y = 2.225541 + (0.1)(2.225541) + \frac{(0.1)^2}{2}(2.225541) + \frac{(0.1)^3}{6}(2.225541) + \frac{(0.1)^4}{24}(2.225541)$

$y(0.9) = 2.459603$

When $y = 2.459603 \quad x = 0.9$

$y' = y \Rightarrow 2.459603$

$y'' = y' \Rightarrow 2.459603$

$y''' = y'' \Rightarrow 2.459603$

$y^{iv} = y''' \quad \Rightarrow 2.459603$

$y = 2.459603 + (0.1)(2.459603) + \frac{(0.1)^2}{2}(2.459603) + \frac{(0.1)^3}{6}(2.459603) + \frac{(0.1)^4}{24}(2.459603)$

$y(1.0) = 2.718281$

$= 1.2155$

$y_{n+1} = 1.2155 + 0.05(1.105 + 1.12155)$

$y(0.2) = 1.221025$

When $y_n = 1.221025$ $x_n = 0.2$ $x_{n+1} = 0.3$

$y_{n+1} = 1.21025 + 0.1(1.221025)$

$= 1.484155888$

$y_{n+1} = 1.349232625 + 0.05(1.349232625 + 1.484115888)$

$y(0.4) = 1.490902$

When $y_n = 1.490902051$ $x_n = 0.3$ $x_{n+1} = 0.4$

$y_{n+1} = 1.490902051 + 0.1(1.490902051)$

$= 1.639992256$

$y_{n+1} = 1.639992256 + 0.05(1.639992256 + 1.639992256)$

$y(0.5) = 1.647447$

When $y_n = 1.647447$ $x_n = 0.5$ $x_{n+1} = 0.6$

$y_{n+1} = 1.647447 + 0.1(1.64744766)$

$= 1.812191443$

$y_{n+1} = 1.64744766 + 0.05(1.64744766 + 1.812191443)$

$y(0.6) = 1.820428676$

When $y_n = 1.820429$ $x_n = 0.6$ $x_{n+1} = 0.7$

$y_{n+1} = 1.820428676 + 0.1(1.820428676)$

$= 2.002471544$

$y_{n+1} = 1.820428676 + 0.05(1.820428676 + 2.002471544)$

$y(0.7) = 2.011574$

When $y_n = 2.011574$ $x_n = 0.7$ $x_{n+1} = 0.8$

$y_{n+1} = 2.011574 + 0.1(2.011574)$

$= 2.212731056$

$y_{n+1} = 2.21273105687 + 0.05(2.212731056 + 2.212731056)$

$y(0.7) = 2.222789$

When $y_n = 2.22278924$ $x_n = 0.8$ $x_{n+1} = 0.9$

$y_{n+1} = 2.222788924 + 0.1(2.222788924)$

$= 2.445067816$

$y_{n+1} = 2.222788924 + 0.05(2.222788924 + 2.212731056)$

$= 2.445067816$

When $y = 0.100333$ $x = 0.1$

$y' = y^2 + 1 \Rightarrow y' = 1.010066711$

$y'' = 2yy' \Rightarrow y'' = 0.202680466$

$y''' = 2(y')^2 + 2yy'' \Rightarrow y'' = 2.081141719$

$y^{iv} = 6y'y'' = 2yy''$ $\Rightarrow$ $y^{iv} = 1.645972955$

$y = 0.10033 + 0.1(0.010066711) + \frac{(0.1)^2}{2}(0.202680466) + \frac{(0.1)^3}{6}(2.081141719) + \frac{(0.1)^4}{24}(1.645972955)$

$y(0.2) = 0.202707$

$y_{n+1} = 2.2278892 + 0.05\,(2.222788924 + 2.44506816)$

$y(0.9) = 2.456182$

when $y_n = 2.45681761$ $\quad x_n = 0.9 \quad x_{n+1} = 1$

$y_{n+1} = 2.45681761 + 0.1(2.456181761)$

$= 2.70179937$

$y_{n+1} = 2.45681761 + 0.05(2.45681761 + 2.701799937)$

$= 2.714081$

TABLE 3.7: THE MODIFIED EULER'S METHOD

N	x_n	y_n	x_n-y_n	Euler's formula	x_{n+1}	Xn+1-y_{n+1}	Modified Euler's	Exact y_{n+1}	Error
1	0.0	1.000000	1.000000	1.100000	0.1	1.1	1.105000	1.105171	1.71×10^{-4}
2	0.1	1.105000	1.015000	1.25500	0.2	1.215500	0.221025	1.221403	3.78×10^{-4}
3.	0.2	1.221025	1.221052	1.348128	0.3	1.343128	1.349233	1.349859	8.26×10^{-4}
4	0.3	1.349233	1.349233	1.484156	0.4	1.484156	1.490902	1.491825	9.23×10^{-4}
5.	0.4	1.490902	1.490902	1.639992	0.5	1.812191	1.647447	1.648721	1.27×10^{-4}
6.	0.5	1.647447	1.647447	1.812191	0.6	1.812191	1.820429	1.822119	1.69×10^{-4}
7.	0.6	1.820429	1.820429	2.002472	0.7	2.002472	2.011574	2.013753	2.179×10^{-4}
8.	0.7	2.011574	2.011574	2.212731	0.8	2.212731	2.222789	2.225541	2.752×10^{-4}
9	0.8	2.222789	2.222789	2.445068	0.9	2.445068	2.456182	2.459603	3.421×10^{-3}
10	0.9	2.456182	2.456182	2.701800	1.0	2.701800	2.714081	2.718282	4.201×10^{-3}
11	1.0	2.714081							

TABLE 3.8: COMPARISON BETWEEN TAYLOR'S SERIES AND MODIFIED EULERS METHODS

N	Exact y_n	y_n Taylor	Error	Modified Euler's y_n	Error	x_n
1.	2	2	0	2.000000	0	0.0
2.	2.205171	2.205171	0	2.205000	3.78x10-4	0.1
3.	2.421403	2.421403	0	2.421025	6.46x10-4	0.2
4.	2.649859	2.649859	0	2.649233	9.23x10-4	0.3
5.	2.891825	2.891825	0	2.890902	1.27x10-3	0.4
6.	3.148721	3.148721	0	3.147447	0.010339	0.5
7.	3.422119	3.422118	1x10-6	3.411780	0.011736	0.6
8.	3.713753	3.713754	1x10-6	3.702017	013312	0.7
9.	4.025541	4.025540	1x10-6	4.012229	0.01509	0.8
10.	4.359603	4.359602	1x10-6	4.344513	0.017095	0.9
11.	4.718282	4.718280	2x10-6	4.721187	1.71x10-4	1.0

TABLE 3.9: COMPARISON BETWEEN TAYLOR'S SERIES AND MODIFIED EULERS METHODS

N	X_n	Taylor's Y_n	Error Taylor's	Modified Euler's	Error M. Euler's	Exact
1.	0.00000	0.00	0	0.000000	-1.65x10-4	0.0
2.	0.100335	0.100333	2x10-6	0.100500	-3.25x10-4	0.1
3.	0.202710	0.202707	3x10-6	0.203035	-4.79x10-4	0.2
4.	0.309336	0.309331	5x10-6	0.309813	-2.48x10-4	0.3
5.	0.422793	0.422785	8x10-6	0.423041	-3.18x10-4	0.4
6.	0.546302	0.546289	1.3x10-5	0.546620	-3.05xx10-4	0.5

7.	0.684132	0.684114	2.3x10-5	0.684442	-1.29x10-4	0.6
8.	0.842288	0.842248	4x10-5	0.842417	3.9x10-4	0.7
9.	1.0295639	1.029564	7.5x10-5	1.029248	1.659x10-3	0.8
10.	1.2600158	1.260010	1.4x10-4	1.258499	4.672x10-3	0.9
11.	1.557408	1.557073	3.25x10-4	1.552736		1.0

TABLE 3.10: COMPARISON BETWEEN TAYLOR'S SERIES AND MODIFIED EULERS METHODS

N	X_n	Taylor's Y_n	Error Taylor's	Modified Euler's	Error M. Euler's	Exact
1.	1.0	1.00	0	1.00	0	1.00
2.	1.2	1.44	0	1.44	0	1.44
3.	1.4	1.96	0	1.96	0	1.96
4.	1.6	2.56	0	2.56	0	2.56
5.	1.8	3.24	0	3.24	0	3.24
6.	2.0	4.00	0	4.00	0	4.00

TABLE 3.11: COMPARISON BETWEEN TAYLOR'S SERIES AND MODIFIED EULERS METHODS

N	X_n	Taylor's Y_n	Error Taylor's	Modified Euler's	Error M. Euler's	Exact
1.	0.0	0.00	0	1.000000	-1.65x10-4	1
2.	0.1	1.105171	0	1.100500	-3.25x10-4	1.105171
3.	0.2	1.221403	0	1.203035	-4.79x10-4	1.221403
4.	0.3	1.349859	0	1.309813	-2.48x10-4	1.349859

5.	0.4	1.491825	0	1.423041	-3.18x10-4	1.491825
6.	0.5	1.648721	0	1.546620	-3.05x10-4	1.648721
7.	0.6	1.822119	0	1.684442	-1.29x10-4	1.822119
8.	0.7	2.013753	0	2.842417	3.9x10-4	2.013753
9.	0.8	2.225541	0	2.029248	1.659x10-3	2.225541
10.	0.9	2.459603	0	2.258499	4.672x10-3	2.459603
11.	1.0	2.718281	1x10-6	2.552736		2.718282

It can be seen that in example 1 the Taylor's series method has a better approximation with error $2x10^6$ while the modified Euler's method has the error of 0.017095.

If we look at example 2 we will find out the Tailors series method with an error of $3.25x10^{-4}$ to $4.67x10^{-3}$ here we notice that the modified Euler's method has a better approximation compared to the previous example. This is because the order of x has increased.

In example 3 we find out that both the Taylor's series method the method Euler's method and the exact has the same result which makes the error be zero (0). This is because the differential equation is of order two and we know that the modified Euler's method is also of order two, so we should expect them to be exact. Since the Taylor's series method is of order four we should not expect less because the equation is of order two.

Finally in example 4. The Taylor's series method has a better approximation with an error of $10x10^{-6}$ to $4.201x10^{-3}$

From all these we are at liberty to conclude that the Taylor's series method has a better approximation and less error compared to the modified Euler's method.

CHAPTER FOUR

4.0 SUMMARY, CONCLUSION AND RECOMMENDATION

4.1 Summary

This project numerical solution of ordinary differential equation using Taylor's series method and modified Euler's methods begins with an introduction, aims and objectives, scope and limitation, some basic definition, types and source of error and some applications of ordinary differential equation. All these were discussed in chapter one.

In chapter two which is the literature review in which journal articles and text books of learner scholars in the field of numerical analysis were reviewed. After many observations and recommendation were made.

Chapter three which is the methodology in with the comparison between the Taylor's series method and the modified Euler's method was made and it was found that the Taylor's series method gives a better approximation compared to the modified Euler's method because it has a limited error.

4.2 Conclusion

In conclusion, since numerical analysis is concerned with solving "hard" problems approximately or more efficiently than analytic method. From the last part of chapter three it was found that the Taylor's series method gives the best approximation with least error.

4.3 Recommendation

Solution of ordinary differential equations are very important to social scientists, engineers and scientist in general yet in this field three are problems which arises especially in the engineering field to which an analytic solution could not be found. However there are numerical methods which analyzed the solution is best obtained.

However we found that the Taylor's series method gives a better approximation compared to the modified Euler's hence the Taylor's series method is recommended.

Hence, it is recommended that further work should be carried out using either Euler's method with Runge-kutta, classical order and check the efficiency, rate of convergence.

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