ON ALGEBRAIC PROPERTIES OF FUNDAMENTAL GROUP

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DECLARATION

I hereby declare that this work is the product of my own research efforts, undertaken under the supervision of AMINA MUHAMMAD LAWAN (Ph.D) and has not been presented to the best of my knowledge elsewhere for the award of a degree or certificate. All sources have been duly acknowledged.

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CERTIFICATION

This is to certify that the research work for this dissertation and the subsequent write-up is for LAARO ABDULLATEEF with Registration Number SPS/14/MMT/00025 and were carried out under my supervision.

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APPROVAL

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DEDICATION

This work, is humbly dedicated to my parents.

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NOTATIONS

- X - Topological space
- p - Point on X
- (X,p) - Pointed topological space
- $\Omega(X,p)$ - Collection of paths in X
- [] - Equivalence class
- $\pi_1(X,p)$ - Collection of equivalence classes of loop in X (known as fundamental group).
- $\Omega(X,p) \times \Omega(X,p)$ - Cartesian product of $\Omega(X,p)$
- $\pi_1(U,p)$ - Subgroup of $\pi_1(X,p)$ (where U is a subspace of X)
- $[f](\pi_1(U,p))$ and $(\pi_1(X,p))[f]$ - left and right coset group, respectively
- $Z(\pi_1(X,p))$ - Ceter of $\pi_1(X,p)$
- $\pi_1(X,p)/\pi_1(X,p)$ - Quotient group
- $*$ - Operation defined on $\Omega(X,p)$
- $\cdot$ - Operation defined on $\pi_1(X,p)$

ABSTRACT

The notion of some basic algebraic properties of a fundamental group were introduced through the study of the algebraic properties of group structure. We give a necessary and sufficient condition for a fundamental group to be abelian, for its subset to be subgroup, for its subgroup to be normal and for an element to be in its center. We also describe the set of centralizers of an element in a fundamental group and the quotient fundamental group.

Those necessary and sufficient conditions were given in form of theorems which expatiate the significance of the study. The theorems explain that some basic algebraic properties that a general group satisfied can also be satisfied by a specific group known as fundamental group.

The scope of the study doesn't encompass some algebraic properties like homomorphism theorems, internal and external direct product and group actions in a fundamental group, therefore further research can be carried out on them.

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION

This chapter gives general overview of the algebraic properties of group. It contains among other things, the background of the study, motivation of the study, the aim and objectives of the study, scope and limitations and definitions of some basic terms.

1.2 BACKGROUND OF THE STUDY

One of the simplest and most basic of all algebraic structures is the group. A group is described to be a set with an operation (let us call it $*$) which is closed, associative, has a identity element (sometimes called neutral element), and for which each element has an inverse.

Group is usually represented by the symbol $(G,*)$. This notation makes it explicit that the group consists of the set G and the operation $*$. In general, there are other possible operations on G, so it may not always be clear which one is the group's operation unless we indicate it. If there is no danger of confusion, the group is simply denoted with the letter G [4].

The groups which come to mind most readily are found in the familiar number systems. Here are a few examples. $\mathbb{Z}$ is the symbol customarily used to denote the set $\{...,-3,-2,-1,0,1,2,3,...\}$ of the integers. The set $\mathbb{Z}$, with the operation of addition, is obviously a group. It is called the additive group of the integers and is represented by the symbol $(\mathbb{Z},+)$, mostly simply denoted by the symbol $\mathbb{Z}$ [31].

$\mathbb{Q}$ designates the set of the rational numbers (that is, quotients of the form m/n of integers, where n, m are integers and n$\neq$0). This set, with the operation of addition, is called the additive group of the rational numbers, $(\mathbb{Q},+)$, mostly simply denoted by the symbol $\mathbb{Q}$ [31].

The symbol $\mathbb{R}$ represents the set of the real numbers. $\mathbb{R}$, with the operation of addition, is called the additive group of the real numbers, and is represented by $(\mathbb{R},+)$, or simply $\mathbb{R}$. The set of all the nonzero rational numbers is represented by $\mathbb{Q}^*$. This set, with the operation of multiplication, is the group $(\mathbb{Q}^*,\cdot)$, or simply $\mathbb{Q}^*$. Similarly, the set of all the nonzero real numbers is represented by $\mathbb{R}^*$. The set $\mathbb{R}^*$ with the operation of multiplication, is the group $(\mathbb{R}^*,\cdot)$, or simply $\mathbb{R}^*$ [31].

Also, $\mathbb{Q}^+$ denotes the group of all the positive rational numbers, with multiplication. $\mathbb{R}^+$ denotes the group of all the positive real numbers, with multiplication [31].

Groups occur abundantly in nature. This statement means that a great many of the algebraic structures which can be discerned in natural phenomena turn out to be groups. Groups are also important because they happen to be one of the fundamental building blocks out of which more complex algebraic structures are made. Especially important in scientific applications are the finite groups, that is, groups with a finite number of elements. It is not surprising that such groups occur often in applications, for in most situations of the real world we deal with only a finite number of objects. The easiest finite groups introduced are those called the groups of integers modulo n (where n is any positive integer greater than 1) [4].

The basic algebraic properties of group includes the monoid, semigroup, groupoid, abelian group, subgroup, center of a group, centralizers of an element in a group, normal subgroup, quotient group, e.t.c.

Algebraic topology can be roughly described as the study of techniques for forming algebraic images of topological spaces. Most often these algebraic images are groups, but more elaborate structures such as rings and modules also arise. The mechanisms that create these images (the lanterns of algebraic topology, one might say) are known formally as functors and have the characteristic feature that they form images not only of spaces but also of maps. Thus, continuous maps between spaces are projected onto homomorphisms between their algebraic images, so topologically related spaces have algebraically related images [1].

With suitably constructed lanterns one might hope to be able to form images with enough detail to reconstruct accurately the shapes of all spaces, or at least of large and interesting classes of spaces. This is one of the main goals of algebraic topology, the lanterns necessary to do this are somewhat complicated pieces of machinery having a certain intrinsic beauty [1].

One of the simplest and most important functors of algebraic topology is fundamental group, which creates an algebraic image of a space from the loops in the space, the paths in the space starting and ending at the same point [14].

This fundamental group can be referred to as an algebraic invariant used to differentiate between topological spaces. By detecting holes in a topological space, the fundamental group of a space gives information about that space's basic structural characteristics [18].

As it is generally known that showing whether two given topological spaces are homeomorphic or not is one of the basic problems of topology. In the literature review, we have not come across any known method for solving this problem in general, but techniques do exist that apply in particular cases [1].

Showing that two spaces are homeomorphic is a matter of constructing a continuous mapping from one to the other having a continuous inverse. Showing two spaces are not homeomorphic is a different matter. For that one must show that a continuous function with continuous inverse does not exist. If one can find some topological properties that hold for one space but not for the other, then the problem is solved i. e., the spaces can not be homeomorphic. The closed interval $[0,1]$ can not be homeomorphic to the open interval $(0,1)$, for instance, because the first space is compact and the second one is not. The real line $\mathbb{R}$ can not be homeomorphic to the "long line" L, because $\mathbb{R}$ has compatible basis and L does not. Nor can real line $\mathbb{R}$ be homeomorphic to the plane $\mathbb{R}^2$; deleting a point from $\mathbb{R}^2$ leaves a connected space remaining, and deleting a point from $\mathbb{R}$ does not [1].

But the topological properties do not provides solution to this problems in general. For instance, how does one show that the plane $\mathbb{R}^2$ is not homeomorphic to three-dimensional space $\mathbb{R}^3$? As one goes down the list of topological properties (compactness, connectedness, local connectedness, metrizability, and so on) one can find no topological property that distinguishes between them. As an example, consider such surfaces as the 2-sphere S^2, the torus T (surface of a doughnut), and the double torus $T\sharp T$; neither of them is simply connected [18].

There is an idea more general than the idea of simple connectedness, an idea that include simple connectedness as a special case. It involves certain group that is called fundamental group of the space. Two spaces that are homeomorphic have fundamental groups that are isomorphic. And the condition of simple connectedness is just the condition that the fundamental group of X is the trivial (one-element) group. Thus the proof that S^2 and T are not homeomorphic can be rephrased by saying that the fundamental group of S^2 is trivial and the fundamental group of T is not. The fundamental group will distinguish between more spaces than the condition of simple connectedness will. It can be used, for example, to show that T and $T\sharp T$ are not homeomorphic; it turns out that T has an abelian fundamental group and $T\sharp T$ does not [1].

Fundamental group is applicable in solving many problems, including the problem of showing that various spaces, such as those already mentioned, are not homeomorphic. Other applications include theorems about fixed points and antipode-preserving maps of the sphere, as well as the well-known fundamental theorem of algebra, which says that every polynomial equation with real or complex coefficients has a root, and also, the famous Jordan curve theorem, which says that every simple closed curve C in the plane separates the plane into two components, of which C is the common boundary [18].

Let $\Omega(X,p)$ be the collection of paths in a topological space X based at point p. Take $f,g \in \Omega(X,p)$ such that $f(1) = g(0)$. Then the concatenation $f * g$ that transverses first f and then g which is defined as:

$$f * g(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{1.2.1}$$

explains the operation on fundamental group $\pi_1(X,p)$. Since the nature of an element in fundamental group can be refered to as the equivalent class of loop (say $[f]$) for every loop $f \in \Omega(X,p)$. The above defines the concatenation map $*: \Omega(X,p) \times \Omega(X,p) \to \Omega(X,p)$ by $(f*g)(t) = f(t)*g(t)$ such that the operation on fundamental group will be defined as $[f*g] = [f]\cdot[g]$ [1].

1.3 MOTIVATION OF THE STUDY

We are motivated by the work of Hatcher [1] who shows that the collection $\pi_1(X,p)$ of equivalence classes of loops with respect to its operation forms a group called fundamental group. His major interest is to study its topological properties. However, algebraic properties have not been covered in the existing literatures. In this dissertation we initiate the study of some basic algebraic properties of this group.

1.4 AIM AND OBJECTIVES

Aim: The study is aimed at characterizing the basic algebraic properties of a fundamental group through the basic algebraic properties of a general group structure.

Objectives: The objectives of this study are to:

(i) give a necessary and sufficient condition for a fundamental group to be abelian;

(ii) give some necessary and sufficient conditions for a subset of fundamental group to be subgroup;

(iii) give a necessary and sufficient conditions for a subgroup of fundamental group to be normal;

(iv) describe the set of centralizers of an element in a fundamental group;

(v) give a necessary and sufficient condition for an element to be in a center of fundamental group;

(vi) describe the quotient fundamental group.

1.5 SCOPE

The research is focused on the study of the basic algebraic properties of fundamental group, these properties includes the abelian, subgroups, centers, quotients and centralizer of an element in a fundamental group.

1.6 DEFINITIONS OF SOME BASIC TERMS

For proper understanding of this work, we give the definitions of some basic terms, which are standard and can be found in [1, 4, 14, 28, 31].

Definition 1.6.1 *A **set** is a well defined collection of objects.*

Definition 1.6.2 *Let G be a set and $*$ be the operation defined on G. Then $(G, *)$ is a **group** if the following axioms are satisfied:*

(G1) G is closed;

(G2) $*$ is associative;

(G3) there is an element e in G such that $a*e = a$ and $e*a = a$ for every a in G;

(G4) for every element a in G, there is an element a^{-1} in G such that $a*a^{-1} = e$ and $a^{-1}*a = e$.

Definition 1.6.3 *A nonempy subset H of G is called a* ***subgroup*** *of G if H itself forms a group with respect to the operations defined on G.*

Definition 1.6.4 *Let $(G,*)$ be a group. Then $(G,*)$ is said to be* ***abelian*** *if it is commutative i.e., $x*y = y*x$ for all x and y in G.*

Definition 1.6.5 *Let N be a subgroup of a group G, then*

(i) for all $g \in G$, the subset gN of G is called the ***left coset*** *of G determined by g i.e., $gN = \{gn : n \in N\}$.*

(ii) for all $g \in G$, the subset Ng of G is called the ***right coset*** *of G determined by g i.e., $Ng = \{ng : n \in N\}$.*

Definition 1.6.6 *Let N be a subgroup of a group G. Then N is said to be* ***normal*** *if the left coset is equal to the right coset. It is denoted by $N \trianglelefteq G$.*

Definition 1.6.7 *Let N be a normal subgroup of a group G. The* ***quotient group*** *of G by N is a set G/N defined by $G/N = \{gN : g \in G\}$ with the operation $(g_1N)(g_2N) = g_1g_2N$ for all g_1 and g_2 in G.*

Definition 1.6.8 *A* ***topology*** *on a set X is a collection τ of subsets of X having the following properties:*

(1) $\emptyset$ and X are in τ;

(2) The arbitrary union of any elements of τ is an element in τ;

(3) The finite intersection of elements of τ is an element in τ.

The pair (X, τ) *is called* ***topological space***.

Definition 1.6.9 *A subset U of a topological space X is called an* ***open set*** *of X if U belongs to* τ. *A complement of an open set is a* ***closed set***.

Definition 1.6.10 *A* ***function*** $f : X \to Y$ *from a set X to a set Y is a rule that assign to each* $x \in X$ *a unique* $f(x) \in Y$, *known as the value of f at x.*

Definition 1.6.11 *Let X and Y be topological spaces. A function* $f : X \to Y$ *is said to be* ***continuous*** *if for each closed subset C of Y, the set* $f^{-1}(C)$ *is a closed subset of X.*

Definition 1.6.12 *Let X and Y be topological spaces and I be the interval* $[0,1]$. *Two continuous functions* $h, i : X \to Y$ *are said to be* ***homotopic*** *(or h is said to be homotopic to i), if there exists a continuous function* $H : X \times I \to Y$ *such that* $H(x,0) = h(x)$ *and* $H(x,1) = i(x)$, $\forall x \in X$. *The map H is called homotopy between h and i, written* $H : h \simeq i$.

Definition 1.6.13 *Let X be a topological space and I be the interval* $[0,1]$. *A continuous function* $f : I \to X$ *such that* $f(0) = x_0$ *and* $f(1) = x_1$, *is called* ***path*** *in X from* x_0 *to* x_1. *The point* x_0 *is called the initial point and the point* x_1 *is called final or terminal point of the path f.*

Definition 1.6.14 *A path f in a topological space X is called a* ***loop*** *if the initial and terminal points p are the same. A collection of paths in X is denoted by* $\Omega(X, p)$.

Definition 1.6.15 *Two paths* $f, g : I \to X$ *are said to be* ***homotopic*** *if they have the same initial point* x_0, *the same final point* x_1 *and there exists a continuous map* $F : I \times I \to X$ *such that* $F(t,0) = f(t)$, $F(t,1) = g(t)$, $\forall t \in I$ *and* $F(0,s) = x_0$, $F(1,s) = x_1$ $\forall s \in I$. *F is called the path homotopy between f and g, written* $F : f \simeq g$.

Definition 1.6.16 *Let X be a non-empty set. A binary relation or simply a* ***relation*** *R is a subset of the cartesian product* $X \times X$ *of X such that* $(x,y) \in R$ *means x is related to y, sometimes written as xRy* $\forall\, x, y \in X$.

Definition 1.6.17 *A relation R on a set X is said to be an* ***equivalence relation*** *if the following conditions are satisfied.*

a. *For all $x \in X$, $(x,x) \in R$ (i.e., R is reflexive).*

b. *For all $x,y \in X$, if $(x,y) \in R$ then $(y,x) \in R$ (i.e., R is symmetric).*

c. *For all $x,y,z \in X$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$ (i.e., R is transitive).*

Definition 1.6.18 *An* ***equivalence class*** *of an element $a \in X$ is the collection of elements in X that are related to a by R i. e., $[a] = \{x \in X : aRx\}$.*

Definition 1.6.19 *Let $\Omega(X,p)$ be the collection of paths. A* ***relation*** *R on $\Omega(X,p)$ is a subset of $\Omega(X,p) \times \Omega(X,p)$ such that $\forall f,g \in \Omega(X,p)$, $(f,g) \in R$ means there is a homotopy between f and g.*

Definition 1.6.20 *The* ***equivalence class of*** *$f \in \Omega(X,p)$ denoted as $[f]$ is the set $\{[g] \in \pi_1(X,p) : (f,g) \in R\}$.*

Definition 1.6.21 *The collection of equivalence classes of loops f i. e.,*

$$\pi_1(X,p) = \{[f] : f \in \Omega(X,p)\}$$

is a group called ***fundamental group****.*

Definition 1.6.22 *Let $\Omega(X,p)$ be the set of all paths in a topological space X, take $f,g \in \Omega(X,p)$ such that $f(0) = g(1)$. Then the* ***concatenation*** *of f and g is the map $* : \Omega(X,p) \times \Omega(X,p) \to \Omega(X,p)$ defined by*

$$f * g(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{1.6.1}$$

Definition 1.6.23 *The* ***operation*** *$\cdot$ on the collection $\pi_1(X,p)$ is defined as*

$$[f] \cdot [g] = [f * g] \ \forall f,g \in \Omega(X,p) \text{ and } \forall \ [f],[g] \in \pi_1(X,p)$$

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, we present review of the related literatures on algebraic properties of a group and further discuss some existing literature on the fundamental group.

2.2 HISTORICAL BACKGROUND OF GROUP THEORY

Although the first clear axiomatic definition of a group was not given until the late 1800s, group-theoretic methods had been employed before this time in the development of many areas of mathematics, including geometry and the theory of algebraic equations [28].

Joseph-Louis Lagrange used group-theoretic methods in a 1770-1771 memoir to study methods of solving polynomial equations. Later, Evariste Galois (1811-1832) succeeded in developing the mathematics necessary to determine exactly which polynomial equations could be solved in terms of the polynomials coefficients. Galois primary tool was group theory [28].

The study of geometry was revolutionized in 1872 when Felix Klein proposed that geometric spaces should be studied by examining those properties that are invariant under a transformation of the space. Sophus Lie, a contemporary of

Klein, used group theory to study solutions of partial differential equations. One of the first modern treatments of group theory appeared in William Burnsides Theory of Groups of Finite Order, first published in 1897 [28].

Clark in 2001, described the abelian property of group to be the commutative property. He explained the condition that a particular group must satisfy before being abelian. Clark gave many examples of non-abelian groups to explain when a group is said to be non-abelian [31].

Pinter in 2010, explained when a subset of a particular group is said to be a subgroup of such group. He explained the effect of a group's operation on its subgroup and gives series of examples on it [4].

2.3 HISTORICAL BACKGROUND OF FUNDAMENTAL GROUP

Fundamental group was first introduced by Poincare in 1892. This was done when he raises the question whether the Betti numbers suffice to determine the topological type of a manifold, and introduces the fundamental group to further illuminate this question. He gave a family of three-dimensional manifolds, obtained as quotients of $\mathbb{R}^3$ by certain groups with a cube as fundamental region, and showed that certain of these manifolds have the same Betti numbers but different fundamental groups. It follows, assuming that the fundamental group is a topological invariant, that the Betti numbers do not suffice to distinguish three-dimensional manifolds [18].
In Analysis situs, Poincare develops these ideas in several directions.

1. He attempted to provide a new foundation for the Betti numbers in a rudimentary homology theory, which introduces the idea of computing with topological objects (in particular, adding, subtracting, testing for linear independence). As Scholz (1980), explained that the first phase of algebraic topology, inaugurated by Poincare, is characterized by the fact that

its algebraic relations and operations always deal with topological objects (sub-manifolds).

2. Using his homology theory, he discovers a duality theorem for the Betti numbers of an n-dimensional manifold:

$$P_m = P_{n-m} \text{ for } m = 1, 2, ..., n-1.$$

In words: ”the Betti numbers equidistant from the ends are equal.” He later called this the fundamental theorem for Betti numbers .

3. He generalizes the Euler polyhedron formula to arbitrary dimensions and situates it in his homology theory.

4. He constructs several three-dimensional manifolds by identifying faces of polyhedra, observing that this leads natural presentations of their fundamental groups by generators and relations.

5. Recognizing that the fundamental group first becomes important for three-dimensional manifolds, Poincare asks whether it suffices to distinguish between them. He was not able to answer this question [18].

Analysis situs is rightly regarded as the origin of algebraic topology, because of Poincare's construction of homology theory and the fundamental group. The fundamental group is the more striking of the two, because it is a blatantly abstract structure and generally non-commutative, yet surprising easy to grasp via generators and relations [18].

After the introduction and development of fundamental group, below are some related literatures.
Salleh and Tap in 1986, introduced the concept of the fundamental group of fuzzy topological spaces based on the definition of fuzzy topology defined by Chang in 1968. They gave the definitions of fuzzy path, fuzzy loop, fuzzy homotopy and explained that the fundamental group of fuzzy topological spaces is the collection of equivalence classes of fuzzy loops. [2].

Salleh and Tap in 1987, introduced the concept of the fundamental groupoid of fuzzy topological spaces based on the definition of fuzzy topology defined by Chang in 1968. He described this as an extension of the above study (the fundamental group of fuzzy topological spaces) [3].

Hatchat in 2002, explained how the collection $\pi_1(X,p)$ form a group called a fundamental group. He defined the path, loop and homotopy which makes the group axioms satisfied [1].

Guner and Balci in 2007, studied on fuzzy sheaf of the fundamental groups. They found out that in constructing the fuzzy sheaf of fundamental group of a space X, there is a covariant functor from category of fuzzy path connected topological spaces and fuzzy continuous mappings to the category of fuzzy sheaves and fuzzy sheaf homomorphism [8].

Fabel in 2007, also studied the fundamental group by considering the metric space with discrete topological fundamental group. He used the related material to offer a counter example to a similar result for a theorem that explained that "with a certain natural topology, the fundamental group of locally path connected metric space X becomes discrete if and only if f is semilocally simply connected" [25].

Saito and Ishibe in 2011, were able to solve some decision problems for the fundamental group, as an application to the study of monoids [19].

Calcut, Gompt and McCarthy in 2012, studied the fundamental group of quotient spaces as presented. They revealed a dual property for certain maps having connected fibers, with applications to orbit spaces of vector field and leaf spaces in general [15].

Duckets, Everaert and Gran in 2012, also carried out a research on description of fundamental group in terms of commutators and closure operators. Their study made it possible to calculate the fundamental groups corresponding to many interesting reflections arising, for instance, in the category of groups, rings, compact group and simplical loops [23].

Az-zo'bi et al. in 2014, introduced fundamental group of intuitionistic fuzzy topological spaces by considering the intuitionistic fuzzy set and intuitionistic fuzzy topological spaces introduced by Atanassove in 1983 and Coker in 1997 respectively. They defined the intuitionistic fuzzy path, loop and homotopy and later introduced fundamental group of intuitionistic fuzzy topological spaces [7].

Madhuri and Amudambigai in 2017, established the concept of fundamental group of fuzzy $\Im^*$-structure by introducing the concepts of fuzzy $\Im^*$-structure homotopy and fuzzy $\Im^*$-structure path homotopy [28].

Remark: As we have seen in all the literatures above, researchers have focused most of their attention on the topological properties of fundamental group. Not much has been done on its algebraic properties. Apart from the work of Hatchat in [1], who shows that fundamental group is indeed a group, other algebraic properties are yet to be in the pipeline in the literature.

CHAPTER THREE

STRUCTURE OF FUNDAMENTAL GROUP

3.1 INTRODUCTION

In this chapter, we present some relevant lemmas and propositions together with proofs that give more explanation on fundamental group.

3.1.1 Gluing Lemma

Lemma 3.1.1 *[6] Let X be a topological space and A, B be closed subsets of X such that X = A $\cup$ B. Let Y be topological space and f : A $\to$ Y and g : B $\to$ Y be continuous maps. If f(x) = g(x) $\forall x \in A \cap B$, then the function h : X $\to$ Y defined by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases} \tag{3.1.1}$$

is continuous.

Proof We are going to show the following;

i. h is well-defined function;

ii. h is continuous.

For (i): Let $x_1, x_2 \in A$.
Suppose $x_1 = x_2$, since f is continuous, $f(x_1) = f(x_2)$

which implies $h(x_1) = h(x_2)$
Let $x_1, x_2 \in B$
Suppose $x_1 = x_2$, since g is continuous, $g(x_1) = g(x_2)$
which implies $h(x_1) = h(x_2)$
Suppose $x_1 \in A$ and $x_2 \in B$ such that $x_1 = x_2$
then $x_1 = x_2 \in A \cap B$
which implies $f(x_1) = g(x_1) = g(x_2) = f(x_2)$
and $h(x_1) = f(x_1) = g(x_1) = g(x_2) = f(x_2) = h(x_2)$. Hence the result follows.
For (ii): Let C be closed in Y and $X = A \cup B$.

$$\begin{aligned} h^{-1}(C) &= X \cap h^{-1}(C) \\ &= (A \cup B) \cap h^{-1}(C) \\ &= (A \cap h^{-1}(C)) \cup (B \cap h^{-1}(C)) \\ &= (A \cap f^{-1}(C)) \cup (B \cap g^{-1}(C)) \\ &= f^{-1}(C) \cup g^{-1}(C) \end{aligned}$$

Since each f and g are continuous, $f^{-1}(C)$ and $g^{-1}(C)$ are both closed in X. Hence $h^{-1}(C)$ is closed in X. Consequently, h is continuous. ■

3.1.2 Equivalence Relation on $\Omega(X, p)$

Recall: Let $R \subseteq \Omega(X, p) \times \Omega(X, p)$. Where R is defined as $(\forall f, g \in \Omega(X, p))$ $(f, g) \in R$ if and only if there is a homotopy between f and g.

Proposition 3.1.2 *[1] The relation R is an equivalence relation.*

Proof

1. Reflexivity. We will show that $(f, f) \in R$ i.e., there is homotopy between f and f.
   Define a function $H : I \times I \to X$ by $H(t, s) = f(t)$. Then
   - H is continuous since f is continuous
   - $f(0) = f(0)$ and $f(1) = f(1)$

- $H(t,0) = f(t)$ and $H(t,1) = f(t) \forall t \in I$
- $H(0,s) = f(0)$ and $H(1,s) = f(1) \forall s \in I$

Hence $(f,f) \in R$.

2. Symmetric. Suppose $(f,g) \in R$, this implies there is homotopy between f and g i.e there exists a continuous function $H : I \times I \to X$ such that

   - $f(0) = g(0)$ and $f(1) = g(1)$
   - $H(t,0) = f(t)$ and $H(t,1) = g(t) \forall t \in I$
   - $H(0,s) = f(0)$ and $H(1,s) = g(1) \forall s \in I$

   Define a function $\bar{H} : I \times I \to X$ by $\bar{H}(x,t) = H(x, 1-s)$. Then

   - $\bar{H}$ is continuous since H is continuous
   - $\bar{H}(t,0) = H(t,1) = g(t) \forall t \in I$
   - $\bar{H}(t,1) = H(t,0) = f(t) \forall t \in I$
   - $\bar{H}(0,s) = H(0,1-s) = g(0) \forall s \in I$
   - $\bar{H}(1,s) = H(1,1-s) = f(1) \forall s \in I$

   therefore $\bar{H}$ is an homotopy between g and f. Hence, $(g,f) \in R$

3. Transitivity. Suppose (f,g) *and* $(g,h) \in R$, this implies there is homotopy between f and g, g and h i. e., there exists continuous functions
   $H_1 : I \times I \to X$ and $H_2 : I \times I \to X$ such that;

   - $f(0) = g(0)$ and $f(1) = g(1)$
   - $H_1(t,0) = f(t)$ and $H_1(t,1) = g(t) \forall t \in I$
   - $H_1(0,s) = f(0)$ and $H_1(1,s) = g(1) \forall s \in I$

   and

   - $g(0) = h(0)$ and $g(1) = h(1)$
   - $H_2(t,0) = g(t)$ and $H_2(t,1) = h(t) \forall t \in I$
   - $H_2(0,s) = g(0)$ and $H_2(1,s) = h(1) \forall s \in I$

Define a function $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} H_1(t,2s), & \text{if } 0 \leq s \leq \frac{1}{2} \\ H_2(t,2s-1), & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{3.1.2}$$

where $H_1(t,1) = H_2(t,0)$. Then H is homotopy between f and h i. e.,

- H is continuous by lemma 3.1.1
- $f(0) = g(0) = h(0)$ *and* $f(1) = g(1) = h(1)$
- $H(t,0) = H_1(t,0) = f(t)$ and $H(t,1) = H_2(t,1) = h(t) \forall t \in I$
- $H(0,s) = H_1(0,2s) = f(0)$ and $H(1,s) = H_2(1,2s-1) = h(1) \forall s \in I$

Therefore $(f,h) \in R$ ■

3.1.3 The Group $\pi_1(X,p)$

Proposition 3.1.3 *[1] The set $\pi_1(X,p)$ together with the operation $\cdot$ as defined above is a group called fundamental group.*

Lemma 3.1.4 *The set $\pi_1(X,p)$ is closed under the operation '$\cdot$' .*

Proof Let $\Omega(X,p)$ be the collection of all loops in X and $\pi_1(X,p)$ be the collection of all equivalence classes of loop in X. Let $R \subseteq \Omega(X,p) \times \Omega(X,p)$ be equivalence relation define on $\Omega(X,p)$.

Goal: The goal is to show that there is $[f] \in \pi_1(X,p)$ such that $[g] \cdot [h] = [f]$ $\forall [g],[h] \in \pi_1(X,p)$.

To show this it is suffices to show that $((g*h),f) \in R \; \forall f,g,h \in \Omega(X,p)$.
Consider the function f defined as

$$f(t) = \begin{cases} g(0), & \text{if } t = 0 \\ h(0), & \text{if } 0 \leq t \leq 1 \end{cases} \tag{3.1.3}$$

and the concatenation of g and h as

$$g*h(t)=\begin{cases} g(2t), & \text{if } 0\leq t\leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1 \end{cases} \tag{3.1.4}$$

where $g(1)=h(0)$. Then the function $H:[0,1]\times[1,0]\to X$ defined by

$$H(t,x)=\begin{cases} g((2t)(1-x)), & \text{if } 0\leq t\leq \frac{1-x}{2} \\ h((2t-1)(1-x)), & \text{if } \frac{1-x}{2}\leq t\leq 1 \end{cases} \tag{3.1.5}$$

is homotopy between f and $(g*h)$ i. e., H is continuous by Lemma 3.1.1,

$$H(t,0)=\begin{cases} g(2t), & \text{if } 0\leq t\leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1 \end{cases} \tag{3.1.6}$$

$H(t,1)=f(t), H(0,x)=(g*h)(0)$ and $H(1,x)=f(t)$. Therefore $((g*h),f)\in R$.
■

Lemma 3.1.5 *The operation $\cdot$ on the set $\pi_1(X,p)$ is associative.*

Proof Let $\Omega(X,p)$ be the collection of all loops in X and $[f],[g]$ and $[h]\in\pi_1(X,p)$.

Goal: The goal is to show that $([f]\cdot[g])\cdot[h]=[f]\cdot([g]\cdot[h])$.

To show this, it is suffices to show that $(((f*g)*h),(f*(g*h)))\in R$ $(\forall f,g,h\in\Omega(X,p))$.
Since $[(f*g)*h]=([f]\cdot[g])\cdot[h]$ and $[f*(g*h)]=[f]\cdot([g]\cdot[h])$, consider
$(f*g)*h=u*h$, where $u=f*g$

$$u*h(t)=\begin{cases} u(2t), & \text{if } 0\leq t\leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1 \end{cases} \tag{3.1.7}$$

where

$$u(2t)=\begin{cases} f(2(2t)), & \text{if } 0\leq 2t\leq \frac{1}{2} \\ g(2(2t)-1), & \text{if } \frac{1}{2}\leq 2t\leq 1 \end{cases} \tag{3.1.8}$$

i. e.,

$$u(2t) = \begin{cases} f(4t), & \text{if } 0 \leq 2t \leq \frac{1}{2} \\ g(4t-1), & \text{if } \frac{1}{2} \leq 2t \leq 1 \end{cases} \tag{3.1.9}$$

then

$$(f*g)*h(t) = \begin{cases} f(4t), & \text{if } 0 \leq t \leq \frac{1}{4} \\ g(4t-1), & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.1.10}$$

where $f(1) = g(0), g(1) = h(0)$.
Also $f*(g*h) = f*v$, where $v = g*h$

$$f*v(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ v(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.1.11}$$

where

$$v(2t-1) = \begin{cases} g(2(2t-1)), & \text{if } 0 \leq (2t-1) \leq \frac{1}{2} \\ h(2(2t-1)-1), & \text{if } \frac{1}{2} \leq (2t-1) \leq 1 \end{cases} \tag{3.1.12}$$

then

$$f*(g*h)(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(4t-2), & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ h(4t-3), & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases} \tag{3.1.13}$$

where $f(1) = g(0), g(1) = h(0)$.
Then we define the homotopy between $(f*g)*h$ and $f*(g*h)$ as
$H : [0,1] \times [1,0] \to X$ by

$$H(t,x) = \begin{cases} f((4t)x + (2t)(1-x)), & \text{if } 0 \leq t \leq (\frac{x}{4} + \frac{1-x}{2}) \\ g((4t-1)x + (4t-2)(1-x)), & \text{if } (\frac{x}{4} + \frac{1-x}{2}) \leq t \leq (\frac{x}{2} + \frac{3(1-x)}{4}) \\ h((2t-1)x + (4t-3)(1-x)), & \text{if } (\frac{x}{2} + \frac{3(1-x)}{4}) \leq t \leq 1 \end{cases} \tag{3.1.14}$$

which implies that

$$H(t,0)=\begin{cases} f(2t), & \text{if } 0\le t\le \frac{1}{2} \\ g(4t-2), & \text{if } \frac{1}{2}\le t\le \frac{3}{4} \\ h(4t-3), & \text{if } \frac{3}{4}\le t\le 1 \end{cases} \tag{3.1.15}$$

and

$$H(t,1)=\begin{cases} f(4t), & \text{if } 0\le t\le \frac{1}{4} \\ g(4t-1), & \text{if } \frac{1}{4}\le t\le \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2}\le t\le 1 \end{cases} \tag{3.1.16}$$

Clearly, H is continuous by Lemma 3.1.1, since $H(t,0)=f*(g*h)(t)$, $H(t,1)=(g*h)*f(t)$ and $H(0,x)=f(0)=h(1)=H(1,x)$ then $(((f*g)*h),(f*(g*h)))\in R$ $(\forall f,g,h\in\Omega(X,p))$.
Thus, the result follows. ■

Lemma 3.1.6 *The element $[\varepsilon_p]\in\pi_1(X,p)$ where ε_p is a constant loop in $\Omega(X,p)$, is an identity element.*

Proof

Goal: We are to show that there exists an identity $[\varepsilon_p]\in\pi_1(X,p)$ such that

$$[\varepsilon_p]\cdot[f]=[f]=[f]\cdot[\varepsilon_p]$$

for all $[f]\in\pi_1(X,p)$.

Let $\varepsilon_p(t)=f(1)=p$ for all $t\in[0,1]$.
To show $[\varepsilon_p]$ is an identity element in $\pi_1(X,p)$, it is suffices to show that:

(i): $(f,(\varepsilon_p*f))\in R$

(ii): $(f,(f*\varepsilon_p))\in R\ \forall\ f\in\Omega(X,p)$

Consider the function f defined from $[0,1]$ to X and let ε_p be a constant function defined from $[0,1]$ to X, then;

(i) The concatenation of ε_p and f is

$$(\varepsilon_p * f)(t) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{3.1.17}$$

Thus, the function $H : [0,1] \times [1,0] \to X$ defined by

$$H(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ f(tx + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{3.1.18}$$

is homotopy between f *and* $(\varepsilon_p * f)$ i. e., H is continuous by Lemma 3.1.1,

$$H(t,0) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.1.19}$$

$H(t,1) = f(t), H(0,x) = (\varepsilon_p * f)(0)$ and $H(1,x) = f(1)$.
Then $(f, (\varepsilon_p * f)) \in R \; \forall f \in \Omega(X,p)$.
(ii) The concatenation of f and ε_p is

$$(f * \varepsilon_p)(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \varepsilon_p(t), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.1.20}$$

thus, the function $K : [0,1] \times [1,0] \to X$ defined by

$$K(t,x) = \begin{cases} f((t)(x) + (2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \leq t \leq 1 \end{cases} \tag{3.1.21}$$

is homotopy between f and $(f * \varepsilon_p)$ i.e., K is continuous by Lemma 3.1.1,

$$K(t,0) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \varepsilon_p(t), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.1.22}$$

$K(t,1) = f(t), K(0,x) = (f * \varepsilon_p)(0)$ and $K(1,x) = f(1)$.
Then $(f, (f * \varepsilon_p)) \in R \; \forall \; f \in \Omega(X,p)$.

Hence $[\varepsilon_p] \cdot [f] = [f] = [f] \cdot [\varepsilon_p]$, implying $[\varepsilon_p]$ is an identity element in $\pi_1(X,p)$.
■

Lemma 3.1.7 *For each $[f] \in \pi_1(X,p)$ there exists $[f^{-1}] \in \pi_1(X,p)$ such that*

$$[f] \cdot [f^{-1}] = [\varepsilon_p] = [f^{-1}] \cdot [f].$$

Proof Let $[f] \in \pi_1(X,p)$.

Goal: we will show that there exists $[f^{-1}] \in \pi_1(X,p)$ such that

$$[f] \cdot [f^{-1}] = [\varepsilon_p] = [f^{-1}] \cdot [f].$$

To prove this, it is suffices to show that

(i): $(\varepsilon_p, (f * f^{-1})) \in R$

(ii): $(\varepsilon_p, (f^{-1} * f)) \in R \forall f \in \Omega(X,p)$

For (i): We define the concatenation of f and f^{-1} as

$$(f * f^{-1})(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{3.1.23}$$

Thus, the function $H : [0,1] \times [1,0] \to X$ defined by

$$H(t,x) = \begin{cases} f((2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ f^{-1}((2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{3.1.24}$$

is a homotopy between ε_p and $(f * f^{-1})$ i. e., H is continuous by Lemma 3.1.1,

$$H(t,0) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \tag{3.1.25}$$

$H(t,1) = \varepsilon_p(t)$, $H(0,x) = (f * f^{-1})(0)$ and $H(1,x) = \varepsilon_p(1)$.
Then $(\varepsilon_p, (f * f^{-1})) \in R \; \forall f \in \Omega(X,p)$.

For (ii): We define the concatenation of f^{-1} and f as

$$(f^{-1} * f)(t) = \begin{cases} f^{-1}(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{3.1.26}$$

Thus, the function $K : [0,1] \times [1,0] \to X$ defined by

$$K(t,x) = \begin{cases} f^{-1}((2t)(1-x)), & \text{if } 0 \le t \le \frac{1+x}{2} \\ f((2t-1)(1-x)), & \text{if } \frac{1+x}{2} \le t \le 1 \end{cases} \tag{3.1.27}$$

is a homotopy between ε_p and $(f^{-1} * f)$ i.e., K is continuous by Lemma 3.1.1,

$$K(t,0) = \begin{cases} f^{-1}(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \tag{3.1.28}$$

$K(t,1) = \varepsilon_p(t)$, $K(0,x) = (f^{-1} * f)(0)$ and $K(1,x) = \varepsilon_p(1)$, then $(\varepsilon_p, (f^{-1} * f)) \in R \; \forall f \in \Omega(X,p)$
Hence $[f^{-1}] \cdot [f] = [\varepsilon_p] = [f] \cdot [f^{-1}]$. ■

In conclusion, Lemma 3.1.4 above implies that $\pi_1(X,p)$ is closed with respect to the operation $\cdot$ defined on it, lemma 3.1.5 implies that the operation $\cdot$ is associative, lemma 3.1.6 implies the existence of identity element in $\pi_1(X,p)$ and lemma 3.1.7 implies the existence of inverse for every element in $\pi_1(X,p)$. Therefore the set $\pi_1(X,p)$ is a group.

CHAPTER FOUR

ALGEBRAIC PROPERTIES OF FUNDAMENTAL GROUP

4.1 INTRODUCTION

This chapter gives a necessary and sufficient condition for a fundamental group to be abelian, for its subset to be subgroup, for its subgroup to be normal and for an element to be in its center. Also, the set of centralizers of an element in a fundamental group and quotient fundamental group will be described.

4.2 NECESSARY AND SUFFICIENT CONDITION FOR A FUNDAMENTAL GROUP TO BE ABELIAN

In this section, we give a necessary and sufficient condition for a fundamental group to be abelian. As it is generally described in [4] that a group G is said to be abelian if $\forall$ a,b $\in$ G, $a*b = b*a$. As an extension of the above many researchers study the conditions for some groups to be abelian. Further applications like; the unitary character group of abelian unipotent groups discovered in [12] and a theorem on the action of abelian unitary groups discussed in [30].
The fundamental group is the more striking of the two, because it is a blatantly abstract structure and generally non-commutative, yet surprisingly easy to grasp via generators and relations [18]. As we have seen in the previous section, the operation in the fundamental group is not direct, and it has been explained by Stillwell in [18] that fundamental group is not generally commutative. Then we

need to give the necessary and sufficient conditions for a fundamental group to be abelian.

Recall: R defined as a subset of $\Omega(X,p)\times\Omega(X,p)$ is an equivalence relation. Thus, throughout the remaining write up, R will denote the equivalence relation on $\Omega(X,p)$.

Theorem 4.2.1 *Let $\pi_1(X,p)$ be a fundamental group, $\Omega(X,p)$ be the collection of all loops in X and R be equivalence relation on $\Omega(X,p)$. Then a fundamental group $\pi_1(X,p)$ is abelian if and only if $(((f*\varepsilon_p)*(\varepsilon_p*g)),((\varepsilon_p*f)*(g*\varepsilon_p)))\in R$ $(\forall f,g\in\Omega(X,p))$, where ε_p is a constant loop in $\Omega(X,p)$.*

Proof $(\Rightarrow)$ Let $\pi_1(X,p)$ be abelian fundamental group and $f,g\in\Omega(X,p)$.

Goal: We will show that $(((f*\varepsilon_p)*(\varepsilon_p*g)),((\varepsilon_p*f)*(g*\varepsilon_p)))\in R$ $(\forall f,g\in\Omega(X,p))$, where ε_p a constant loop in $\Omega(X,p)$.

To show this, it is suffices to show that there exists loop $\sigma\in\Omega(X,p)$ such that:

(i) $(\sigma,((f*\varepsilon_p)*(\varepsilon_p*g)))\in R$

(ii) $(\sigma,((\varepsilon_p*f)*(g*\varepsilon_p)))\in R$

Consider the following concatenations $f*\varepsilon_p$, ε_p*g, $g*\varepsilon_p$ and ε_p*f defined below

$$(f*\varepsilon_p)(t)=\begin{cases} f(2t), & \text{if } 0\le t\le \frac{1}{2}\\ \varepsilon_p(t), & \text{if } \frac{1}{2}\le t\le 1\end{cases} \tag{4.2.1}$$

where $f(1)=\varepsilon_p(0)$,

$$(\varepsilon_p*g)(t)=\begin{cases} \varepsilon_p(t), & \text{if } 0\le t\le \frac{1}{2}\\ g(2t-1), & \text{if } \frac{1}{2}\le t\le 1\end{cases} \tag{4.2.2}$$

where $\varepsilon_p(1)=g(0)$,

$$(g*\varepsilon_p)(t)=\begin{cases} g(2t), & \text{if } 0\le t\le \frac{1}{2}\\ \varepsilon_p(t), & \text{if } \frac{1}{2}\le t\le 1\end{cases} \tag{4.2.3}$$

where $g(1) = \varepsilon_p(0)$,

$$(\varepsilon_p * f)(t) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{4.2.4}$$

where $\varepsilon_p(1) = f(0)$. We define $\varepsilon_p : [0,1] \to X$ by $\varepsilon_p(t) = f(1)$ $\forall t \in [0,1]$.

For (i): The function $H_1 : [0,1] \times [1,0] \to X$ defined by

$$H_1(t,x) = \begin{cases} f(tx + (2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \leq t \leq 1 \end{cases} \tag{4.2.5}$$

is a homotopy between f and $(f * \varepsilon_p)$ i. e., H_1 is continuous by Lemma 3.1.1, $H_1(t,0) = (f * \varepsilon_p)(t)$, $H_1(t,1) = f(t)$, $H_1(0,x) = f * \varepsilon_p(0)$ and $H_1(1,x) = f(1)$, Then $(f, (f * \varepsilon_p)) \in R$.

Also, the function $H_2 : [0,1] \times [1,0] \to X$ defined by

$$H_2(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ g(tx + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{4.2.6}$$

is a homotopy between f and $(\varepsilon_p * g)$ i. e., H_2 is continuous by Lemma 3.1.1, $H_2(t,0) = (g * \varepsilon_p)(t)$, $H_2(t,1) = g(t)$ and $H_2(0,x) = (g * \varepsilon_p)(0), H_2(1,x) = g(1)$, then $(g, (g * \varepsilon_p)) \in R$. Thus, the function $H : [0,1] \times [1,0] \to X$ defined by $H(t,x) = H_1(t,x) * H_2(t,x)$ is a homotopy between σ and $(f * \varepsilon_p) * (\varepsilon_p * g)$ i.e., H is the concatenation of two continuous functions, $H(t,0) = (f * \varepsilon_p) * (\varepsilon_p * g)(t), H(t,1) = (f * g)(t) = \sigma(t)$ and $H(0,x) = H_1(0,x) * H_2(0,x) = H_1(1,x) * H_2(1,x) = H(1,x)$. Therefore $\sigma = (f * g)$, then $(\sigma, ((f * \varepsilon_p) * (\varepsilon_p * g))) \in R$.

For (ii): The function $K_1 : [0,1] \times [1,0] \to X$ defined by

$$K_1(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ f(tx + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{4.2.7}$$

is a homotopy between $(\varepsilon_p * f)$ and f i. e., K_1 is continuous by Lemma 3.1.1, $K_1(t,0) = (\varepsilon_p * f)(t)$, $K_1(t,1) = f(t)$ and $K_1(0,x) = (\varepsilon_p * f)(0), K_1(1,x) = f(1)$, then $((\varepsilon_p * f), f) \in R$.
Also the function $K_2 : [0,1] \times [1,0] \to X$ defined by

$$K_2(t,x) = \begin{cases} g(tx + (2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \leq t \leq 1 \end{cases} \tag{4.2.8}$$

is a homotopy between g and $(g * \varepsilon_p)$ i. e., K_2 is continuous by Lemma 3.1.1, $K_2(t,0) = (g * \varepsilon_p)(t)$, $K_2(t,1) = g(t)$, $K_2(0,x) = (g * \varepsilon_p)(0)$ and $K_2(1,x) = g(1)$, then $((g * \varepsilon_p), g) \in R$.
Hence the function $K : [0,1] \times [1,0] \to X$ defined by $K(t,x) = K_1(t,x) * K_2(t,x)$ is a homotopy between σ and $(\varepsilon_p * f) * (g * \varepsilon_p)$ i. e., K is the concatenation of two continuous functions, $K(t,0) = (\varepsilon_p * f) * (f * \varepsilon_p)(t)$, $K(t,1) = (f * g)(t) = \sigma(t)$ and $K(0,x) = K_1(0,x) * K_2(0,x) = K_1(1,x) * K_2(1,x) = K(1,x)$.
Therefore $\sigma = (f * g)$, then $(\sigma, ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R$.
Thus, the result follows.
($\Leftarrow$) Suppose $(((f * \varepsilon_p) * (\varepsilon_p * g)), ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R \; \forall f, g \in \Omega(X,p)$ and ε_p a constant loop in $\Omega(X,p)$.
Since $(f * g) = (f * \varepsilon_p) * (\varepsilon_p * g)$ *and* $(\varepsilon_p * f) * (g * \varepsilon_p) = (g * f)$.
Then $((f * g), (g * f)) \in R$ which implies $[f * g] = [g * f]$ and $[f] \cdot [g] = [g] \cdot [f]$ $\forall [f], [g] \in \pi_1(X,p)$. Thus, $\pi_1(X,p)$ is abelian.

∎

4.3 NECESSARY AND SUFFICIENT CONDITIONS FOR A SUBSET OF FUNDAMENTAL GROUP TO BE A SUBGROUP

In this section, we give necessary and sufficient conditions for a subset of fundamental group to be a subgroup. Some necessary and sufficient conditions for a subset of a general group to be subgroup have been explained in [31], that is given a group G with a particular binary operation, if a non-empty subset H of G is closed and every elements in it has an inverse under that particular operation, then we say H is a subgroup of G. As an extension of this, many researchers study the subgroup of some groups. This includes: the maximal subgroups of symplectic groups stabilizing spreads in [26]; 3-permutable subgroups and p-nilpotency of finite group II in [33]; fuzzy subgroups of nilpotent groups in [22]; subgroup of nilpotent group in [16]; two theorems about nilpotent subgroup in [21]. Now, considering the facts gathered from the above, it is discovered that the study of subgroups of fundamental group is very essential. Therefore we have the following theorem:

Theorem 4.3.1 *Let $\pi_1(X,p)$ be fundamental group and U be a subspace of a topological space X such that $\Omega(X,p)$ and $\Omega(U,p)$ are collections of all loops in X and U, respectively. Let R be equivalence relation on $\Omega(X,p)$. Then $\pi_1(U,p)$ is a subgroup of $\pi_1(X,p)$ if and only if the following conditions hold:*

(i) $(f,(f*\varepsilon_p))$ *and* $(f,(\varepsilon_p*f)) \in R$ $(\forall f \in \Omega(U,p))$, *where ε_p is a constant loop in $\Omega(U,p)$;*

(ii) $\forall g,h \in \Omega(U,p) \exists f \in \Omega(U,p)$ *such that* $((g*h),f) \in R$*;*

(iii) $(\varepsilon_p,(f*f^{-1}))$ *and* $(\varepsilon_p,(f^{-1}*f)) \in R$ $\forall f \in \Omega(U,p)$, *where ε_p and f^{-1} are constant loop and inverse of loop f in $\Omega(U,p)$, respectively.*

Proof $(\Rightarrow)$ suppose $\pi_1(U,p)$ is a subgroup of $\pi_1(X,p)$ and $\Omega(U,p)$ is the collection of all paths in U.

Goal: We will show that the three conditions above are satisfied.

For (i): We show that the following are satisfied

(a): $(f,(f*\varepsilon_p)) \in R\ \forall\ f \in \Omega(U,p)$, where ε_p is a constant loop in $\Omega(U,p)$,

(b): $(f,(\varepsilon_p*f)) \in R\ \forall\ f \in \Omega(U,p)$, where ε_p is a constant loop in $\Omega(U,p)$.

Consider the function f which is defined from $[0,1]$ to X such that $f[0,1] \subseteq U \subseteq X$ where U is a subspace of X. Thus, f defined from $[0,1]$ to U.

For (a): Consider the concatenation of ε_p and f defined below

$$(\varepsilon_p * f)(t) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.3.1}$$

where $\varepsilon_p(1) = f(0)$.
Then the function $K_1 : [0,1] \times [1,0] \to X$ defined by

$$K_1(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \le t \le \frac{1-x}{2} \\ f((t)(x) + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \le t \le 1 \end{cases} \tag{4.3.2}$$

is the homotopy between $(\varepsilon_p * f)$ and f i. e., K_1 is continuous by Lemma 3.1.1, $K_1(t,0) = (\varepsilon_p * f)(t)$, $K_1(t,1) = f(t)$, $K_1(0,x) = (\varepsilon_p * f)(0)$ and $K_1(1,x) = f(1)$. Thus, $(f, (\varepsilon_p * f)) \in R$.
For (b): Consider the concatenation of f and ε_p defined by

$$(f * \varepsilon_p)(t) = \begin{cases} f(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ \varepsilon_p(t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.3.3}$$

where $f(0) = \varepsilon_p(1) = f(1) = \varepsilon_p(0)$.
Then the function $K_2 : [0,1] \times [1,0] \to X$ defined by

$$K_2(t,x) = \begin{cases} f((t)(x) + (2t)(1-x)), & \text{if } 0 \le t \le \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \le t \le 1 \end{cases} \tag{4.3.4}$$

is the homotopy between f and $(f * \varepsilon_p)$ i. e., K_2 is continuous by Lemma 3.1.1, $K_2(t,0) = (f * \varepsilon_p)(t)$, $K_2(t,1) = f(t)$, $K_2(0,x) = (f * \varepsilon_p)(0)$ and $K_2(1,x) = f(1)$. Then $(f, (f * \varepsilon_p)) \in R$.
For (ii): Let $g, h \in \Omega(U, p)$.

Goal: The goal is to show that there is $f \in \Omega(U, p)$ such that $((g * h), f) \in R$.

Consider the function f defined as

$$f(t) = \begin{cases} g(0), & \text{if } t = 0 \\ h(0), & \text{if } 0 < t \leq 1 \end{cases} \tag{4.3.5}$$

and the concatenation of g and h defined as

$$(g*h)(t) = \begin{cases} g(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{4.3.6}$$

Where $g(1) = h(0)$.
Then the function $H : [0,1] \times [1,0] \to X$ defined by

$$H(t,x) = \begin{cases} g((2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ h((2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{4.3.7}$$

is homotopy between f and $(g*h)$ i. e., H is continuous by Lemma 3.1.1,

$$H(t,0) = \begin{cases} g(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ h(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{4.3.8}$$

$H(t,1) = f(t), H(0,x) = (g*h)(0)$ and $H(1,x) = f(1)$. Therefore $((g*h), f) \in R$ $(\forall f, g, h \in \Omega(U,p))$.
For (iii): To prove this, we will show the following:

(a): $(\varepsilon_p, (f*f^{-1})) \in R$

(b): $(\varepsilon_p, (f^{-1}*f)) \in R$ $(\forall f \in \Omega(U,p))$, where ε_p is a constant loop in $\Omega(U,p)$ and f^{-1} is the inverse of $f \in \Omega(U,p)$.

For (a): Consider the composition of f and f^{-1} defined below

$$(f*f^{-1})(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{4.3.9}$$

where $f(1) = f^{-1}(1)$.
Then the function $H : [0,1] \times [1,0] \to X$ defined by

$$H(t,x) = \begin{cases} f((2t)(1-x)), & \text{if } 0 \le t \le \frac{1-x}{2} \\ f^{-1}((2t-1)(1-x)), & \text{if } \frac{1-x}{2} \le t \le 1 \end{cases} \tag{4.3.10}$$

is the homotopy between ε_p and $(f * f^{-1})$ i. e., H is continuous by Lemma 3.1.1,

$$H(t,0) = \begin{cases} f(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \tag{4.3.11}$$

$H(t,1) = \varepsilon_p(t), H(0,x) = (f * f^{-1})(0)$ and $H(1,x) = \varepsilon_p(1)$. Then $(\varepsilon_p, (f * f^{-1})) \in R$.
For (b): Consider the composition of f^{-1} and f defined below

$$(f^{-1} * f)(t) = \begin{cases} f^{-1}(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.3.12}$$

where $f^{-1}(1) = f(0)$.
Then the function $K : [0,1] \times [1,0] \to X$ defined by

$$K(t,x) = \begin{cases} f^{-1}((2t)(1-x)), & \text{if } 0 \le t \le \frac{1+x}{2} \\ f((2t-1)(1-x)), & \text{if } \frac{1+x}{2} \le t \le 1 \end{cases} \tag{4.3.13}$$

is homotopy between ε_p and $(f^{-1} * f)$ i. e., K is continuous by Lemma 2.7,

$$K(t,0) = \begin{cases} f^{-1}(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \tag{4.3.14}$$

$K(t,1) = \varepsilon_p(t)$, $K(0,x) = (f^{-1} * f)(0)$ and $K(1,x) = \varepsilon_p(1)$, then $(\varepsilon_p, (f^{-1} * f)) \in R$
Therefore $(\varepsilon_p, (f * f^{-1}))$ and $(\varepsilon_p, (f^{-1} * f)) \in R$.
$(\Leftarrow)$ Suppose the three conditions above are satisfied.

Goal: We will show that $\pi_1(U,p)$ is a subgroup of $\pi_1(X,p)$.

From (i): Since $(f,(f*\varepsilon_p))$ *and* $(f,(\varepsilon_p*f)) \in R$ then we have $[f*\varepsilon_p]=[f]=[\varepsilon_p*f]$, which implies $[f]\cdot[\varepsilon_p]=[f]=[\varepsilon_p]\cdot[f]$. Therefore the identity $[\varepsilon_p]$ exist in $\pi_1(U,p)$.
From (ii): Since $((g*h),f) \in R$ then we have $[g*h]=[f]$, which implies $[g]\cdot[h]=[f] \in \pi_1(U,p)$. Therefore $\pi_1(U,p)$ is closed $\forall f,g,h \in \Omega(U,p)$ and $\forall [f],[g],[h] \in \pi_1(U,p)$.
From (iii): Since $(\varepsilon_p,(f*f^{-1}))$ and $(\varepsilon_p,(f^{-1}*f)) \in R$ then we have $[f^{-1}*f]=[\varepsilon_p]=[f*f^{-1}]$, which implies $[f^{-1}]\cdot[f]=[\varepsilon_p]=[f]\cdot[f^{-1}]$. Therefore the inverse $[f^{-1}]$ exist in $\pi_1(U,p)$ $\forall [f] \in \pi_1(U,p)$.
Now, clearly from the above $\pi_1(U,p)$ is a subgroup of $\pi_1(X,p)$. ∎

4.3.1 Description of Left and Right Coset of Fundamental Group

Definition 4.3.2 *Let $\pi_1(U,p)$ be a subgroup of a fundamental group $\pi_1(X,p)$, then*

(i) for [f]$\in \pi_1(X,p)$, the subset [f]($\pi_1(U,p)$) of $\pi_1(X,p)$ is called the left coset of $\pi_1(U,p)$ determined by $[f]$ i. e.,

$$[f](\pi_1(U,p)) = \{[f]\cdot[g] : [g] \in \pi_1(U,p)\}.$$

(ii) for [f]$\in \pi_1(U,p)$ the subset $(\pi_1(U,p))$[f] of $(\pi_1(X,p))$ is called the right coset of $(\pi_1(U,p))$ determined by $[f]$ i. e.,

$$(\pi_1(U,p))[f] = \{[g]\cdot[f] : [g] \in \pi_1(U,p)\}.$$

4.3.2 Necessary and Sufficient Conditions for a Subgroup of Fundamental Group to be Normal

Theorem 4.3.3 *Let $\pi_1(U,p)$ be a subgroup of $\pi_1(X,p)$ such that $\Omega(X,p)$ and $\Omega(U,p)$ are collections of all loops in X and U, respectively.*
Then $\pi_1(U,p) \trianglelefteq \pi_1(X,p)$ if and only if $f*g*f^{-1} \in \Omega(U,p)$ $\forall f \in \Omega(X,p)$, $g \in \Omega(U,p)$, where f^{-1} is the inverse of f in $\Omega(X,p)$.

Proof (⇒): Let $\pi_1(U,p)$ be a subgroup of $\pi_1(X,p)$ such that $\Omega(X,p)$ and $\Omega(U,p)$ are collections of all loops in X and U, respectively. Let $\pi_1(U,p) \trianglelefteq \pi_1(X,p)$ and R be a relation define on $\Omega(U,p)$.

Goal: We are to show that $f*g*f^{-1} \in \Omega(U,p)$ $\forall f \in \Omega(X,p)$, $g \in \Omega(U,p)$ and f^{-1} is the inverse of f in $\Omega(X,p)$.

To show this, it is suffices to show that $\exists\, g^{'} \in \Omega(U,p)$ such that $((f*g*f^{-1}),g^{'}) \in R$.
Consider the concatenation of functions f,g and f^{-1} defined below

$$(f*g*f^{-1})(t) = \begin{cases} f(3t), & \text{if } 0 \le t \le \frac{1}{3} \\ g(3t-1), & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ f^{-1}(3t-2), & \text{if } \frac{2}{3} \le t \le 1 \end{cases} \tag{4.3.15}$$

Where $f(1) = g(0) = f^{-1}(0)$.
Define $g^{'}$ by

$$g^{'}(t) = \begin{cases} f(0), & \text{if } t = 0 \\ g(t), & \text{if } 0 \le t \le 1 \\ f^{-1}(0), & \text{if } t = 1 \end{cases} \tag{4.3.16}$$

where $f(0) = g(0)$ *and* $g(1) = f^{-1}(0)$
Then we can define the homotopy between $f*g*f^{-1}$ and $g^{'}$ as a function $H : [0,1]\times[1,0] \to X$ by

$$H(t,x) = \begin{cases} f((3t)(x)), & \text{if } 0 \le t \le \frac{x}{3} \\ g((3t-1)(x)+(t)(1-x)), & \text{if } \frac{x}{3} \le t \le \frac{3-x}{3} \\ f^{-1}((3t-2)(x)), & \text{if } \frac{3-x}{3} \le t \le 1 \end{cases} \tag{4.3.17}$$

i.e., H is continuous by Lemma 3.1.1,

$$H(t,1) = \begin{cases} f(3t), & \text{if } 0 \le t \le \frac{1}{3} \\ g(3t-1), & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ f^{-1}(3t-2), & \text{if } \frac{2}{3} \le t \le 1 \end{cases} \tag{4.3.18}$$

$H(t,0) = g^{'}(t)$ and $H(0,x) = f(0) = f(1-x) = f^{-1}(x) = H(1,x)$ $\forall x \in [0,1]$. Then $((f * g * f^{-1}), g^{'}) \in R$.
($\Leftarrow$): Suppose $f * g * f^{-1} \in \Omega(U,p)$.

Goal: Is to show that $\pi_1(U,p) \trianglelefteq \pi_1(X,p)$.

Let $g^{'} \in \Omega(U,p)$ such that $((f * g * f^{-1}), g^{'}) \in R$.
Clearly, there exists $\varepsilon_p \in \Omega(U,p)$ such that

$$
\begin{aligned}
&(g^{'}, (g^{'} * \varepsilon_p)) \in R \\
&\Rightarrow [g^{'}] = [g^{'}] \cdot [\varepsilon_p] \\
&\quad = [g^{'}] \cdot [f * f^{-1}] \\
&\quad = [g^{'}] \cdot ([f] \cdot [f^{-1}]) \\
&\quad = [g^{'}] \cdot ([f^{-1}] \cdot [f]) (\text{by Lemma 3.1.8}) \\
&\quad = ([g^{'}] \cdot [f^{-1}]) \cdot [f] (\text{by Lemma 3.1.5})
\end{aligned}
$$

then

$$
\begin{aligned}
[f * g * f^{-1}] = [g^{'}] &= ([g^{'}] \cdot [f^{-1}]) \cdot [f] \\
[f] \cdot ([g * f^{-1}]) &= ([g^{'}] \cdot [f^{-1}]) \cdot [f] \\
[f] \cdot ([g] \cdot [f^{-1}]) &= ([g^{'}] \cdot [f^{-1}]) \cdot [f]
\end{aligned}
$$

therefore since $\pi_1(U,p)$ is a subgroup of $\pi_1(X,p) \exists [g], [g^{'}] \in \pi_1(U,p)$ $\forall$ $[f^{-1}] \in \pi_1(X,p)$ such that $([g] \cdot [f^{-1}]), ([g^{'}] \cdot [f^{-1}]) \in \pi_1(U,p)$.
Thus,

$$[f](\pi_1(U,p)) = (\pi_1(U,p))[f]$$

Hence, we conclude that $\pi_1(U,p) \trianglelefteq \pi_1(X,p)$. ∎

4.4 DESCRIPTION OF CENTRALIZER OF AN ELEMENT IN FUNDAMENTAL GROUP

In this section, we deduce the centralizer of an element in a fundamental group from general description of centralizer of an element in a group. As it is generally described that given a general group G, the centralizer of an element $x \in G$ is defined as a set of elements in G that commutes with x i.e.,

$$C(x) = \{y \in G : xy = yx\}$$

As an extension of the above, many researchers studied the centralizers of some group's elements. This includes; centralizers of nilpotent group's elements in semisimple algebraic groups in [10]; the isomorphism type of the centralizer of an element in a lie group [11]; centralizers of semisimple elements in the finite classical groups in [31]. Now, we have the following theorem:

Theorem 4.4.1 *Let $\Omega(X, p)$ be the collection of all loops in X. Let $\pi_1(X, p)$ be a fundamental group and R be equivalence relation defined on $\Omega(X, p)$. Then a set $C([f])$ is a centralizer of $[f] \in \pi_1(X, p)$ if and only if*
$(((f * \varepsilon_p) * (\varepsilon_p * g)), ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R$
$\forall f, g \in \Omega(X, p)$.

Proof ($\Rightarrow$): Let $\pi_1(X, p)$ be a fundamental group, σ be an arbitrary loop in $\pi_1(X, p)$ and $C([f])$ be a centralizer of $[f] \in \pi_1(X, p)$.

Goal: We will show that $(((f * \varepsilon_p) * (\varepsilon_p * g)), ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R$ $\forall f, g \in \Omega(X, p)$, where ε_p is a constant path in $\Omega(X, p)$.

To show this, it is suffices to show that there exists a loop σ in $\Omega(X, p)$ such that:

(i) $(\sigma, ((f * \varepsilon_p) * (\varepsilon_p * g))) \in R$

(ii) $(\sigma, ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R$ $\forall g \in \Omega(X, p)$, where ε_p is a constant path in $\Omega(X, p)$.

Consider the following concatenations $f * \varepsilon_p$, $\varepsilon_p * g$, $g * \varepsilon_p$ and $\varepsilon_p * f$ defined below

$$(f * \varepsilon_p)(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \varepsilon_p(t), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{4.4.1}$$

where $f(1) = \varepsilon_p(0)$,

$$(\varepsilon_p * g)(t) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.4.2}$$

where $\varepsilon_p(1) = g(0)$,

$$(g * \varepsilon_p)(t) = \begin{cases} g(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ \varepsilon_p(t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.4.3}$$

where $g(1) = \varepsilon_p(0)$,

$$(\varepsilon_p * f)(t) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \le t \le \frac{1}{2} \\ f(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{4.4.4}$$

where $\varepsilon_p(1) = f(0)$.
We define $\varepsilon_p : [0,1] \to X$ by $\varepsilon_p(t) = f(1)\ \forall t \in [0,1]$.
For (i): Clearly, the function $H_1 : [0,1] \times [1,0] \to X$ defined by

$$H_1(t,x) = \begin{cases} f((t)(x) + (2t)(1-x)), & \text{if } 0 \le t \le \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \le t \le 1 \end{cases} \tag{4.4.5}$$

is the homotopy between f and $(f * \varepsilon_p)$ i. e., H_1 is continuous by Lemma 3.1.1, $H_1(t,0) = (f * \varepsilon_p)(t)$, $H_1(t,1) = f(t)$, $H_1(0,x) = (f * \varepsilon_p)$ and $H_1(1,x) = f(1)$. Thus, $(f,(f * \varepsilon_p)) \in R$.
Also, the function $H_2 : [0,1] \times [1,0] \to X$ defined by

$$H_2(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \le t \le \frac{1-x}{2} \\ g((t)(x) + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \le t \le 1 \end{cases} \tag{4.4.6}$$

is the homotopy between g *and* $(\varepsilon_p * g)$ i. e., H_2 is continuous by Lemma 3.1.1, $H_2(t,0) = (g * \varepsilon_p)(t)$, $H_2(t,1) = g(t)$, $H_2(0,x) = (g * \varepsilon_p)(0)$ and $H_2(1,x) = g(1)$, so $(g,(g * \varepsilon_p)) \in R$. Thus, the function $H : [0,1] \times [1,0] \to X$ defined by

$H(t,x) = H_1(t,x) * H_2(t,x)$ is the homotopy between σ and $(f * \varepsilon_p) * (\varepsilon_p * g)$ i. e., H is the concatenation of two continuous functions,
$H(t,0) = (f * \varepsilon_p) * (\varepsilon_p * g)(t), H(t,1) = (f * g)(t) = \sigma(t)$ and
$H(0,x) = H_1(0,x) * H_2(0,x) = H_1(1,x) * H_2(1,x) = H(1,x)$.
Therefore $\sigma = (f * g)$, then $(\sigma, ((f * \varepsilon_p) * (\varepsilon_p * g))) \in R$
For (ii): The function $K_1 : [0,1] \times [1,0] \to X$ defined by

$$K_1(t,x) = \begin{cases} \varepsilon_p(t), & \text{if } 0 \leq t \leq \frac{1-x}{2} \\ f((t)(x) + (2t-1)(1-x)), & \text{if } \frac{1-x}{2} \leq t \leq 1 \end{cases} \tag{4.4.7}$$

is the homotopy between $(\varepsilon_p * f)$ and f i. e., K_1 is continuous by Lemma 3.1.1, $K_1(t,0) = (\varepsilon_p * f)(t)$, $K_1(t,1) = f(t)$, $K_1(0,x) = (\varepsilon_p * f)(0)$ and $K_1(1,x) = f(1)$, so $((\varepsilon_p * f), f) \in R$.
Also the function $K_2 : [0,1] \times [1,0] \to X$ defined by

$$K_2(t,x) = \begin{cases} g((t)(x) + (2t)(1-x)), & \text{if } 0 \leq t \leq \frac{1+x}{2} \\ \varepsilon_p(t), & \text{if } \frac{1+x}{2} \leq t \leq 1 \end{cases} \tag{4.4.8}$$

is homotopy between g and $(g * \varepsilon_p)$ i. e., K_2 is continuous by Lemma 3.1.1,
$K_2(t,0) = (g * \varepsilon_p)(t)$, $K_2(t,1) = g(t)$, $K_2(0,x) = (g * \varepsilon_p)(0)$ and $K_2(1,x) = g(1)$.
Thus $((g * \varepsilon_p), g) \in R$.
Hence the function $K : [0,1] \times [1,0] \to X$ defined by $K(t,x) = K_1(t,x) * K_2(t,x)$ is the homotopy between σ and $(\varepsilon_p * f) * (g * \varepsilon_p)$ i. e., K is the concatenation of two continuous functions, $K(t,0) = (\varepsilon_p * f) * (f * \varepsilon_p)(t)$, $K(t,1) = (f * g)(t) = \sigma(t)$ and $K(0,x) = K_1(0,x) * K_2(0,x) = K_1(1,x) * K_2(1,x) = K(1,x)$.
Therefore $\sigma = (f * g)$, then $(\sigma, ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R$. Thus, the result follows.
($\Leftarrow$): Suppose $(((f * \varepsilon_p) * (\varepsilon_p * g)), ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R \; \forall g \in \Omega(X,p)$ and ε_p is a constant path in $\Omega(X,p)$.

Goal: We will show that $C([f])$ is a centralizer of $[f] \in \pi_1(X,p)$.

Since $f * g = (f * \varepsilon_p) * (\varepsilon_p * g)$ *and* $(\varepsilon_p * f) * (g * \varepsilon_p) = g * f$ then we have

$$((f * g), (g * f)) \in R$$
$$[f * g] = [g * f]$$
$$[f] \cdot [g] = [g] \cdot [f]$$

Hence by definition we conclude that $C([f])$ is a centralizer of $[f] \in \pi_1(X, p)$ $\forall [g] \in \pi_1(X, p)$. ■

4.5 NECESSARY AND SUFFICIENT CONDITION FOR AN ELEMENT TO BE IN THE CENTER OF FUNDAMENTAL GROUP

In this section, we give a necessary and sufficient condition for an element to be in the center of a fundamental group. The condition necessary and sufficient for an element to be in the center of a general group as been described in [30] that given a group G, the center of G is defined as the set of element $x \in G$ that commutes with every element of G i.e.,

$$Z(G) = \{x \in G : xy = \text{yx } \forall y \in G\}$$

Now, follow from the above descriptions we have the following theorem:

Theorem 4.5.1 *Let $\Omega(X, p)$ be the collection of all loops in X. Let $Z(\pi_1(X, p))$ be the center of $\pi_1(X, p)$ and R be the equivalence relation on $\Omega(X, p)$. Then $[f] \in Z(\pi_1(X, p))$ if and only if $(((f * \varepsilon_p) * (\varepsilon_p * g)), ((\varepsilon_p * f) * (g * \varepsilon_p))) \in R \; \forall g \in \Omega(X, p)$, where ε_p is a constant loop in $\Omega(X, p)$.*

Proof The prove follows directly from theorem 4.4.1. ■

4.6 QUOTIENT FUNDAMENTAL GROUP

In this section, we deduce the quotient fundamental group from the description of general quotient group. As it was described in [30] that given a group G and a normal subgroup $N\ of\ G$, a set G/N defined by

$$G/N = \{gN : g \in G\}$$

is called a quotient group with the binary operation $aN.bN = abN$ for $a,b \in G$. As an extension of the study of quotient group many researchers study the quotient group of some particular groups which includes the quotient of a nilpotent group considered in [31]. Clearly from the above the study of quotient fundamental group is essential. Now we have the following lemma and corollary.

Lemma 4.6.1 *Let $\pi_1(X,p)$ be fundamental group and $\pi_1(U,p)$ be a subgroup of $\pi_1(X,p)$. Then the set $\pi_1(X,p)/\pi_1(U,p)$ is a quotient fundamental group if and only if $[f * g * f^{-1}] \in \pi_1(U,p)\ \forall f,g \in \Omega(X,p)$.*

Proof The proof follows directly from theorem 4.3.5. ■

Corollary 4.6.2 *Let $\pi_1(X,p)$ be fundamental group and $Z(\pi_1(X,p))$ be center of $\pi_1(X,p)$. Then the set $\pi_1(X,p)/Z(\pi_1(X,p))$ is a quotient fundamental group.*

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 INTRODUCTION

This chapter gives the summary and conclusions of the entire work, together with some recommendations for further research.

5.2 SUMMARY

In this study, we were able to give the algebraic properties of fundamental group. We began with the introduction of group, fundamental group and the algebraic properties of group.

Literature review was given to explain what has been done on this very aspect. We give lemmas and Propositions that explained the continuity of the composition of continuous functions, the homotopy equivalence as an equivalence relation and how fundamental group satisfied all the group axioms.

We give a necessary and sufficient condition for a fundamental group to be abelian, for its subset to be subgroup and for its subgroup to be normal. We also give a necessary and sufficient condition for an element to be in a center of fundamental group and we described the set of centralizers of an element and the quotient fundamental group. These necessary and sufficient conditions were given in theorems 4.2.1, 4.3.1, 4.3.3, 4.4.1, and 4.5.1. Finally, the proofs of these theorems were given in a very clear form.

5.3 CONCLUSION

We have investigated some basic algebraic properties of fundamental group. These properties include abelian, subgroups, normality, quotients, center and centralizers of an element in a fundamental group. Considering the fact that the operation on fundamental group is not straight forward as defined in definition 1.6.23, we discovered that many of the basic algebraic properties applicable to general group can also be inherited by fundamental group.

5.4 RECOMMENDATIONS

We recommend further research on other algebraic properties like homomorphism theorems, internal and external direct product and group actions in a fundamental group .

We also recommend extension of this study to the fundamental group of intuitionistic fuzzy topological spaces as a specific case of this study and further extend this study to intuitionistic fuzzy ring defined by Zhang [36].

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