A STUDY ON THE PROPERTIES AND APPLICATIONS OF LOMAX-GOMPERTZ DISTRIBUTION

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NIGERIA

MAY, 2018

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B. Sc. STATISTICS (2005)

P14SCMT8004

A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER OF SCIENCE DEGREE IN STATISTICS

DEPARTMENT OF STATISTICS,

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

MAY, 2018

DECLARATION

I declare that the work in this dissertation titled "Study on the Properties and Applications of Lomax-Gompertz Distribution" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_________________ _________________

OMALE Aisha Date

CERTIFICATION

This dissertation titled A STUDY ON THE PROPERTIES AND APPLICATION OF LOMAX-GOMPERTZ DISTRIBUTION by OMALE Aisha meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

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Prof. S.Z Abubakar Dean, School of Postgraduate Studies	Signature	Date

DEDICATION

This work is dedicated to Allah Subuhanahu wa Ta'ala, my family, both immediate and extended, and all those who walk in the path of knowledge.

ACKNOWLEDGEMENTS

There is no power or might except with Allah beside Whom none has the right to be worshiped. All praise and adoration belongs to Him who has seen me through my studies and enabled me to carry out this research. I wish to express my sincere gratitude to my supervisors, Dr A. Yahaya and Prof. O. E. Asiribo for their tireless effort to improve this work. I appreciate very much the time and effort which has been invested in this work to enable it see the light of day. My sincere and unreserved thanks go to my husband, Jibrin A. Shuaibu, to my children, Ummulkhair, Abdullah, Abdulrahman and Rahama. Thank you all for your endless patients and support. I also appreciate my parents, brothers and sisters for their supports and prayers. I appreciate the contributions of the Head of Department of Statistics, Dr H. G. Dikko and the entire staff members and students of the Department of Statistics for all the support and encouragement given to me all through the period of my study. I sincerely appreciate the contributions of the Departments of Mathematics and Computer Science and my colleagues in learning. I also sincerely thank Mallam Sulaiman and his family for their hospitality all through my stay. Thank you all very much and May Allah reward you all.

ABSTRACT

The Gompertz distribution can be skewed to the right or to the left. This dissertation introduces a new positively skewed Gompertz model known as Lomax-Gompertz Distribution (LGD). This extension was possible with the aid of a Lomax generator. Some basic statistical properties of the new distribution such as moments, moment generating function, characteristics function, reliability analysis, quantile function and distribution of order statistics were derived. A plot of the probability density function (pdf) of the distribution revealed that it is positively skewed. The model parameters have been estimated using the method of maximum likelihood estimation. The plot for the survival function indicates that the Lomax-Gompertz Distribution could be used to model time or age-dependent variables, where probability of survival decreases with time or age. The performance of the Lomax-Gompertz Distribution has been compared to the Generalized Gompertz, Transmuted Gompertz, Odd Generalized Exponential Gompertz and the Gompertz distributions by some applications to three real-life data sets. The results show that the proposed distribution outperformed the Generalized Gompertz, Transmuted Gompertz, Odd Generalized Exponential Gompertz and the Gompertz distributions in two of the datasets. The model should be used to model positively skewed datasets with various peaks where the sample size is large.

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CHAPTER ONE

BACKGROUND TO THE STUDY

1.1 Introduction

Lomax (1954) pioneered the study of a distribution used for modeling business failure data called the Lomax or Pareto II distribution. This distribution has found wide application in a variety of fields such as income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. It has been applied to model data obtained from income and wealth (Harris, 1968), firm size (Corbellini *et al.,* 2007), size distribution of computer files on servers (Holland *et al.,*1989), reliability and life testing (Hassan and Al-Ghamdi, 2009), receiver operating characteristic curve analysis (Campbell and Ratnaparkhi, 1993) and Hirsch-related statistics (Gl'anzel, 2008). It is known as a special form of Pearson type VI distribution. In the lifetime context, the Lomax model belongs to the family of decreasing failure rate (Chahkandi and Ganjali, 2009) and arises as a limiting distribution of residual lifetimes at great age (Balkema and de Hann, 1974). This distribution has been suggested as heavy-tailed alternative to the exponential, Weibull and gamma distributions (Bryson, 1974). Further, it is related to the Burr family of distributions (Tadikamalla, 1980) and as a special case obtained from compound gamma distributions (Durbey, 1970). Some details about the Lomax distribution and Pareto family are given in (Arnold, 1983) as well as (Johnson *et al.,* 1994). In record value theory, some properties and moments for the Lomax distribution have been discussed in (Balakrishnan and Ahsanullah, 1994) as well as (Amin, 2011). The moments and inference for the order statistics and generalized order statistics are given in (Saran and Pushkarna. 1999) as well as (Moghadam *et al.,* 2012). The estimation of parameters in case of progressive and hybrid censoring have been investigated in (Asgharzadan and

Valiollahi, 2011) as well as (Ashour *et al.,* 2011). The problem of Bayesian prediction bounds for future observation based on uncensored and type-I censored sample from the Lomax model are dealt with in (Abd-Ellah, 2003) and (Al-Hussaini *et al*., 2001). Furthermore, the Bayesain and non-Bayesian estimators of the sample size in case of type-I censored samples for the Lomax distribution are obtained in (Abd-Elfattah *et al*., 2007), and the estimation under step-stress accelerated life testing for the Lomax distribution is considered in (Hassan and Al-Ghamdi, 2009). The parameter estimation through generalized probability weighted moments (*PWMs*) is addressed in (Abd-Elfattah and Alharby, 2010). More recently, the second-order bias and bias-correction for the maximum likelihood estimators (MLEs) of the parameters of the Lomax distribution are determined in (Giles *et al.,* 2013). Cordeiro *et al.* (2014d) introduced a new family of distributions based on the Lomax distribution, called the Lomax-G generator. The Lomax-G generator adds two additional positive parameters to an existing continuous distribution. It allows for greater flexibility of its tails and can be widely applied to many areas of Engineering and Biology. This study takes advantage of this generator to introduce Lomax-Gompertz Distribution, by generalizing the Gompertz Distribution. The resultant Lomax-Gompertz Distribution (LGD) will have four parameters, two from the baseline Gompertz Distribution and two additional positive parameters from the Lomax-G generator. This will increase the flexibility of the Gompertz distribution and also widen its areas of applications in Engineering and Biology.

The Gompertz distribution (*GD*) can be skewed to the right or to the left. It is a generalization of the exponential distribution (*ED*) and is commonly used in many applied lifetime data analysis (Johnson *et al*., 1995). The *GD* is applied in the analysis of survival, in some sciences such as Gerontology (Brown and Forbes, 1974), Computer (Ohishi *et al.,* 2009), Biology (Economos, 1982), and Marketing science (Bemmaor and Glady, 2012). The hazard rate function of Gompertz distribution is an increasing function, which makes it applicable to describe the distribution of adult life spans by actuaries and demographers (Willemse and Koppelaar, 2000).

1.2 Statement of the Problem

One of the major problems in distribution theory and applications is that some datasets do not follow any of the existing and well known probability distributions appropriately and hence create anomalies in the process of statistical analysis.

Despite the applicability of the Gompertz distribution, limited work has been done in extending the distribution to increase its flexibility. Also to the best of our knowledge, there has been no research that extends the Gompertz distribution based on a Lomax link function, therefore, we expect that the proposed distribution will perform better in fitting real datasets and also improve the flexibility of the distribution as reported by previous studies in the literature.

1.3 Aim and Objectives of the Study

The aim of this work is to generalize a Gompertz distribution using the Lomax link function. The specific objectives for achieving the stated aim are, to;

i. define the proposed Lomax-Gompertz distribution.
ii. derive some statistical properties of the proposed distribution such as the moments, the moment generating function, characteristics function, survival function, hazard function, quantile function and distribution of order statistics.
iii. estimate the parameters of the proposed distribution using the method of Maximum Likelihood Estimation (MLE).
iv. evaluate the performance of the proposed distribution when compared to other generalizations of Gompertz distribution.

1.4 Significance of the Study

The main significance of this study is increasing the flexibility of the Gompertz Distribution by the addition of two positive parameters, thus making it better able to fit datasets. Also this will widen the areas that the distribution can be applied. The proposed distribution has been compared to some existing generalizations of the Gompertz distribution using some real life datasets to evaluate its performance.

1.5 Limitation

In this study, we focus mainly on generalizing the Gompertz distribution, deriving some statistical properties of the proposed distribution, studying and interpreting some plots of the proposed distribution and estimating the parameters of the model using only the method of maximum likelihood estimation. The maximum likelihood Estimation method employed in this study yielded non-linear system of equations which could not be solved analytically to obtain the maximum likelihood estimates. R software was utilized to solve for the Estimates of the parameters in the model using datasets. Other suitable software includes Python, SAS etc.

1.6 Definition of Terms

1.6.1 Probability distribution

A random variable is a variable whose value changes from one subject to the other. Probability is used to describe the likelihood or the chances that these random variables will equal specific values or be within a given range of specific values. A probability density function is a mathematical expression that approximately agrees with the frequencies of possible events of a random variable. A random variable is said to be continuous if its range contains an interval of real numbers. For a continuous random variable X, the respective cumulative distribution function (cdf) and probability density function (pdf) are defined as;

$$F(x) = \Pr[X \geq x] = \alpha$$

$$F(x) = \int_{-\infty}^{x} f(u)du \tag{1.1}$$

Where α is a real number between 0 and 1.

$$f(x) = \frac{d(F(x))}{dx} \tag{1.2}$$

The pdf has the following properties

1. $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

3. $P(a < X < b) = \int_{a}^{b} f(x)dx$

1.6.2 Moments

Moments are used to study some of the most important features and characteristics of a random variable such as mean, variance, skewness and kurtosis. Let X be a continuous random variable, the n^{th} moment of X about the origin can be defined as;

$$\mu'_n = E\left[X^n\right] = \int_{-\infty}^{\infty} x^n f(x)dx \tag{1.3}$$

1.6.3 Moment generating function

The moment generating function of a continuous random variable X can be defined as;

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{1.4}$$

Where $f(x)$ is the pdf of a continuous distribution.

This function is used for generating other moments as it is a general function for all the moments.

1.6.4 Characteristics function

The characteristics function of a random variable *X* can be defined as

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \tag{1.5}$$

It has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in the analysis of linear combination of independent random variables.

1.6.5 Reliability analysis

1.6.5.1 Survival function

Survival function is the probability function that a system or an individual will survive beyond a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{1.6}$$

where $F(x)$ is Cumulative distribution function (*cdf)* of a baseline distribution

1.6.5. Hazard function

Hazard function is also called the failure or risk function and is the probability that a component will fail or die within an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \tag{1.7}$$

where *F(x)* and *f(x)* are the *cdf and pdf* of a baseline distribution.

1.6.6 Order statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}(x)$ of the *i^th^* order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \tag{1.8}$$

Order statistics are used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analyses of censored samples, quality control, reliability analysis and strength of materials.

1.6.7 Maximum likelihood method

Let $x_1, x_2, ..., x_n$ be a random sample from a population *X* with probability density function $f(x;\theta),$ where parameter θ is unknown. The likelihood function, $L(\theta)$, is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$. That is,

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{1.9}$$

The sample statistic that maximizes the likelihood functions $L(\theta)$, is called the maximum likelihood estimator of θ and is denoted by $\hat{\theta}$.

1.6.8 Lifetime data

Datasets from real happenings or normal life occurrences are called real life datasets. They are observations or records in our day to day activities. Lifetime data are data collected on living subjects.

CHAPTER TWO

LITERATURE REVIEW

2.1 The Gompertz Distribution

The Gompertz distribution was formulated by (Gompertz, 1825) to fit mortality tables. The Gompertz distribution is widely used to describe the distribution of adult lifespan. Makeham (1860) examined the fit to actuarial data provided by the Gompertz distribution and observed with specific examples there could be an improvement to the fit with the modification now known as the Gompertz-Makeham distribution.However, using the generalized integro-exponential function, exact formulae can be derived for its central moments and moment-generating function (Milgram, 1985). For low levels of infant (and young adult) mortality, the Gompertz force of mortality extends to the whole life span of populations with no observed mortality deceleration (Vaupel, 1986). The Gompertz distribution is important in describing the pattern of lifespan and deaths (Wetterstrand, 1981) as well as (Gavrilov and Gavrilova, 1991). It has received considerable attention from demographers and actuaries. Pollard and Valkovics (1992) were the first to study the Gompertz distribution thoroughly. Johnson *et al* (1994) noted that the Gompertz distribution is a truncated extreme value distribution. Its properties were studied by (Johnson *et al.,* 1995) as well as (Garg *et al*., 1970) and they both obtained the Maximum Likelihood Estimates of the parameters. The Gompertz distribution is a generalization of the exponential (E) distribution and is applied in life time data analysis (Johnson *et al*, 1995). Kunimura (1998) arrived at the same conclusions as (Pollard and Valkovics, 1992) above. They both defined the moment generating function of the Gompertz distribution in terms of the incomplete or complete gamma function and their results are either approximate or left in an integral form. Exponential distributions are limits of sequences of Gompertz distribution, therefore it can be viewed as extensions of exponential distribution. Willemse and Koppelaar (2000) reformulated the Gompertz force of mortality and derived relationships for this new formulation. Willekens (2002) provided connections between the Gompertz, the Weibull and other Type I extreme value distributions. The Gompertz distribution is a flexible distribution which can be skewed to the left or to the right. The hazard rate function (hrf) of Gompertz distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers (Willemse & Koppelaar, 2000). Later, (Marshall and Olkin, (2007) described the negative Gompertz distribution; a Gompertz distribution with a negative rate of aging parameter. Burga *et al*. (2009) discussed the stress-strength reliability problem in Gompertz case. In demographic or actuarial applications, Maximum Likelihood Estimation is often used to determine the parameters of the Gompertz distribution. The Gompertz distribution is considered for the analysis of survival, in sciences such as: Gerontology (Brown and Forbes, 1974); Computer (Ohishi *et al.,* 2009); Biology (Economos, 1982) and Marketing Science (Bemmaor & Glady, 2012). Previous works concentrated on formulating approximate relationships to characterize it. By solving the maximum-likelihood estimates analytically, the dimension of the optimization problem can be reduced to one, both in the case of discrete and continuous data (Lenart, 2014).

2.2 Generalization of Distributions

The way Probability distributions or models are defined differs nowadays compared to the past two decades. New models or their classes are proposed, extended or generalized mainly to explain phenomenon in life which arises in different areas of human endeavor such as Computer sciences, Public Health, Physics, Engineering and many others. Some distributions which are well-known and fundamental such as the gamma, exponential and weibull are limited in their characteristics and as such are not able to show wide flexibility (Tahir and Cordeiro, 2016). Some well-established characteristics which a life time model may possess include closed form of cumulative distribution function, goodness of fit, parameter estimations both in censored and uncensored cases, skewness and kurtosis features, quantile functions, entropy measures, reliability analysis and a host of others (Tahir and Cordeiro, 2016). The failure rate behavior for complex phenomenon in Biological surveys, Engineering modeling, reliability studies and human mortality studies, can be bath tub, upside down bath tub and may take other shapes but usually not monotone increasing or monotone decreasing. Thus, several generalizations or G-classes are introduced by researchers to cope with failure rate that are both monotonic and non-monotonic (Tahir and Cordeiro, 2016). These G-classes are flexible to study important properties of models and fitting the models. The parameter(s) induction has been proved to be useful in tail property exploration and also improving the goodness-of-fit of proposed family of Generators (Cordeiro *et al.,* 2014d). Two main generalization approaches have been adopted and practiced in the last two decades. These are the G- classes and the compounding approaches.

2.2.1 The G-classes approach

Shape parameter(s) induction into baseline distributions is carried out in this approach. This is done in order to explore properties of tails and also improve goodness of fit of the proposed G family of distributions. (Tahir and Nadarajah, 2015). Also, G families have the ability to fit skewed data better than existing distributions (Pescim *et al.*, 2010). Some well-known generalized (or G-) classes as given by (Tahir and Cordeiro, 2016) are: Marshall-Olkin-G (Marshall and Olkin, 1997), exponentiated-G (Gupta *et al.*, 1998), beta-G (Eugene *et al.*, 2002), KumaraswamyG (Cordeiro and de-Castro, 2011), McDonald-G (Alexander *et al.*, 2012), ZBgammaG (Zografos and Balakrishnan, 2009; Amini *et al.*, 2014), RBgamma-G (Ristic and Balakrishnan, 2012; Amini *et al.*, 2014), odd-gamma-G (Torabi and Montazari, 2012), Kummer-beta-G (Pescim *et al.*, 2012), beta extended Weibull-G (Cordeiro *et al.*, 2012b), odd exponentiated generalized-G (Cordeiro *et al.*, 2013a), truncated exponential-G (Barreto-Souza and Simas, 2013), logistic-G (Torabi and Montazari, 2014), gamma extended Weibull-G

(Nascimento *et al*., 2014), odd Weibull-G (Bourguignon *et al*., 2014a), exponentiated-half-logistic-G (Cordeiro *et al*., 2014a), Libby-Novick beta-G (Cordeiro *et al*. 2014b; Ristic *et al*., 2015), Lomax-G (Cordeiro *et al*., 2014d), Harris-G′ (Batsidis and Lemonte, 2015; Pinho *et al*., 2015), modified beta-G (Nadarajah *et al*., 2014b), odd generalized-exponential-G (Tahir *et al*., 2015), Kumaraswamy odd log-logistic-G (Alizadeh *et al*., 2015b), beta odd log-logistic-G (Cordeiro *et al*., 2016), KumaraswamyMarshall-Olkin-G (Alizadeh *et al*., 2015c), beta-Marshall-Olkin-G (Alizadeh *et al*., 2015a), Weibull-G (Tahir *et al*., 2016b), exponentiated-Kumaraswamy-G (da-Silva *et al*.,2016), ZBgamma-odd-loglogistic-G (Cordeiro *et al*., 2015a) and Tukey's g- and h-G (Jiménez *et al*., 2015). For further reading, please refer to (Tahir and Nadarajah, 2015). T-gamma{Y} (Alzaatreh *et al*., 2016a), T-Cauchy {Y} (Alzaatreh *et al*.,2016b), Gumbel-X (Al-Aqtash, 2013; Al-Aqtash *et al.,* 2014) and logistic-X (Tahir *et al*., 2016). There is also the class based on the quadratic rank transmutation map (QRTM) pioneered by (Shaw and Buckley, 2009) and highlighted by (Aryal and Tsokos, 2 0 0 9, 2 0 1 1). Bourguignon *et al*., (2016a) and (Das, 2015) obtained the general properties of the transmuted family. The transmuted family was further extended as the exponentiated transmuted-G type 1 using the Lehmann alternative type 1 (LA1) class (due to Gupta *et al*., 1998) by (Nofal *et al*., 2016) and (Alizadeh *et al*., 2016a), the exponentiated transmutedG type 2 using the Lehmann alternative type 2 (LA2) class (due to Gupta *et al*., 1998) by (Merovci *et al*., 2016), and the transmuted exponentiated generalized-G by (Yousof *et al*., 2015).

This thesis is interested in the Generator proposed by (Cordeiro *et al.,* 2014d). They proposed a new class of distributions called the Lomax generator with two extra positive parameters to generalize any continuous baseline distribution. Some special models such as the Lomax-normal, Lomax–Weibull, Lomax-log-logistic and Lomax–Pareto distributions were discussed. Some mathematical properties of the new generator including ordinary and incomplete moments, quantile and generating functions mean and median deviations, distribution of the order statistics and some entropy measures were presented. They discussed the estimation of the model parameters using the method of maximum likelihood. They also proposed a "minification" process based on the marginal Lomax-exponential distribution. They also defined a log-Lomax–Weibull regression model for censored data. The importance of the new generator was illustrated by means of three real data sets. The pdf of the Lomax-G generator allows for greater flexibility of its tails and also has wide applicability in areas such as Engineering and Biology. Hence we introduce the Lomax-Gompertz distribution because the Lomax-G family extends some well-Known distributions such as the Log-Normal, Weibull, Pareto and the Log-Logistic distributions.

2.2.2 Compounding approach

This approach of generalization involves compounding of discrete models, which includes binomial, geometric, logarithmic, Poisson, binomial, negative-binomial, power series and Conway-MaxwellPoisson (COMP) with continuous lifetime models (Tahir and Cordeiro, 2016).

2.3 Some Generalizations of the Gompertz Distributions

El-Gohary *et al*. (2013) made a generalization of the exponential, Gompertz, and generalized exponential distributions. This distribution is called the generalized Gompertz distribution (*GGD*). The main advantage of this new distribution is that it has increasing or constant or decreasing or bathtub curve failure rate depending upon the shape parameter. This property makes *GGD* very useful in survival analysis. Some statistical properties such as moments, mode, and quantiles were derived. The failure rate function was also derived. The maximum likelihood estimators of the parameters were derived using a simulations study. Real data set was used to determine whether the *GGD* better than other well-known distributions in modeling lifetime data. The proposed distribution was applied to data using Monte Carlo simulation method. The findings indicate that the proposed distribution fits the data better than the other distributions.

Jafari *et al.* (2014) developed a new generalized Gompertz model having four-parameters, which is called Beta-Gompertz (BG) distribution. It includes some well-known lifetime distributions such as Beta-exponential and generalized Gompertz distributions as special sub-models. This new distribution is quite flexible and can be used effectively in modeling survival data and reliability problems. It can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters. Some mathematical properties of the new distribution, such as moment generating function, hazard rate function, quantile measure, the k^{th} order moment and Shannon entropy were derived. They discussed maximum likelihood estimation of the *BGD* parameters from one observed sample and derived the observed Fisher's information matrix. A simulation study was also performed in order to investigate the properties of the proposed estimators. Lastly, an application using a real data set was presented. They presented hypotheses for comparing the proposed distribution with other distributions using the Likelihood Ratio Test (LRT) and it was concluded that the Beta Gompertz Distribution is an adequate Model.

El-Damcese *et al*. (2015) proposed the Odd Generalized Exponential-Gompertz (*OGEG*) distribution and studied its different properties. The quantile, median, mode and moments of *OGEG* were derived explicitly. They also discussed the distribution of order statistics. Both point and asymptotic confidence interval estimates of the parameters were derived using the maximum likelihood method and they obtained the observed Fisher information matrix. The new distribution was applied on set of real data to compare it with other known distributions such as Exponential (E), Generalized Exponential (GE), Gompertz (G), Generalized Gompertz (GG) and Beta-Gompertz (BG). Applications on set of real data showed that the OGE-G is the best distribution for fitting these data sets compared with other distributions considered in their study.

Abdul-Moniem and Seham (2015) introduced a new generalization of the Gompertz distribution called transmuted Gompertz distribution and presented its theoretical properties. The estimation of parameters was done by using the method of maximum likelihood. They also used the likelihood ratio statistic to compare the model with its baseline model. After the application of the transmuted Gompertz distribution to real data, the results show that the new distribution is better than the Gompertz distribution.

Khan *et al* (2016) introduced a three parameter Transmuted Gompertz Distribution using two parameter Gompertz Distribution. They investigated the potential usefulness of the distribution for modeling lifetime data. Various structural properties of the Transmuted Gompertz Model such as moments, moment generating function, incomplete moment, probability weighted moment, entropies, mean deviation. Bonferonni and Lorenz's curves and order statistics. Parameter estimation was carried out using Maximum Likelihood Estimation and evaluating the performance of the Maximum Likelihood Estimates using simulation. The Distribution was compared to the Gompertz (G), Generalised Exponential (GE) and Generalised Weibull (GE) using the Windshield data. The Transmuted Gompertz Distribution outperformed the other distributions and provides more flexibility for fitting Windshield data.

2.4 Some Generalizations of the Lomax Distributions

Special Lomax-G distributions such as the Lomax-normal (LN), Lomax–Weibull (LW), Lomax-log-logistic (LLL) and Lomax–Pareto (LPa) distributions were introduced and discussed by Cordeiro *et al* (2014d). The importance of the proposed distributions was evaluated using three datasets. The fits of the Lomax Normal and Lomax Weibull distributions were compared with those of the beta normal (BN), Kumaraswamy normal (KwN), Mc Donald

Normal McN, Gamma Normal (GN), skewnormal (SN), Weibull, beta Weibull (BW), Kumaraswamy Weibull (KwW), Lomax Exponential (LE), Burr type II, exponentiated Pareto (EPa), beta Pareto (BPa) and their baseline distributions themselves. The LN had the best fit when compared to Normal Distribution and some of its generalizations. LW distribution yielded a better fit when compared to the Weibull and other distributions.

Gupta *et al*. (2015) introduced a new statistical distribution constructed by composition of the cumulative density functions (*cdf*) of Frechet probability distribution and Lomax probability distribution which was named as the Lomax-Frechet distribution. It may have wider applicability in engineering, order statistics and other fields as it has more number of parameters. They derived expressions for its moments, characteristic function, hazard rate function and survival function. They plot some graphs for its probability density function (*pdf*) using the software 'Mathematica' and also investigated the variation of the skewness and kurtosis and discuss estimation by the method of maximum-likelihood.

Gupta *et al*. (2016) developed a new technique for constructing a family of new statistical distributions by combining the cdf of two known statistical distributions. They defined and studied a new distribution by combining the *cdf* of Lomax and Gumbel distributions and named it as Lomax-Gumbel distribution. This distribution has four parameters, two from Lomax and two from Gumbel and was found to be more flexible compared to its constituent distributions. They derived explicit expressions for the moments, characteristic function, hazard rate function, and survivor function for the Lomax-Gumbel distribution. They also used the software 'Mathematica' to plot some graphs for its *pdf* which show the effect of variation of different parameters occurring in the definition and investigated the variation of the hazard rate function. The parameters of the distribution were estimated by the method of maximum-likelihood.

Golzar *et al.* (2016) proposed a new extension of the exponential distribution called Lomax-Exponential Distribution (LED). This was achieved by generalizing the exponential distribution with the aid of T-X family of distibutions introduced by Alzaatreh *et al.* (2013a). They studied the relationship between Lomax-Exponential distribution and the Lomax distribution. They provided results for moment, limit behavior, hazard function, Shannon entropy and Order Statistics. The parameters of the model were estimated with Maximum Likelihood Estimation and Bayes Estimation. Two datasets were used to illustrate the applicability of LED and the distribution provided a better fit to the datasets than other models considered.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 The Definition of the Lomax-Gompertz Distribution

Cordeiro *et al*, (2014d) defined the respective *cdf* and *pdf* of the Lomax-G family of distribution for any continuous distribution as follows:

$$F(x) = \int_0^{-\log[1-G(x)]} \alpha\beta^{\alpha} \frac{dt}{(\beta+t)^{\alpha+1}} \qquad = 1 - \left\{ \frac{\beta}{\beta - \log[1-G(x)]} \right\}^{\alpha} \tag{3.1}$$

and

$$f(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1-G(x)]\{\beta - \log[1-G(x)]\}^{\alpha+1}}, \tag{3.2}$$

where $g(x)$ and $G(x)$ are the *pdf* and *cdf* of any continuous distribution to be generalized respectively and $\alpha>0$ and $\beta>0$ are the two new additional parameters responsible for the scale and shape of the distribution respectively.

The Gompertz distribution with parameters $\theta>0$ and $\gamma>0$ has the respective cumulative distribution function (*cdf*) and probability density function (*pdf*) given by:

$$G(x) = 1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)} \tag{3.3}$$

and

$$g(x) = \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)} \tag{3.4}$$

For $x \geq 0$, $\theta > 0$, $\gamma > 0$, where θ and γ are the model parameters.

3.1.1 The pdf and cdf of Lomax-Gompertz distribution

Using equation (3.3) and (3.4) in (3.1) and (3.2) and simplifying, we obtain the *cdf* and *pdf* of the Lomax-Gompertz distribution as follows:

3.1.1.1 Pdf of Lomax-Gompertz distribution

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right]^{\alpha}}$$

$$F(x) = 1 - \beta^{\alpha}\left\{\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right\}^{-\alpha}$$

$$F(x) = 1 - \beta^{\alpha}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)\right\}^{-\alpha} \tag{3.5}$$

3.1.1.2 Cdf of Lomax-Gompertz distribution

$$f(x) = \frac{\alpha\beta^{\alpha}\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]}$$

$$f(x) = \frac{\alpha\beta^{\alpha}\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta + \frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right)^{-(\alpha+1)}}{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}}$$

$$f(x) = \alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)\right\}^{-(\alpha+1)} \tag{3.6}$$

Hence equation (3.5) and (3.6) are the cdf and pdf of the Lomax-Gompertz distribution.

Where $x \geq 0$, $\alpha > 0$, $\beta > 0$, $\theta > 0$, and $\gamma > 0$.

Model validity check

Recall that for any continuous probability distribution to be valid,

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{3.7}$$

Proof

Considering the *pdf* of the Lomax-Gompertz distribution, where

$$f(x) = \frac{\alpha\beta^{\alpha}\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]}$$

$$\int_{0}^{\infty} f(x)dx = 1$$

$$\int_{0}^{\infty} \frac{\alpha\beta^{\alpha}\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]} dx$$

$$\int_{0}^{\infty} \frac{\alpha\beta^{\alpha}\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\beta^{-(\alpha+1)}\left(1 - \beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]} dx \tag{3.8}$$

Now let

$$y = 1 - \beta^{-1} \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]$$

$$\frac{dy}{dx} = \frac{\theta e^{\gamma x}\left(1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right)}{\beta\left(1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right)}$$

$$dx = \frac{\beta\left(1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right) dy}{\theta e^{\gamma x}\left(1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right)}$$

Substituting for *dx* in (3.8) and simplifying the resulting expression, we obtain

$$\int_0^\infty \alpha y^{-(\alpha+1)} dy = -\left[y^{-\alpha}\right]_1^\infty$$

$$= -[\infty^{-\alpha}] - [1^{-\alpha}] = -[0 - 1] = 1$$

Thus the pdf of the Lomax-Gompertz Distribution is valid.

3.1.2 Plot of pdf and cdf of the Lomax-Gompertz distribution

The following is a graphical representation of the *pdf* and *cdf* of the Lomax-Gompertz distribution. Given some values for the parameters *α, β, ɵ* and *γ*, we provide some possible shapes for the *pdf* and the *cdf* of the *LGD* as shown in figure 3.1 and 3.2 below:

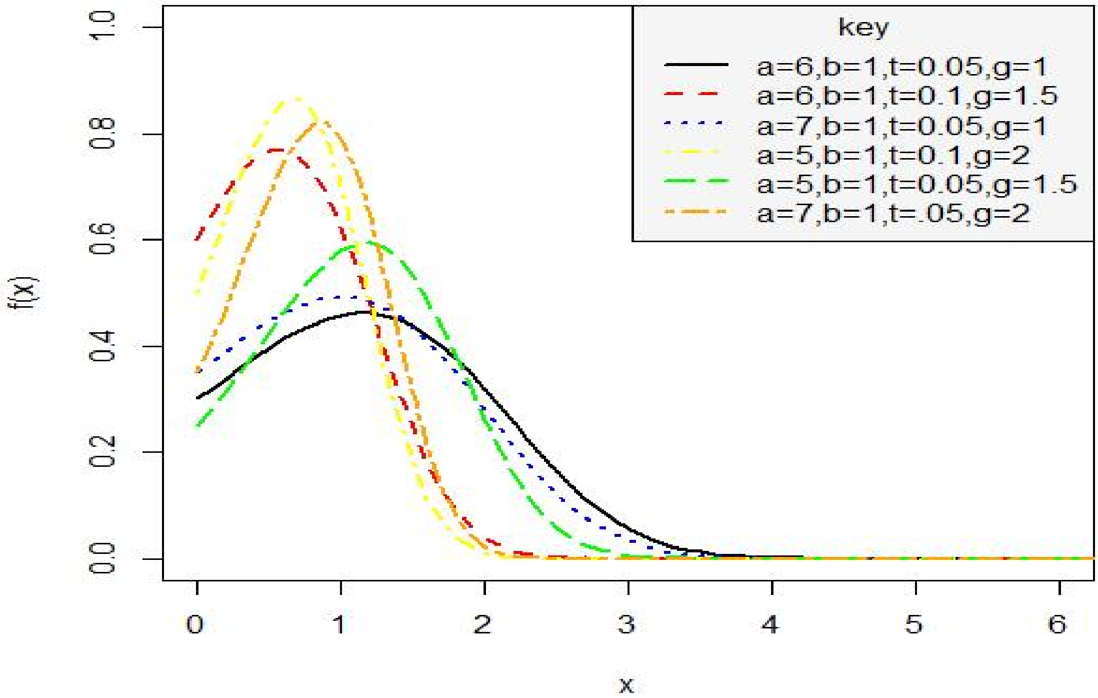

Figure 3.1: *PDF* of the *LGD* for different values of $a = \alpha, b = \beta, t = \theta$ *and* $g = \gamma$.

Figure 3.1 indicates that the *LGD* is a skewed distribution and such skewed to the right. This means that distribution can be very useful for datasets that are positively skewed.

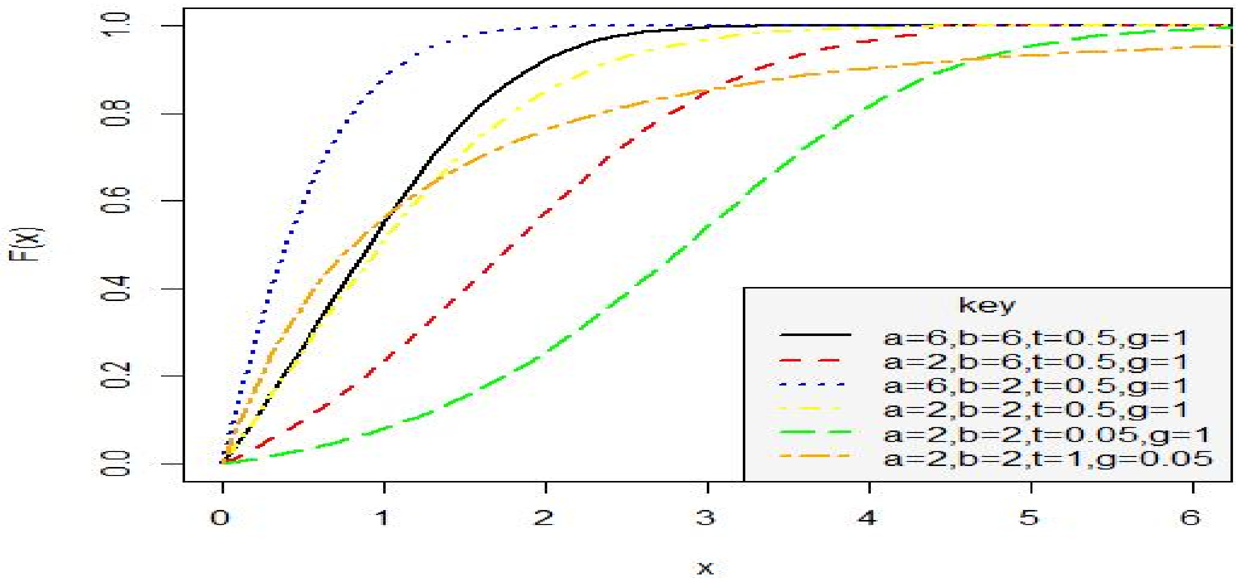

Figure 3.2: *CDF* of the *LGD* for different values of $a = \alpha, b = \beta, t = \theta$ *and* $g = \gamma$.

From the Figure 3.2, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3.2 Some Properties of the Lomax-Gompertz Distribution

3.2.1 Moments

Moments of a random variable are very important in distribution theory because some moments are used to study some of the most important features and characteristics of a random variable such as mean, variance, skewness and kurtosis.

Let X denote a continuous random variable, the n^{th} moment of X about the origin is given by;

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx \tag{3.12}$$

Taking $f(x)$ to be the *pdf* of the Lomax-Gompertz distribution as given in equation (3.9) and substituting in equation (3.12), we get the following results.

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx$$

$$f(x) = \frac{\alpha\beta^\alpha \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]}$$

Expansion and simplification of the pdf

$$f(x) = \frac{\alpha\beta^\alpha \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]}$$

$$f(x) = \alpha\beta^\alpha \theta e^{\gamma x} \beta^{-(\alpha+1)}\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}$$

$$f(x) = \frac{\alpha\theta e^{\gamma x}}{\beta\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{\alpha+1}} \tag{3.13}$$

Let

$$A=\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{\alpha+1}$$

Using the generalized binomial theorem on *A* above, from (Cordeiro et al., 2014), we have:

$$\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{(\alpha+1)}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha+1}{i}\beta^{-i}\left(\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{i} \tag{3.14}$$

Now, consider the following formula from (Tahir *et al.*, 2016) which holds for $i\geq 1$, then we can write the last term in (3.14) above as

$$\left(\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{i}=\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]^{l} \tag{3.15}$$

Where for (for $j\geq 0$) $P_{j,0}=1$ and (for k=1,2,…..)

$$P_{j,k}=k^{-1}\sum_{m=1}^{k}(-1)^{m}\frac{\left[m(j+1)-k\right]}{(m+1)}P_{j,k-m} \tag{3.16}$$

Combining equation (3.14) and (3.15) and inserting the above power series in equation (3.13), we have:

$$f(x)=\frac{\alpha\theta e^{\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]^{l}}$$

$$f(x)=\frac{\alpha\theta e^{\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right]^{l}}$$

$$f(x)=\frac{\alpha\theta e^{\gamma x}e^{\frac{\theta l}{\gamma}\left(e^{\gamma x}-1\right)}}{\beta\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}}$$

$$f(x)=\frac{\alpha\theta e^{-\frac{\theta l}{\gamma}}e^{\gamma x}e^{\frac{\theta l}{\gamma}e^{\gamma x}}}{\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}} \tag{3.17}$$

Using power series expansion on the last term in the numerator part of equation (3.17), we have:

$$e^{\frac{\theta l}{\gamma}e^{\gamma x}}=\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{r\gamma x} \tag{3.18}$$

Now, substituting equation (3.18), the power series expansion in equation (3.17) above, one gets:

$$f(x)=\frac{\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{\gamma x}e^{r\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}}$$

$$f(x)=\frac{\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{\gamma(1+r)x}}{\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}} \tag{3.19}$$

Simplifying (3.19) above results in the following:

$$f(x)=\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}\left(\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\right)^{-1}e^{\gamma(1+r)x}$$

$$f(x)=W_{i,j,k,l,r}e^{\gamma(1+r)x} \tag{3.20}$$

Where

$$W_{i,j,k,l,r}=\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}\left(\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\right)^{-1}$$

Hence,

$$\mu_n' = E\left[X^n\right] = \int_0^\infty x^n f(x)dx = \int_0^\infty W_{i,j,k,l,r}\, x^n e^{\gamma(1+r)x} dx \tag{3.21}$$

Also, using integration by substitution method in equation (3.21); we obtain the following:

Let $-u = \gamma(1+r)x \Rightarrow x = -\frac{u}{\gamma(1+r)}$

$$-\frac{du}{dx} = \gamma(1+r)$$

$$dx = \frac{-du}{\gamma(1+r)}$$

Substituting for u and dx in equation (3.21) and simplifying; we have:

$$\mu_n' = E\left[X^n\right] = \int_0^\infty x^n f(x)dx = W_{i,j,k,l,r}\left[\frac{-1}{\gamma(1+r)}\right]^{n+1} \int_0^\infty u^{n+1-1} e^{-u} du \tag{3.22}$$

Again recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma(k+1)$

Thus we obtain the n^{th} ordinary moment of a Lomax-Gompertz distributed random variable given by:

$$\mu_n' = E\left[X^n\right] = W_{i,j,k,l,r}\left[\frac{-1}{\gamma(1+r)}\right]^{n+1} \Gamma(n+1) \tag{3.23}$$

3.2.1.1 Mean

The mean of the *LGD* can be obtained from the n^{th} moment of the distribution when $n=1$ as follows:

$$\mu_n' = E\left[X^n\right] = W_{i,j,k,l,r}\left[\frac{-1}{\gamma(1+r)}\right]^{n+1} \Gamma(n+1)$$

$$\mu_1' = E\left[X^1\right] = \frac{W_{i,j,k,l,r}}{\left[\gamma(1+r)\right]^2} \tag{3.24}$$

Also the second moment of the *LGD* is obtained from the n^{th} moment of the distribution when $n=2$ as

$$E\left[X^2\right]=\frac{-2W_{i,j,k,l,r}}{\left[\gamma\left(1+r\right)\right]^3} \tag{3.25}$$

3.2.1.2 Variance

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\mu_n=E\left[X-\mu_1^{'}\right]^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\mu_1^{'i}\mu_{n-i}^{'} \tag{3.26}$$

The variance of *X* for *LGD* is obtained from the central moment when $n=2,$ that is,

$$Var(X)=E[X^2]-\{E[X]\}^2 \tag{3.27}$$

$$Var(X)=\frac{-2W_{i,j,k,l,r}}{\left[\gamma\left(1+r\right)\right]^3}-\left\{\frac{W_{i,j,k,l,r}}{\left[\gamma\left(1+r\right)\right]^2}\right\}^2 \tag{3.28}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2.2 Moment generating function

The moment generating function (*mgf*) is a simple way of arranging all the respective moments in a single function. It produces all the moments of the random variable by way of differentiation i.e., for any real number say k, the k^{th} derivative of $M_X(t)$ evaluated at $t=0$ is the k^{th} moment $\mu_k^{'}$ of *X*.

The *mgf* of a continuous random variable *X* is defined as

$$M_x(t) = E\left[e^{tx}\right] = \int_0^{\infty} e^{tx} f(x)dx \tag{3.29}$$

Recall that by power series expansion,

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n \tag{3.30}$$

Using the result in equation (3.30) and simplifying the integral in (3.29), therefore we have;

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} x^n f(x)dx$$

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu_n^{'} \tag{3.31}$$

where n and t are constants, t is a real number and μ'_n denotes the n^{th} ordinary moment of X and can be obtained from equation (3.23) as stated previously.

3.2.3 Characteristics function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in the analysis of linear combination of independent random variables.

The characteristics function of a random variable X is given by;

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \tag{3.32}$$

Recall from power series expansion that

$$\cos(tx) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} x^{2n}$$

$$E[\cos(tx)] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n}$$

And also that

$$\sin(tx) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$E[\sin(tx)] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu'_{2n+1}$$

Simple algebra and power series expansion proves that

$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu'_{2n+1} \quad (3.33)$$

Whereμ'_{2n} and μ'_{2n+1} are the moments of X for $n=2n$ and $n=2n+1$ respectively and can be obtained from μ'_n in equation (3.23)

3.2.4 Quantile function

This function is derived by inverting the cdf of any given continuous probability distribution. It is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question. Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$.

Taking *F(x)* to be the *cdf* of the Lomax-Gompertz distribution and inverting it as above will give us the quantile function as follows.

$$F(x) = 1 - \beta^{\alpha} \left\{ \beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)} \right) \right] \right\}^{-\alpha}$$

Inverting $F(x) = u$

$$1-\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}=u \tag{3.34}$$

Simplifying equation (3.26) above, we obtain:

$$1-u=\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}$$

$$\left[\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right]^{\alpha}=\frac{\beta^{\alpha}}{1-u}$$

$$\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]=\left(\frac{\beta^{\alpha}}{1-u}\right)^{\frac{1}{\alpha}}=\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}$$

$$\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)=\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}$$

$$\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)=\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta$$

$$\left(e^{\gamma x}-1\right)=\frac{\gamma}{\theta}\left(\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta\right)$$

$$e^{\gamma x}=\left[1+\frac{\gamma}{\theta}\left(\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta\right)\right]$$

$$\gamma x=\log\left[1+\frac{\gamma}{\theta}\left(\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta\right)\right]$$

$$Q(u) = X_q = \frac{1}{\gamma}\left\{\log\left[1+\frac{\gamma}{\theta}\left(\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta\right)\right]\right\} \tag{3.35}$$

3.2.5 Skewness and kurtosis

In this dissertation, the quantile based measures of skewness and kurtosis will be employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{3.36}$$

And the Moors' (1988) kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{4}\right)} \tag{3.37}$$

3.3 Reliability Analysis

3.3.1 Survival function

Survival function is the likelihood that a system or an individual will not fail after a given time. It tells us about the probability of success or survival of a given product or component. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{3.38}$$

Where $F(x)$ is *cdf* of the Lomax-Gompertz distribution, we have:

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right]^{\alpha}}$$

$$F(x) = 1 - \beta^{\alpha}\left\{\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right\}^{-\alpha}$$

$$S(x) = 1 - \left\{1 - \beta^{\alpha}\left\{\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right\}^{-\alpha}\right\}$$

$$S(x) = \beta^{\alpha}\left\{\beta - \log\left[1 - \left(1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)}\right)\right]\right\}^{-\alpha}$$

$$S(x) = \beta^{\alpha}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)\right\}^{-\alpha} \tag{3.39}$$

Below is a plot of the survival function at chosen parameter values in figure **3.3**

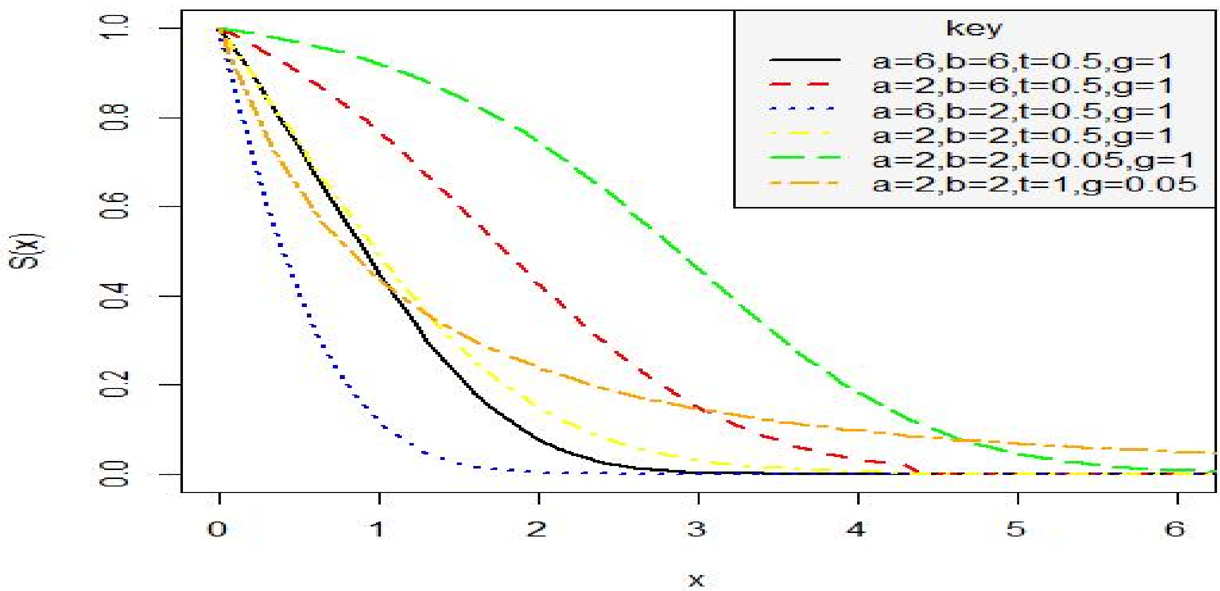

Figure 3.3: The survival function of the *LGD* for different values of a=α, b=β, t=θ and g=γ.

Interpretation: The Figure 3.3 revealed that the probability of survival for any random variable following a Lomax-Gompertz distribution decreases as the values of the random variable increases, that is, as time or age grows, probability of life or survival decreases. This implies that the Lomax-Gompertz distribution can be used to model random variables whose survival rate decreases as their age grows or time increases.

3.3.2 Hazard function

Hazard function, also called risk function gives the probability that a component will fail or die within an interval of time. The hazard function is defined mathematically as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{3.40}$$

Taking $f(x)$ and $F(x)$ to be the respective *pdf* and *cdf* of the proposed Lomax-Gompertz distribution given as

$$f(x)=\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}$$

and

$$F(x)=1-\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}$$

Substituting for *f(x)* and $F(x)$ in equation (3.36) and simplifying gives the following results.

$$h(x)=\frac{\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-(\alpha+1)}}{1-\left(1-\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}\right)}$$

$$h(x)=\frac{\alpha\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}}{\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}}$$

$$h(x)=\alpha\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-1} \tag{3.41}$$

Figure 3.4 A plot of the hazard function is presented below at chosen parameter values.

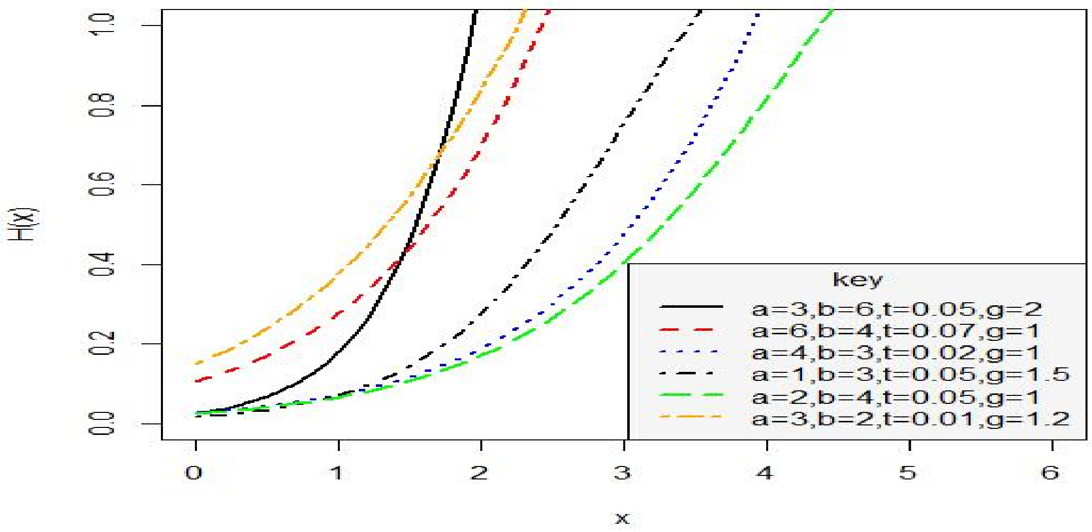

Figure 3.4: The hazard function of the *LGD* for different values of a=α, b=β, t=θ and g=γ.

Interpretation: Figure 3.4 revealed that the probability of failure for any random variable following a Lomax-Gompertz distribution increases as the values of the random variable increases, that is, as the values of the variable gets larger or increases, the probability of death increases.

3.4 Order Statistics

Order statistics are used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analyses of censored samples, reliability analysis, quality control and strength of materials. Suppose $X_1, X_2, \ldots\ldots, X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1:n}, X_{2:n}, \ldots\ldots, X_{i:n}$ denote the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \tag{3.42}$$

Where $f(x)$ and $F(x)$ are the *pdf* and *cdf* of the Lomax-Gompertz distribution respectively.

By using binomial expansion on the last term of equation 3.42 above we obtain the following

$$\left[1-F(x)\right]^{n-i} = \sum_{k=0}^{n-\mathrm{i}} (-1)^k \binom{n-\mathrm{i}}{k} F(x)^k \tag{3.43}$$

Replacing Equ (3.43) into Equ (3.42) yields

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1} \sum_{k=0}^{n-\mathrm{i}} (-1)^k \binom{n-\mathrm{i}}{k} F(x)^k \tag{3.44}$$

Further simplifying,

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{k=0}^{n-\mathrm{i}} (-1)^k \binom{n-\mathrm{i}}{k} F(x)^{i+k-1} \tag{3.45}$$

Using Equations (3.5), (3.6) and (3.45) the *pdf* of the i^{th} order statistics $X_{i:n}$, can be expressed as;

$$f_{i:n}(x) = \frac{n!}{(\mathrm{i}-1)!(n-\mathrm{i})!} \sum_{k=0}^{n-\mathrm{i}} (-1)^k \binom{n-\mathrm{i}}{k} \left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right] * \left[1-\beta^{\alpha}\left\{\beta+\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{\mathrm{i}+k-1} \tag{3.46}$$

Hence, the *pdf* of the respective minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *LGD* are given by;

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right] * \left[1-\beta^{\alpha}\left\{\beta+\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{k} \tag{3.47}$$

and

$$f_{n:n}(x) = n \left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right] \left[1-\beta^{\alpha}\left\{\beta+\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{n-1} \tag{3.48}$$

3.5 Estimation of Parameters

Let $X_1, ..., X_n$ be a sample of size 'n' independently and identically distributed random variables from the *LGD* with unknown parameters *α, β, θ* and *γ* defined previously. Recall the *pdf* of the *LGD* as given in equation 3.6

$$f(x)=\alpha\beta^{\alpha}\theta e^{\gamma x}\left(\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}$$

The likelihood function is given by;

$$L\left(X_1,X_2,.....,X_n/\theta,\gamma,\alpha,\beta\right)=\prod_{i=1}^{n}\alpha\beta^{\alpha}\theta e^{\gamma x_i}\left(\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right)^{-(\alpha+1)} \quad (3.49)$$

$$L\left(X_1,X_2,.....,X_n/\theta,\gamma,\alpha,\beta\right)=\left(\alpha\beta^{\alpha}\theta\right)^{n}e^{\gamma\sum_{i=1}^{n}x_i}\sum_{i=1}^{n}\left(\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right)^{-(\alpha+1)} \quad (3.50)$$

Taking the natural logarithm of the likelihood function, Let,

$$l\left(n\right)=\log L\left(X_1,X_2,.....,X_n/\theta,\gamma,\alpha,\beta\right),$$ such that

$$l\left(n\right)=\log\left\{\left(\alpha\beta^{\alpha}\theta\right)^{n}e^{\gamma\sum_{i=1}^{n}x_i}\sum_{i=1}^{n}\left(\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right)^{-(\alpha+1)}\right\} \quad (3.51)$$

$$l(n)=n\log\alpha+n\alpha\log\beta+n\log\theta+\gamma\sum_{i=1}^{n}x_i-(\alpha+1)\sum_{i=1}^{n}\log\left(\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right) \quad (3.52)$$

Differentiating $l(n)$ partially with respect to *Θ, γ, α* and *β* respectively gives;

$$\frac{\partial l\left(n\right)}{\partial\theta}=\frac{n}{\theta}-\frac{(\alpha+1)}{\gamma}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(e^{\gamma x_i}-1\right)}{\left(\beta+\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} \quad (3.53)$$

$$\frac{\partial l\left(n\right)}{\partial\gamma}=\sum_{i=1}^{n}x_i-\frac{\theta(\alpha+1)}{\gamma^{2}}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(1-e^{\gamma x_i}\right)}{\left(\beta+\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} \quad (3.54)$$

$$\frac{\partial l(n)}{\partial \alpha} = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}\log\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right) \tag{3.55}$$

$$\frac{\partial l(n)}{\partial \beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\left\{\frac{1}{\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right)}\right\} \tag{3.56}$$

Equating equations (3.53), (3.54), (3.55) and (3.56) each to zero we have the following equations

$$\frac{n}{\theta} - \frac{(\alpha+1)}{\gamma}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(e^{\gamma x_i}-1\right)}{\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} = 0 \tag{3.57}$$

$$\sum_{i=1}^{n} x_i - \frac{\theta(\alpha+1)}{\gamma^2}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(1-e^{\gamma x_i}\right)}{\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} = 0 \tag{3.58}$$

$$\frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}\log\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right) = 0 \tag{3.59}$$

$$\frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\left\{\frac{1}{\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right)}\right\} = 0 \tag{3.60}$$

solving the non-linear system of equations 3.57, 3.58, 3.59 and 3.60 above will give us the maximum likelihood estimates of parameters $\theta, \gamma, \alpha, and, \beta$ respectively. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, etc. when data sets are given.

CHAPTER FOUR

ANALYSIS AND DISCUSSION

4.1 Application to real life datasets

In this chapter, we present three data sets, their descriptive statistics, graphics and applications to some selected generalizations of the Gompertz distribution. We have compared the performance of the proposed distribution, Lomax-Gompertz distribution to the Gompertz distribution (*GD*) and other generalizations of the Gompertz distribution such as Generalized Gompertz distribution (*GGD*), odd generalized Exponential-Gompertz distribution (*OGEGD*), and Transmuted Gompertz distribution (*TGD*).

The Generalized Gompertz Distribution (*GGD*)

The *pdf* of the *GGD* distribution is given as;

$$f(x;\theta,\gamma,c) = c\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left[1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right]^{c-1} \tag{4.1}$$

The Odd Generalized Exponential Gompertz Distribution (*OGEGD*)

The *pdf* of the *OGEGD* is given as;

$$f(x;\alpha,\beta,\theta,\gamma) = \frac{\alpha\beta\theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]^{2}} \exp\left\{-\alpha\left[\frac{1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}}{1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)}\right]\right\}\left(1-\exp\left\{-\alpha\left[\frac{1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}}{1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)}\right]\right\}\right)^{\beta-1} \tag{4.2}$$

The Transmuted Gompertz Distribution (*TGD*)

The *pdf* of the *TGD* is given by;

$$f(x;\theta,\gamma,\lambda)=\theta e^{\gamma x}e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left[1+\lambda-2\lambda\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]$$ Khan *et al* (2016) (4.3)

The Gompertz Distribution (*GD*)

The *pdf* of the Gompertz distribution is given by

$$f(x;\theta,\gamma)=\theta e^{\gamma x}e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)} \tag{4.4}$$

4.2 Datasets

The following are the datasets used for analysis and applications in this dissertation

Dataset I: This data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by (Ghitany *et al*., 2013) for fitting the Lindley distribution.

Table 4.1: Dataset I

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5

DataSet II: This data set is the strength data of glass of the aircraft window reported by (Fuller *et al*., 1994).

Table 4.2: Dataset II

18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Dataset III: This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by (Gross and Clark, 1975) and has been used by (Shanker *et al*., 2016).

Table 4.3: Dataset III

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

The following table gives the summary descriptive statistics for the three data sets above.

Table 4.4: Summary Statistics for the three data sets

Parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values for data set I	100	0.80	4.675	8.10	13.020	9.877	38.500	52.3741	1.4953	5.7345
Values for data set II	31	18.83	25.51	29.90	35.83	30.81	45.38	52.61	0.43	2.38
Values for data set III	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.7198	5.9241

We also provide some histograms and densities for the three datasets as shown in Figures 4.1, 4.2 and 4.3 below respectively.

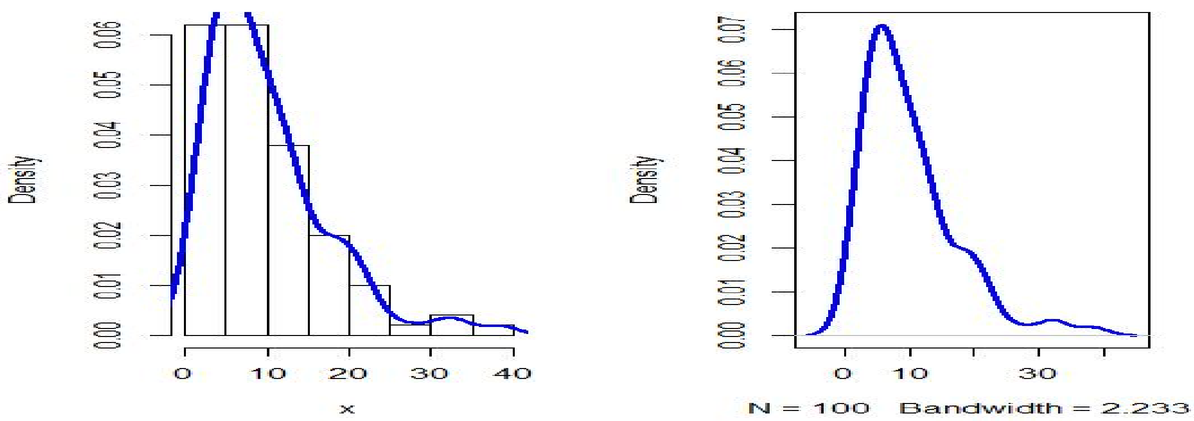

Figure 4.1: A Histogram and density plot for Data set I

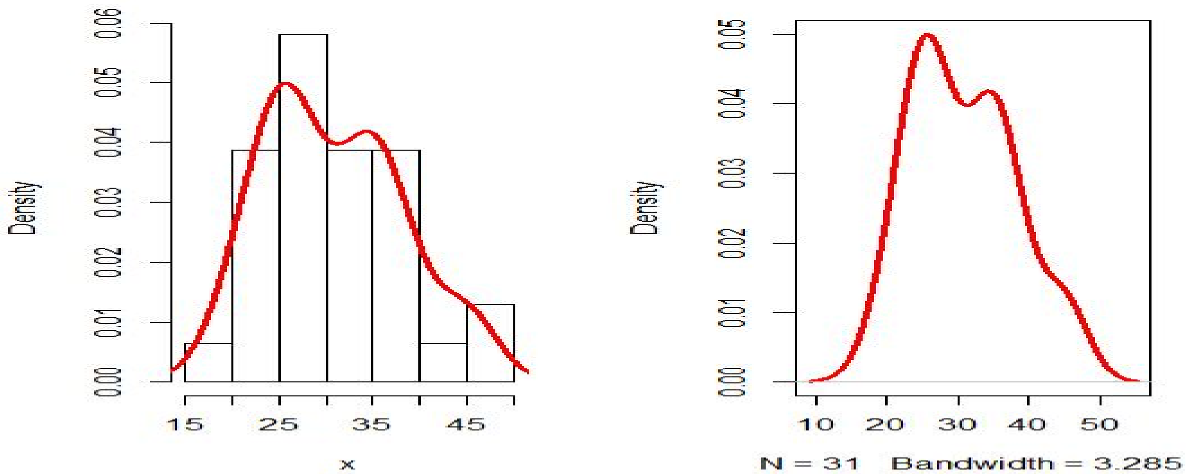

Figure 4.2: A Histogram and density plot for Data set II

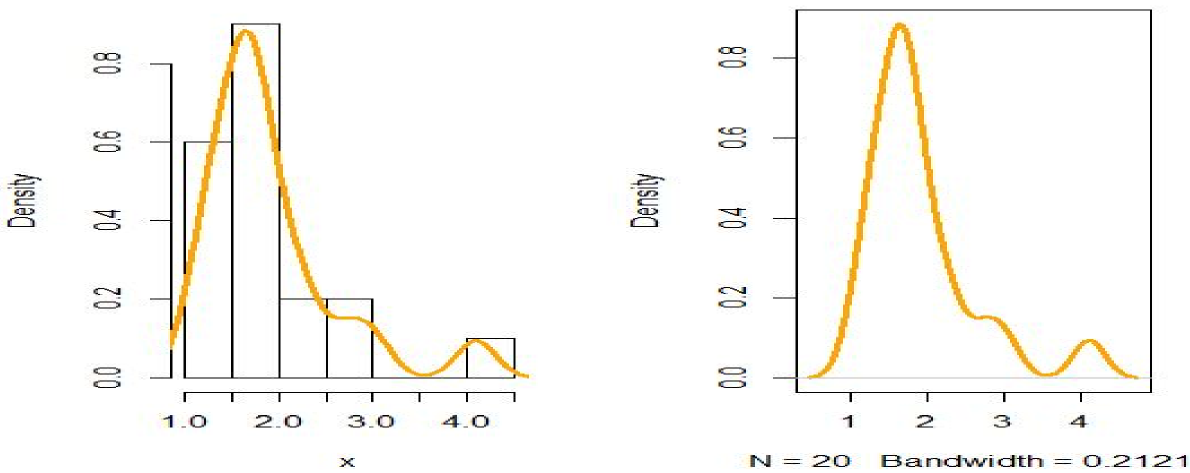

Figure 4.3: A Histogram and density plot for Data set III

From the descriptive statistics in tables 4.3 and the histograms and densities shown above in figures 4.1, 4.2 and 4.3 for the three data sets respectively, we observed that the three data sets are positively skewed; however, the third data set has a higher skewness coefficient followed by the first and then the second with a very low peak.

4.3 Information Criteria for Comparison of Distributions

For us to fit and assess the performance of the models listed above, we made use of some criteria: the *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (Hannan Quin information criterion). The formulas for these statistics are given as follows:

$$AIC = -2\mathrm{L} + 2k$$

$$BIC = -2\mathrm{L} + k\log(n),$$

$$CAIC = -2\mathrm{L} + \frac{2kn}{(n-k-1)}$$

$$HQIC = -2\mathrm{L} + 2k\log\left[\log(n)\right]$$

Where *L* denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

The R Software will be used for analysis of data.

Decision bench mark: The model with the lowest values of these statistics would be chosen as the best model to fit the data.

4.4 Results

Table 4.5: *MLEs*, L, *AIC*, *CAIC*, *BIC and HQIC* values of the models for data set I

Distributions	*MLEs*	L	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models performance
LGD	$\hat{\theta}$=0.2593 $\hat{\gamma}$=0.4411 $\hat{\alpha}$=2.3755 $\hat{\beta}$=3.1367	347.8476	703.6952	704.1162	714.1159	707.9126	**1**
TGD	$\hat{\theta}$=0.1950 $\hat{\gamma}$=0.0217 $\hat{\lambda}$=-0.1190	365.8488	737.6975	737.9475	745.5130	740.8606	**2**
OGEGD	$\hat{\theta}$=0.0347 $\hat{\gamma}$=0.0063 $\hat{\alpha}$=7.5647 $\hat{\beta}$=1.5793	659.9827	1327.9650	1328.3870	1338.3860	1332.1830	**3**
GGD	$\hat{\theta}$=0.2215 $\hat{\gamma}$=0.0932 $\hat{c}$=0.3262	739.5045	1485.0090	1485.2590	1492.8240	1488.1720	**4**
GD	$\hat{\theta}$=2.0907 $\hat{\gamma}$=0.0433	2894.2880	5792.575	5792.6990	5797.7850	5794.6840	**5**

Table 4.6: *MLEs*, L, *AIC*, *CAIC*, *BIC and HQIC* values of the models for data set II

Distributions	*MLEs*	L	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models performance
LGD	$\hat{\theta}$=0.1808 $\hat{\gamma}$=0.0108 $\hat{\alpha}$=7.0269 $\hat{\beta}$=8.2813	193.1088	394.2177	395.7562	399.9536	396.0875	**1**
GGD	$\hat{\theta}$=0.2824 $\hat{\gamma}$=0.0019 $\hat{c}$=3.0485	281.3734	568.7469	569.6358	573.0489	570.1492	**2**
OGEGD	$\hat{\theta}$=0.0545 $\hat{\gamma}$=0.0373 $\hat{\alpha}$=2.0383 $\hat{\beta}$=0.2229	443.9031	895.8062	897.3447	901.5422	897.6760	**3**
TGD	$\hat{\theta}$=0.5276 $\hat{\gamma}$=0.0122 $\hat{\lambda}$=0.7111	665.7328	1337.4060	1338.3540	1341.7670	1338.8680	**4**
GD	$\hat{\theta}$=0.4463 $\hat{\gamma}$=0.0588	780.4185	1564.837	1564.961	1570.047	1566.946	**5**

Table 4.7: *MLEs*, L, *AIC*, *CAIC*, *BIC and HQIC* values of the models for data set III

Distributions	*MLEs*	L	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models performance
GGD	$\hat{\theta}$=0.9839 $\hat{\gamma}$=0.3899 $\hat{c}$=7.1231	19.2364	44.4729	45.9729	47.4601	45.0559	**1**
TGD	$\hat{\theta}$=0.1472 $\hat{\gamma}$=0.8821 $\hat{\lambda}$=0.1998	24.6575	55.3151	56.8151	58.3023	55.8982	**2**
GD	$\hat{\theta}$=0.2765 $\hat{\gamma}$=0.5845	25.8436	55.6873	56.3932	57.6787	56.0760	**3**
LGD	$\hat{\theta}$=0.2646 $\hat{\gamma}$=1.0598 $\hat{\alpha}$=2.9677 $\hat{\beta}$=8.5964	25.1072	58.2143	60.8809	62.1972	58.9918	**4**
OGEGD	$\hat{\theta}$=0.1094 $\hat{\gamma}$=0.3918 $\hat{\alpha}$=2.9711 $\hat{\beta}$=4.4035	186.5786	381.1572	383.8238	385.1401	381.9347	**5**

4.5 Discussion of Results

Using the values of the parameter MLEs and the corresponding values of L, *AIC*, *BIC*, *CAIC* and *HQIC* for each model as shown in table 4.5, we can understand that the *LGD* performs better with smaller values of the information criteria compared to the performance of the other models which are, the *GGD*, *TGD, OGEGD* and *GD*. The above performance can be traced to the fact that the proposed distribution is heavily skewed to the right with a high peak and the first data set is also positively skewed with a large coefficient of kurtosis.

Table 4.6 presents the parameter estimates to each of the five fitted distributions for data set II, values of *L*, *AIC*, *BIC*, *CAIC* and *HQIC* of the fitted models evaluated at their corresponding *MLEs*. The values in Table 4.6 indicate that the *LGD* has better performance with the lowest values of *AIC*, *CAIC*, *BIC* and *HQIC* followed by the *GGD*, *TGD, OGEGD* and *GD*. Again the reason behind this outperformance is that, the second data set has a low degree of kurtosis and skewness to the right meanwhile, our proposed model has various shapes with both moderate and higher peak all skewed to the right.

Table 4.7 presents the parameter estimates and the values of L, *AIC*, *BIC*, *CAIC* and *HQIC* for the five fitted models for dataset III. However, the values in the above table show that the *GGD* has better performance with the lowest values of *AIC*, *CAIC*, *BIC* and *HQIC* compared to the other four models including the proposed distribution. This exceptional case could be attributed to the fact that the third dataset has a small sample size.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

This dissertation proposes a new probability distribution named a Lomax-Gompertz distribution. Some properties of the proposed distribution such as its ordinary moments, moment generating function, characteristics function, reliability analysis, quantile function and order statistics are being derived and studied adequately. The estimation of parameters of the new model has been done using the method of maximum likelihood estimation. The performance of the new distribution is tested by means of some applications to three real life data sets with quality recommendations.

5.2 Conclusions

This dissertation introduced a new distribution called Lomax-Gompertz distribution. It studied some mathematical and statistical properties of the proposed distribution with some graphical demonstration appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done effectively. Some plots of the distribution revealed that it is positively skewed and its degree of kurtosis depends on the values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Lomax-Gompertz distribution could be used to model time-dependent events or variables whose survival decreases as time grows or where survival rate decreases with time. The results of the three applications showed that the proposed distribution performs better than some extensions of the Gompertz distribution however, depending on the nature of the datasets. It was revealed that this new distribution has better performance for positively skewed data sets with larger sample sizes.

5.3 Recommendations

We recommend based on the findings of this dissertation that the proposed distribution should be used to model positively skewed datasets with large sample sizes. We also recommend that the new model should be used for analyzing time or age dependent variables based on the behavior of the survival and the hazard functions.

5.4 Contribution to Knowledge

In this research, we have proposed a new distribution which has not been carried out before. The distribution is useful for modeling positively skewed data sets with different peaks and large sample sizes. Expressions for some of its properties which are useful for many applications have been derived and studied effectively. The model parameters have been estimated adequately by using the method of Maximum Likelihood Estimation.

This dissertation has compared the performance of the new distribution with some existing generalizations of the parent distribution and the parent distribution itself and it was discovered that it is better than other models mostly when dealing with positively skewed datasets with large sample sizes.

5.5 Areas of Further Research

This study dwelt much on the mathematical properties of the new distribution with emphasis on a single classical method of estimation, subsequent studies should deal with other methods of estimation of the parameters of the model, classical or non-classical methods. Also estimation of confidence intervals for the parameters of the proposed distribution can be carried out. There is need for further studies on the effects of small sample size on the proposed distribution. We also call for more applications of the proposed distribution based on our recommendations.

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