DEVELOPMENT OF EXPLICIT RATIONAL RUNGE-KUTTA SCHEME FOR THE SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS.

BY

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CERTIFICATION

This is to certify that the thesis, entitled "Development of Explicit Rational Runge-kutta Scheme for the solution of second order ordinary differential equations" has been duly written by Mrumun Comfort Soomiyol (M.Sc/MA/06/0060) of the department of Mathematics and Computer Science, Federal University of Technology Yola.

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DEDICATION

This work is dedicated to the Almighty and most gracious God, and to my loving husband and children.

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ABSTRACT

The explicit rational Runge-Kutta methods are a sub class of well known family of explicit Runge-Kutta methods and have application in the efficient numerical solution of ordinary differential equations. In this project, the explicit rational Runge-Kutta scheme was developed for second order initial value problems. The developed scheme was found to be consistent and convergent. The results obtained when compared with other known Runge-Kutta methods were encouraging.

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CHAPTER ONE

INTRODUCTION

1.0 BACKGROUND OF THE STUDY

In Numericalanalysis, the Runge-Kutta methods(developed by the German Mathematicians C. Rungeand M. W.Kutta around the 1900s) are important family of iterative methods for the approximation of solutions of ordinary differential equations(ODEs).

The Runge-Kutta methods can be implicit, semi implicit or explicit depending on the definition of parameters. Explicit Runge-Kutta methods are popular as each stage can be calculated with one function evaluation. This makes an explicit scheme less expensive to implement than the implicit schemes which usuallyinvolve solving a non-linear system of equations in order to evaluate the stages.

Let an initial value problem for first order ODE be specified as follows

$$y' = f(x, y), y(x_0) = y_0. \quad (1.1)$$

Then the fourth order Runge-Kutta method for (1.1) popularly known as the classical Runge-Kutta method is given by the following equations

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

wherey_{n+1} is the Runge-Kutta approximation for $y(x_{n+1})$ and

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$

$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$

$k_4 = f(x_n + h, y_n + hk_3)$.

The family of explicit Runge-Kutta methods is a generalisation of the fourth order Runge-Kutta method. It is given as

$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$(1.2)

where

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + c_2 h, y_n + a_{21} hk_1)$

.(1.3)

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$k_s = f(x_n + c_s h, y_n + a_{s1} hk_1 + a_{s2} hk_2 + \cdots + a_{ss-1} hk_{s-1})$.

To specify a particular method, one needs to provide the integer s (the number of stages) and the coefficients$a_{ij} (for\ 1 \leq j < i \leq s)$, $b_i (for\ i = 1,2, \dots, s)$and $c_i (for\ i = 2,3, \dots, s)$.

The Runge-Kutta method is consistent if

$$\sum_{j=1}^{i-1} a_{ij} = c_i \quad for\ i = 2, \dots, s.$$

Other requirements also follow if we require the method to have a certain order **p,** meaning that the truncation error is$o(h^{p+1})$. These can be derived from the definition of the truncation error itself. For example a 2-stage method has order 2 if $b_1 + b_2 = 1$, $b_2 c_2 = \frac{1}{2}$and $b_2 a_2 = \frac{1}{2}$ (Wikipedia 2010).

Although these formulae look complicated at first sight and involve time and tedious manipulations they are actually easy to deal with.

Okunbor (1987) investigated the use of the rationalexplicit Runge-Kutta scheme

$$y_{n+1} = \frac{y_n + h\sum_{i=1}^{s} W_i K_i}{1 + h y_n \sum_{j=1}^{s} V_i H_i} \qquad (1.4a)$$

where

$$K_i = f(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{i-1,j} K_j)\ ,\ i = 1(1)s$$

$$H_i = g(x_n + d_i h, z_n + h\sum_{j=1}^{s} b_{i-1,j} H_j), i = 1(1)s \qquad (1.4b)$$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)andz_n = {}^{1}/{}_{y_n}$

When $a_{ij} = 0,\ for\ j \geq i$ we obtain an explicit method with relatively small interval of absolute stability, which renders them unsuitable for stiff initial value problems.

Ademiluyiand Babatola(2000) derived an implicit rational Runge-Kutta scheme for approximation of first order ordinary differential equations with large response characteristics.

Special methods for second-order differential equations were proposed by Nystrom (1925).

The higher order equations can be solved by considering an equivalent system of first order equations. However, it is also possible to develop direct singlestep methods to solve higher order equations.

The Runge-Kutta method for general second order equations is given as

$$y_{n+1} = y_n + h y_n' + \sum_{i=1}^{s} W_i' k_i$$

$$y'_{n+1} = y'_n + \frac{1}{h}\sum_{i=1}^{s} W'_i k_i \qquad (1.5a)$$

where

$$K_i = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{i-1} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{i-1} b_{ij} K_j\right) \quad (1.5b)$$

with

$$c_i = \sum_{j=1}^{i-1} a_{ij} = \frac{1}{2}\sum_{j=1}^{i-1} b_{ij}$$

where$c_i, a_{ij}, b_{ij}, W_i, W'_i$ are arbitrary constants to be determined.

1.1 STATEMENT OF THE PROBLEM

Okunbor(1987) worked on the Rational Explicit Runge-Kutta scheme for the solution of first order ordinary differential equations; the scheme performed well and thus the desire to derive a scheme that can solve a second order initial value ordinary differential equation.

1.2 AIMS AND OBJECTIVES OF THE STUDY

1. To develop an explicit rational Runge-Kutta scheme for the solution of second order ordinary Differential Equations with initial conditions.
2. To investigate the consistency and convergence and of the scheme.
3. To illustrate the applicability of the scheme with specific examples.
4. To compare results with some existing schemes which also solve second order differential equations.

5. To suggest any further research for improvement where necessary.

1.3 SIGNIFICANCE OF THE STUDY

Most of the studies carried out on explicit rational Runge-Kutta methods are for the solution of first order ordinary differential equations. In this work, we shall derive Rational Explicit Runge-Kutta methods for the solution of second order ordinary differential equations. Then we shall check it's response to the tests for convergence and consistency. We shall implement the schemes developed using hypothetical problems and check the local errors to see their performance.

1.4 SCOPE OF THE STUDY

This work involves the derivation of an Explicit Rational Runge-Kutta method for the solution of second order ordinary differential equations with initial conditions.

1.5 BASIC CONCEPTS AND SOME DEFINITIONS

The following concepts and definitions are relevant for proper understanding of this work.

Gupta (1978) explains that:

- The rate of change of one variable with respect to another is called derivative.
- Equations expressing a relationship among these variables and thier derivatives are known as differential equations.

- An equation involving$x$, the function $f(x)$ which defines a function of x, and one or more of it's derivatives is called an ordinary differential equation (ODE), otherwise it is called a partial differential equation (PDE).
- The order of an ODE is the order of the highest derivative involved in the equation.
- The degree of the ODE is the highest power of the derivative involved.
- An ODE is said to be linear if none ofits term involving derivative is of degree more than one and no product of the dependent variable with the derivatives, otherwise it is called a non-linear ODE.

Definition 1.1

The Runge-Kutta method is said to be consistent with the IVP (1.1) if

$\emptyset(x, y, 0) \equiv f(x, y)$. (1.6)

If the method is consistent with the IVP (1.1) then

$$y(x+h) - y(x) - h\emptyset(x, y(x), h)$$

$$= y(x) + hy'(x) + 0(h^2) - y(x) - h\emptyset(x, y(x), 0)$$

$= hy'(x_n) - hy'(x_n) + 0(h^2) = O(h^2)$.

Since $y'(x) = f\big(x, y(x)\big) = \emptyset(x, y(x), 0)$ by equation (1.6). It is a consistent method and has order at least one.

Definition 1.2

The Runge-Kuttamethod is said to have order p if p is the largest integer for which

$$y(x+h) - y(x) - h\emptyset(x, y(x), h) = O(h^{p+1}) \quad (1.7)$$

holds.

The Taylor algorithm of order p is

$$y(x+h) = y(x) + hy^{(1)}(x) + \frac{h^2}{2!}y_{(x)}^{(2)} + \cdots + \frac{h^p}{p!}y^{(p)}(x) + \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(\xi),$$

$0<\xi<1$ (1.7a)

Definition 1.3

An s-stage Runge-Kutta method involves sfunction evaluations per step. Each of the functions$f(x, y, h)$, $s = 1,2,3, \dots, S$ may be interpreted as an aproximation to the derivative $y'(x)$, and the function $\emptyset(x, y, h)$ as a weighted mean of those approximations.

Runge-Kutta method of higher order involves tedious manipulations, as such, to reduce the cumbersome nature of the work the following notations are adopted (Lambert, 1973):

$$f = f(x,y), \qquad f_x = \frac{\partial f(x,y)}{\partial x}, \qquad f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2}$$

$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y}, \; f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2} \quad (1.8)$$

For a function of two variables, $f(x, y)$, the rate of change of the function can be due to change in either x or y. The derivatives of f can be expressed in terms of the partial derivatives in equation (1.8).

For the expression in the neighbourhood of the point (a, b),

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \cdots$ (1.9)

Now expanding k in Taylor series about the point (x, y), we have

$$k_1 = f$$

$$k_2 = f + ha_2F + \frac{1}{2}h^2a_2^2G + O(h^3)$$

$k_3 = f + ha_3F + h^2\left(a_2b_{32}Ff_y + \frac{1}{2}a_3^2G\right) + O(h^3)$(1.10)

where

$$F = f_y + ff_y$$

and

$G = f_{xx} + 2ff_{xy} + f^2f_{yy}.$

CHAPTER TWO

LITERATURE REVIEW

2.0 INTRODUCTION

Runge-Kutta methods were developed to avoid the computation of high-order derivatives which the Taylor method involves. In place of these derivatives extra values of the given function$f(x, y)$ are used in a way which duplicates the accuracy of the Taylor polynomial.

Their self-starting property makes them unique amongst other methods. The application of this method runs through engineering, physics, heat and matter transfer, kinetic theory, electric transmission network and many others.

Implicit rational Runge-Kutta schemes have been developed, which are capable of handling numerous problems with large response characteristics. Classes of such schemes are stablebut often are difficult to solve numerically because of their fast responding component which tends to control the stability of the system.

Odekunle (2000) discovered a set of semi-implicit rational Runge-Kutta schemes that are used in the solution of some stiff initial value problems. The idea was further expanded by Ademiluyi, Babatola and Odekunle (2002) to develop a class of implicit rational Runge-Kutta Schemes which were found to perform very well in stiff systems though highly inadequate when applied to problems with points of singularity. This challenge motivated Odekunle, oye and Adee (2004) to come up with a class of inverse Runge-Kutta Schemes for the numerical integration of singular problems. These inverse Runge-Kutta Schemes were found to be efficient for solving problems with singularity points.

2.1 RATIONALISED RUNGE-KUTTA METHOD

The rational Runge-Kutta Schemes for the solution of first order ordinary differential equations with initial values as defined by Ademiluyi, Babatola and Odekunle (2002) is

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{s} W_i K_i}{1 + y_n \sum_{i=1}^{s} V_i H_i} \qquad (2.2a)$$

where

$$K_i = hf(x_n + c_i h_i, y_n + \sum_{j=i}^{s} a_{i-1,j} K_j), \;\; i = 1(1)s$$

and

$H_i = hg(x_n + d_i h_i, z_n + \sum_{j=i}^{s} b_{i-1,j} \, H_j) \; i = 1(1)s$ (2.2b)

$g(x_n, z_n) = z_n^2 f(x_n, y_n), \rightarrow z_n = {}^{1}\!/_{y_n}$.

The scheme (2.2a) is said to be

i. Explicit if $b_{ij} = 0, j \geq i$
ii. Semi-implicit if $b_{ij} = 0, j > i$
iii. Implicit if $b_{ij} \neq 0$ for at least one $j > i$, h is the step size and the constraint

$d_i = \sum_{j=1}^{i} b_{ij}, i = 1(1)s$ (2.2c)

is imposed to ensure consistency of the method.

2.2 EXPLICIT RUNGE-KUTTA METHOD FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Let an initial value problem for second order ordinary differential equation be specified as follows

$y'' = f(x, y, y'), \quad y(x_0) = y_0 y'(x_0) = y_0'$. (2.3)

Then according to Jain(1984), the Runge-Kutta method for (2.3) is given by

$$y_{n+1} = y_n + hy_n' + \sum_{i=1}^{s} W_i k_i$$

$$y_{n+1} = y_n' + \frac{1}{h}\sum_{i=1}^{s} W_i' k_i \qquad (2.3a)$$

where

$$k_i = \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{i} a_{ij} K_j, y_n' + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j) \qquad (2.3b)$$

with

$$c_i = \sum_{j=1}^{i-1} a_{ij} = \frac{1}{2}\sum_{j=1}^{i-1} b_{ij}$$

where$c_i, a_{ij}, b_{ij}, W_i, W_i'$ are arbitrary constants to be determined.

CHAPTER THREE

DERIVATION OF THE EXPLICIT RATIONAL RUNGE KUTTA SCHEME FOR THE SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS

3.0 INTRODUCTION

We shall define the general s-stage explicit rational Runge-Kutta scheme for the solution of second order ODEs as

$$y_{n+1} = \frac{y_n + hy_n' + \sum_{i=1}^{S} W_i K_i}{1 + y_n \sum_{i=1}^{S} V_i H_i} \qquad (3.1)$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{i=1}^{s} W_i' K_i'}{1 + \frac{1}{h} y_n' \sum_{i=1}^{s} V_i' H_i} \quad (3.2)$$

where

$$K_i = \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{i} a_{ij} K_j, y_n' + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j) \quad (3.3a)$$

$$H_i = \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z_n' + \sum_{j=1}^{i} \alpha_{ij} H_j, z_n' + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j) \quad (3.3b)$$

h is the step size (or grid spacing) with the following constraints,

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij}$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j=1}^{i} \beta_{ij} \quad (3.3c)$$

$z_n = {}^{1}/_{y_n}$, $g(x_n, z_n, z_n') = -z_n^2 f(x_n, y_n, y_n')$.

where $c_i, a_{ij}, b_{ij}, d_i, \alpha_{ij}, \beta_{ij}, W_i, W_i', V_i, V_i'$ are arbitrary constants to be determined.

In this work we shall consider one stage, two stage and three-stage methods of (3.1)

3.1 DERIVATION OF THE ONE-STAGE SCHEME

Set s=1 in equations (3.1), (3.2), (3.3a) and (3.3b) to have

$$y_{n+1} = \frac{y_n + hy_n' + W_1 K_1}{1 + y_n V_1 H_1} \quad (3.4)$$

$$y'_{n+1} = \frac{y_n' + \frac{1}{h} W_1' K_1}{1 + \frac{1}{h} y_n' V_1' H_1} \quad (3.5)$$

where

$K_1 = \frac{h^2}{2} f(x_n + c_1 h, y_n + hc_1 y_n^1 + a_{11} k_{1,} y_n^1 + \frac{1}{h} b_{11} k_1)$ (3.6a)

$H_1 = \frac{h^2}{2} g(x_n + d_1 h, Z_n + hd_1 Z_n' + \alpha_{11} H_1, Z_n' + \frac{1}{h} \beta_{11} H_1)$ (3.6b)

with$g(x_n, z_n) = -Z_n^2 f(x_n, y_n)$

$Z_n = {}^{1}/{y_n}$ (3.6c)

and constraints

$c_1 = a_{11} = b_{11} = 0$ (3.6i)

and

$d_1 = \alpha_{11} = \beta_{11} = 0$(3.6ii)

whereW_1, W_1', V_1, V_1' are all constants to be determined

Using the binomial expansion in the right hand side of (3.4) and ignoring any order higher than one, we obtain

$$y_{n+1} = (y_n + hy_n' + W_1 K_1)(1 + y_n V_1 H_1)^{-1}$$

$$y_{n+1} = (y_n + hy_n' + W_1 K_1)(1 - y_n V_1 H_1 - \cdots)$$

$y_{n+1} = y_n + hy_n' + W_1 K_1 - y_n^2 V_1 H_1 \ldots$. (3.7a)

Again using the binomialexpansion on the right hand side of equation (3.5) and ignoring any order higher than one, we obtain

$$y_{n+1}' = (y_n' + \frac{1}{h} W_1' K_1)(1 + \frac{1}{h} y_n' V_1' H_1)^{-1}$$

$$y'_{n+1} = (y'_n + \frac{1}{h} W_1' K_1)(1 - \frac{1}{h} y'_n V_1' H_1 - \cdots)$$

$y'_{n+1} = y'_n + \frac{1}{h} W_1' K_1 - \frac{1}{h} y_n'^2 V_1' H_1 \ldots$. (3.7b)

The Taylor series of y_{n+1} is

$y_{n+1} = \mathrm{y_n} + \mathrm{hy'_n} + \frac{\mathrm{h}^2}{2} \mathrm{y''_n} + \frac{h^3}{6} y'''_n + 0(h^4)$ (3.8a)

and that of y'_{n+1} is

$y'_{n+1} = y'_n + h y''_n + \frac{h^2}{2} y'''_n + \frac{h^3}{4} y_n^{\prime v} + 0(h^4)$. (3.8b)

weshall adopt the differential notations

$$f = f(x, y), f_x = \frac{\partial f(x, y)}{\partial x}, f_{xx} = \frac{\partial^2 f(x, y)}{\partial x^2}$$

$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y}, f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2}, \ldots$

$\Delta f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = y'\,, \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} = f_x + f f_y$

$$y'_n = f(x_n, y_n) = f_n$$

$$y''_n = f_x + f_n f_y = \Delta f_n$$

$y'''_n = \frac{\partial}{\partial x}\left(f_x + f_n f_y\right) + \frac{\partial}{\partial y}\left(f_x + f_n f_y\right) f_n$ (3.9)

$$= f_{xx} + f_n f_{xy} + f_x f_y + (f_{xy} + f_n f_{yy} + f_y^2) f_n$$

$$y'''_n = f_{xx} + 2 f_n f_{xy} + f_n^2 f_{yy} + f_y (f_x + f_n f_y)$$

$$= \Delta^2 f_n + f_y \Delta f_n$$

Using these notations in (3.9)on the Taylor series (3.8a) and (3.8b) we have,

$y_{n+1} = y_n + h y'_n + \frac{h^2}{2} f_n + \frac{h^3}{3!} \Delta f_n + \cdots$ (3.10a)

$$y'_{n+1} = y'_n + hf_n + \frac{h^2}{2}\Delta f_n + \frac{h^3}{3!}\left(\Delta^2 f_n + f_{y'}\Delta f_n + f_n f_y\right) + \cdots \quad (3.10b)$$

with the constraints in equation (3.6i) and (3.6ii), K_1 and H_1 becomes

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y'_n) = \frac{h^2}{2} f_n \quad (3.11a)$$

$$H_1 = \frac{h^2}{2} g(x_n, Z_n, Z'_n) = \frac{h^2}{2} g_n \quad (3.11b)$$

Putting (3.11a) and (3.11b) into (3.7a) and (3.7b) respectively we get

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}(W_1 f_n - y_n^2 V_1 g_n) \quad (3.12a)$$

and

$$y'_{n+1} = y'_n + \frac{h}{2}(W'_1 f_n - y_n'^2 V'_1 g_n) \quad (3.12b)$$

Comparing corresponding coefficients of powers of h in equations (3.10a) and(3.10b) with equations (3.12a) and (3.12b) respectively we get

$$W_1 f_n - y_n^2 V_1 g_n = f_n \quad (3.13a)$$

and

$$W'_1 f_n - y_n'^2 V'_1 g_n = f_n. \quad (3.13b)$$

From equation (3.6c) we have that

$$g_n = -\frac{f_n}{y_n^2} \quad (3.14)$$

So equations (3.13a) and (3.13b) becomes

$$W_1 + V_1 = 1 \quad (3.15a)$$

$W_1' + V_1' = 1 \quad .$ (3.15b)

We now have 4 unknowns W_1, V_1, W_1', V_1' in two equations to be solved in order to determine the particular scheme in question.Equations (3.15a) and (3.15b) can be solved by

i. Choosing the parameters

$$W_1 = 0, V_1 = 1, W_1' = 0, V_1' = 1$$

equations (3.4) and (3.5) become

$$y_{n+1} = \frac{y_n + hy_n'}{1 + y_n H_1}$$

and

$$y_{n+1}' = \frac{y_n'}{1 + \frac{1}{h} y_n' H_1}$$

where

$H_1 = \frac{h^2}{2} g(x_n, Z_n, Z_n').$

ii. Choosing the parameters

$W_1 = \frac{1}{2}, V_1 = \frac{1}{2}, W_1' = \frac{1}{2}, V_1' = \frac{1}{2},$

equations (3.4) and (3.5) become

$$y_{n+1} = \frac{y_n + hy_n' + \frac{1}{2} K_1}{1 + \frac{1}{2} y_n H_1}$$

and

$$y_{n+1}' = \frac{y_n' + \frac{1}{2} hK_1}{1 + \frac{1}{2} hy_n' H_1}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y_n')$$

$$H_1 = \frac{h^2}{2} g(x_n, Z_n, Z'_n)$$

iii. Choosing the parameters

$$W_1 = \frac{1}{3}, V_1 = \frac{2}{3}, W'_1 = \frac{1}{3}, V'_1 = \frac{2}{3}$$

equation (3.4) and (3.5) become

$$y_{n+1} = \frac{y_n + hy'_n + \frac{1}{3}K_1}{1 + \frac{2}{3}y_n H_1}$$

$$and$$

$$y'_{n+1} = \frac{y'_n + \frac{1}{3}hK_1}{1 + \frac{2}{3}hy'_n H_1}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y'_n)$$

$H_1 = \frac{h^2}{2} g(x_n, Z_n, Z'_n)$.

Note that the parameters in all three cases were chosen arbitrarily but making sure that they satisfy the equations(3.15a) and (3.15b).

3.2 DERIVATION OF THE TWO-STAGE SCHEME

If we set $s = 2$ in equations (3.1) and (3.2) we get

$$y_{n+1} = \frac{y_n + hy'_n + W_1K_1 + W_2K_2}{1 + y_n(V_1H_1 + V_2)} \quad (3.16a)$$

$$y'_{n+1} = \frac{y'_n + \frac{1}{h}(W'_1K_1 + W'_2K_2)}{1 + \frac{1}{h}y'_n(V'_1H_1 + V'_1H_2)} \quad (3.16b)$$

where

$$K_1 = \frac{h^2}{2} f_n \text{and} H_1 = \frac{h^2}{2} g_n \quad (3.17)$$

$$K_2 = \frac{h^2}{2} f(x_n + c_2 h, y_n + hc_2 y_n^1 + a_{21} k_{1,} y_n^1 + \frac{1}{h} b_{21} k_1) \qquad (3.17a)$$

$$H_2 = \frac{h^2}{2} g(x_n + d_2 h, Z_n + hd_2 Z_n' + \alpha_{21} H_1, Z_n' + \frac{1}{h} \beta_{21} H_1)\ . \qquad (3.17b)$$

Adopting the Binomial expansion on the right hand side of equations (3.16a) and (3.16b) we get

$$y_{n+1} = (y_n + hy_n' + W_1 K_1 + W_2 K_2)(1 + y_n V_1 H_1 + y_n V_2 H_2)^{-1}$$

$$y_{n+1} = (y_n + hy_n' + W_1 K_1 + W_2 K_2)(1 - y_n V_1 H_1 - y_n V_2 H_2 \ldots)$$

$$y_{n+1} = y_n + hy_n' + W_1 K_1 + W_2 K_2 - y_n^2 V_1 H_1 - y_n^2 V_2 H_2 \ldots\ . \qquad (3.18a)$$

Again using the binomialexpansion on the right hand side of equation (3.16b) , we obtain

$$y_{n+1}' = (y_n' + \frac{1}{h} W_1' K_1 + \frac{1}{h} W_2' K_2)(1 + \frac{1}{h} y_n' V_1' H_1 + \frac{1}{h} y_n' V_2' H_2)^{-1}$$

$$y_{n+1}' = (y_n' + \frac{1}{h} W_1' K_1 + \frac{1}{h} W_2' K_2)(1 - \frac{1}{h} y_n' V_1' H_1 - \frac{1}{h} y_n' V_2' H_2 \ldots)$$

$$y_{n+1}' = y_n' + \frac{1}{h} W_1' K_1 + \frac{1}{h} W_2' K_2 - \frac{1}{h} y_n'^2 V_1' H_1 - \frac{1}{h} y_n'^2 V_2' H_2 \ldots. \qquad (3.18b)$$

The Taylor series expansion of K_2 about (x_n, y_n, y_n') gives

$$K_2 = \frac{h^2}{2}\Big\{f_n + \Big[c_2 h f_x + (c_2 h y_n' + a_{21} K_1) f_y + \frac{1}{h} b_{21} K_1 f_{y'}\Big] + \frac{1}{2!}[c_2^2 h^2 f_{xx} + 2c_2 h(c_2 h y_n' + a_{21} K_1) f_{xy} + 2c_2 h. \frac{1}{h} b_{21} K_1 f_{xy'} + \frac{2}{h} b_{21} K_1 (c_2 h y_n' + a_{21} K_1) f_{yy'} + c_2^2 h^2 y_n'^2 f_{yy} + 2a_{21} K_1 c_2 h y_n' f_{yy} + \frac{1}{h^2} b_{21}^2 K_1^2 f_{y'y'} \ldots\Big\}.$$

Further, applying the differential notations in equation (3.9) we get

$$K_2 = \frac{h^2}{2} f_n + \frac{h^3}{2} c_2 \Delta f_n + \frac{h^4}{4}\left(c_2^2 \Delta^2 f_n + a_{21} f_n f_y\right) + 0(h^5)\ . \qquad (3.19a)$$

A similar expansion of H_2 about (x_n, Z_n, Z_n') with application of the differential notations in equation (3.9) yields

$$H_2 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_2 \Delta g_n + \frac{h^4}{4}\left(d_2^2 \Delta^2 g_n + \alpha_{21} g_n g_y\right) + 0(h^5) \qquad (3.19b)$$

Using(3.17), (3.19a) and (3.19b) on (3.18a) and (3.18b) we get the following results:

$$y_{n+1} = y_n + hy_n' + W_1 \frac{h^2}{2} f_n + W_2 \left(\frac{h^2}{2} f_n + \frac{h^3}{2} c_2 \Delta f_n\right) - y_n^2 V_1 \frac{h^2}{2} g_n - y_n^2 V_2 \left(\frac{h^2}{2} g_n + \frac{h^3}{2} d_2 \Delta g_n\right) \dots$$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}(W_1 f_n + W_2 f_n - y_n^2 V_1 g_n - y_n^2 V_2 g_n) + \frac{h^3}{2}(W_2 c_2 \Delta f_n - y_n^2 V_2 d_2 \Delta g_n) + \cdots \qquad (3.20a)$$

and

$$y_{n+1}' = y_n' + \frac{1}{h} W_1' \frac{h^2}{2} f_n + \frac{1}{h} W_2' \left(\frac{h^2}{2} f_n + \frac{h^3}{2} c_2 \Delta f_n\right) - \frac{1}{h} y_n'^2 V_1' \frac{h^2}{2} g_n - \frac{1}{h} y_n'^2 V_2' \left(\frac{h^2}{2} g_n + \frac{h^3}{2} d_2 \Delta g_n\right) \dots$$

$$y_{n+1}' = y_n' + \frac{h}{2}(W_1' f_n + W_2' f_n - y_n'^2 V_1' g_n - y_n'^2 V_2' g_n) + \frac{h^2}{2}(W_2' C_2 \Delta f_n - y_n'^2 V_2' d_2 \Delta g_n) + \cdots \qquad (3.20b)$$

Comparing corresponding coefficients of powers of h in equations (3.20a) and (3.20b) with the Taylor series in equations (3.10a) and (3.10b) respectively we get

$$W_1 f_n + W_2 f_n - y_n^2 V_1 g_n - y_n^2 V_2 g_n = f_n$$

$$W_2 c_2 \Delta f_n - y_n^2 V_2 d_2 \Delta g_n = \frac{\Delta f_n}{3}$$

$$W_1' f_n + W_2' f_n - y_n'^2 V_1' g_n - y_n'^2 V_2' g_n = 2 f_n$$

$$W_2' c_2 \Delta f_n - y_n'^2 V_2' d_2 \Delta g_n = \Delta f_n$$

which gives the equations

$$W_2 + W_2 + V_1 + V_2 = 1$$

$$W_2c_2 + V_2d_2 = \frac{1}{3}$$

$W_1' + W_2' + V_1' + V_2' = 2$ 3.21

$W_2'c_2 + V_2'd_2 = 1.$

We now have four equations with ten unknowns $W_1, W_2, W_1', W_2', V_1, V_2, V_1', V_2', c_2, d_2$ to be solved in order to determine the particular scheme in question. We can solve the equations in (3.21) by

i. Choosing the parameters

$$W_1 = W_2 = 0, V_1 = \frac{2}{3}, V_2 = \frac{1}{3}, W_1' = W_2' = V_1' = V_2' = \frac{1}{2}, c_2 = d_2 = 1$$

and putting in the values in equations (3.17), (3.19a) and (3.19b) into equations (3.16a) and (3.16b) respectively we get

$$y_{n+1} = \frac{y_n + hy_n'}{1 + y_n(\frac{2}{3}H_1 + \frac{1}{3}H_2)}$$

and

$$y_{n+1}' = \frac{y_n' + \frac{1}{h}\left(\frac{1}{2}K_1 + \frac{1}{2}K_2\right)}{1 + \frac{1}{h}y_n'\left(\frac{1}{2}H_1 + \frac{1}{2}H_2\right)}$$

where

$K_1 = \frac{h^2}{2}f_n, H_1 = \frac{h^2}{2}g_n$

$$K_2 = \frac{h^2}{2}f_n + \frac{h^3}{2}\Delta f_n$$

$$H_2 = \frac{h^2}{2}g_n + \frac{h^3}{2}\Delta g_n$$

ii. Choosing the parameters

$$W_1 = \frac{2}{3}, W_2 = \frac{1}{3}, V_1 = V_2 = 0, W_1' = \frac{1}{3}, W_2' = \frac{1}{4}, V_1' = \frac{2}{3}, V_2' = \frac{3}{4}$$

$$c_2 = d_2 = 1$$

and putting in the values in equations (3.17), (3.19a) and (3.19b) into equations (3.16a) and (3.16b) respectively we get

$$y_{n+1} = y_n + hy_n' + \frac{2}{3}K_1 + \frac{1}{3}K_2$$

and

$$y_{n+1}' = \frac{y_n' + \frac{1}{h}(\frac{1}{3}K_n + \frac{1}{4}K_{2)}}{1 + \frac{1}{h}y_n'(\frac{2}{3}H_1 + \frac{3}{4}H_2)}$$

where

$$K_1 = \frac{h^2}{2}f_n, H_1 = \frac{h^2}{2}g_n$$

$$K_2 = \frac{h^2}{2}f_n + \frac{h^3}{2}\Delta f_n$$

$$H_2 = \frac{h^2}{2}g_n + \frac{h^3}{2}\Delta g_n.$$

The parameters in both cases were chosen arbitrarily but making sure that they satisfy the equations in (3.21)

3.3 DERIVATION OF THE THREE-STAGE SCHEME

If we set $s = 3$ in equations (3.1) and (3.2) we have

$$y_{n+1} = \frac{y_n + hy_n' + W_1K_1 + W_2K_2 + W_3K_3}{1 + y_n(V_1H_1 + H_2V_2 + V_3H_3)} \quad (3.22a)$$

$$y_{n+1}' = \frac{y_n' + \frac{1}{h}(W_1'K_1 + W_2'K_2 + W_3'K_3)}{1 + \frac{1}{h}y_n'(V_1'H_1 + V_1'H_2 + V_3'H_3)} \quad (3.22b)$$

where

$K_1 = \frac{h^2}{2} f_n$, $H_1 = \frac{h^2}{2} g_n$

$K_2 = \frac{h^2}{2} f(x_n + c_2h, y_n + hc_2y_n^1 + a_{21}k_{1,}y_n^1 + \frac{1}{h} b_{21}k_1)$

$$H_2 = \frac{h^2}{2} g(x_n + d_2h, Z_n + hd_2Z_n' + \alpha_{21}H_1, Z_n' + \frac{1}{h}\beta_{21}H_1)$$

$K_3 = \frac{h^2}{2} f(x_n + c_3h, y_n + hc_3y_n^1 + a_{31}k_1 + a_{32}k_2, y_n^1 + \frac{1}{h} b_{31}k_1 + \frac{1}{h} b_{32}k_2)$

$H_3 = \frac{h^2}{2} g(x_n + d_3h, Z_n + hd_3Z_n' + \alpha_{31}H_1 + \alpha_{32}H_2, Z_n' + \frac{1}{h}\beta_{31}H_1 + \frac{1}{h}\beta_{32}H_2)$

Adopting the Binomial expansion on the right hand side of equations (3.22a) and (3.22b) we get

$$y_{n+1} = (y_n + hy_n' + W_1K_1 + W_2K_2 + W_3K_3)(1 + y_nV_1H_1 + y_nV_2H_2 + y_nV_3H_3)^{-1}$$

$$y_{n+1} = (y_n + hy_n' + W_1K_1 + W_2K_2 + W_3K_3)(1 - y_nV_1H_1 - y_nV_2H_2 - y_nV_3H_3 \ldots)$$

$$y_{n+1} = y_n + hy_n' + W_1K_1 + W_2K_2 + W_3K_3 - y_n^2V_1H_1 - y_n^2V_2H_2 -$$

$y_n^2V_3H_3 - \cdots,$ (3.23a)

Again using the binomialexpansion on the right hand side of equation (3.22b) , we obtain

$$y_{n+1}' = (y_n' + \frac{1}{h}W_1'K_1 + \frac{1}{h}W_2'K_2 + \frac{1}{h}W_3'K_3)(1 + \frac{1}{h}y_n'V_1'H_1 + \frac{1}{h}y_n'V_2'H_2 + \frac{1}{h}y_n'V_3'H_3)^{-1}$$

$$y'_{n+1} = (y'_n + \frac{1}{h}W_1'K_1 + \frac{1}{h}y'_nW_2'K_2 + \frac{1}{h}y'_nW_3'K_3)(1 - \frac{1}{h}y'_nV_1'H_1 + \frac{1}{h}y'_nV_2'H_2 + \frac{1}{h}y'_nV_3'H_3 \dots)$$

$$y'_{n+1} = y'_n + \frac{1}{h}W_1'K_1 + \frac{1}{h}W_2'K_2 + \frac{1}{h}W_3'K_3 - \frac{1}{h}y_n'^2V_1'H_1 - \frac{1}{h}y_n'^2V_2'H_2 - \frac{1}{h}y_n'^2V_3'H_3 \dots \quad (3.23b)$$

The Taylor series expansion of K_3 about (x_n, y_n, y'_n) gives

$$K_3 = \frac{h^2}{2}\Big\{f_n + c_3hf_x + c_3hy'_nf_y + a_{31}K_1f_y + a_{32}K_2f_y + \frac{1}{h}b_{31}K_1f_{y'} + \frac{1}{h}b_{32}K_2f_{y'} + \frac{1}{2}\Big[c_3^2h^2f_{xx} + h^2c_3^2y_n'^2f_{yy} + a_{31}^2K_1^2f_{yy} + a_{32}^2K_2^2f_{yy} + 2hc_3y'_na_{31}K_1f_{yy} + 2hc_3y'_na_{32}K_2f_{yy} + 2a_{31}a_{32}K_1K_2f_{yy} + \frac{1}{h^2}b_{31}^2K_1^2f_{y'y'} + \frac{2}{h^2}b_{31}b_{32}K_1K_2f_{y'y'} + \frac{1}{h^2}b_{32}^2K_2^2f_{y'y'} + 2c_3^2h^2y'_nf_{xy} + 2c_3ha_{31}K_1f_{xy} + 2c_3ha_{32}K_2f_{xy} + 2c_3b_{31}K_1f_{xy'} + 2c_3b_{32}K_2f_{xy'} + 2c_3y'_nb_{31}K_1f_{yy'} + 2c_3y'_nb_{32}K_2f_{yy'} + \frac{2}{h}a_{31}b_{31}K_1^2f_{yy'} + \frac{2}{h}a_{31}b_{32}K_1K_2f_{yy'} + \frac{2}{h}a_{32}b_{31}K_1K_2f_{yy'} + \frac{2}{h}a_{32}b_{32}K_2^2f_{yy'}\Big] + \cdots\Big\}$$

Further simplification and applying the constraints in equations (3.3c) and the differential notations in equation (3.9) and putting in the values of K_1, H_1, K_2, H_2 we arrive at

$$K_3 = \frac{h^2}{2}f_n + \frac{h^3}{2}c_3\Delta f_n + \frac{h^4}{4}\Big\{c_3^2\Delta^2 f_n + c_3f_nf_y + \frac{b_{32}}{2}c_2\Delta f_nf_{y'}\Big\} + 0(h^5) \quad (3.24a)$$

Similar operations carried out on H_3 will give us

$$H_3 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_3 \Delta g_n + \frac{h^4}{4}\left\{d_3^2 \Delta^2 g_n + d_3 g_n g_y + \frac{\beta_{32}}{2} c_2 \Delta g_n g_{y'}\right\} + 0(h^5) \qquad (3.24b)$$

$$\begin{aligned} y_{n+1} = {} & y_n + h y_n' + W_1 \frac{h^2}{2} f_n + W_2 \frac{h^2}{2} f_n + W_2 \frac{h^3}{2} c_2 \Delta f_n + W_3 \frac{h^2}{2} f_n + W_3 \frac{h^3}{2} c_3 \Delta f_n \\ & + W_3 \frac{h^4}{4}\left\{c_3^2 \Delta^2 f_n + c_3 f_n f_y + \frac{b_{32}}{2} c_2 \Delta f_n f_{y'}\right\} - y_n^2 V_1 \frac{h^2}{2} g_n - y_n^2 V_2 \frac{h^2}{2} g_n \\ & - y_n^2 V_2 \frac{h^2}{2} d_2 \Delta g_n - y_n^2 V_3 \frac{h^2}{2} g_n - y_n^2 V_3 \frac{h^3}{2} d_3 \Delta g_n \\ & - y_n V_3 \frac{h^4}{4}\left\{d_3^2 \Delta^2 g_n + d_3 g_n g_y + \frac{\beta_{32}}{2} c_2 \Delta g_n g_{y'}\right\} \dots \end{aligned}$$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} f_n (W_1 + W_2 + W_3 + V_1 + V_2 + V_3) + \frac{h^3}{2} \Delta f_n (W_2 c_2 + W_3 c_3 + V_2 d_2 + V_3 d_3) + \frac{h^4}{4}\left[(W_3 c_3^2 + V_3 d_3^2)\Delta^2 f_n + (W_3 c_3 + V_3 d_3) f_n f_y + (W_3 \frac{b_{32}}{2} c_2 + V_3 \frac{\beta_{32}}{2} d_2) \Delta f_n f_{y'}\right] + 0(h^5) \qquad (3.25a)$$

$$\begin{aligned} y_{n+1}' = {} & y_n' + \frac{1}{h} W_1' \frac{h^2}{2} f_n + \frac{1}{h} W_2' \frac{h^2}{2} f_n + \frac{1}{h} W_2' \frac{h^3}{2} c_2 \Delta f_n + \frac{1}{h} W_3' \frac{h^2}{2} f_n + \frac{1}{h} W_3' \frac{h^3}{2} c_3 \Delta f_n \\ & + \frac{1}{h} W_3' \frac{h^4}{4}\left[c_3^2 \Delta^2 f_n + c_3 f_n f_y + \frac{b_{32}}{2} c_2 \Delta f_n f_{y'}\right] - \frac{1}{h} y_n'^2 V_1' \frac{h^2}{2} g_n \\ & - \frac{1}{h} y_n'^2 V_2' \frac{h^2}{2} g_n - \frac{1}{h} y_n'^2 V_2' \frac{h^3}{2} d_2 \Delta g_n - \frac{1}{h} y_n'^2 V_3' \frac{h^2}{2} g_n - \frac{1}{h} y_n'^2 V_3' \frac{h^3}{2} d_3 \Delta g_n \\ & - \frac{1}{h} y_n'^2 V_3' \frac{h^4}{4}\left[d_3^2 \Delta^2 g_n + d_3 g_n g_y + \frac{\beta_{32}}{2} c_2 \Delta g_n g_{y'}\right] + \cdots \end{aligned}$$

$$y_{n+1}' = y_n' + \frac{h}{2} f_n (W_1' + W_2' + W_3' + V_1' + V_2' + V_3') + \frac{h^2}{2} \Delta f_n (c_2 W_2' + c_3 W_3' + d_2 V_2' + d_3 V_3') + \frac{h^4}{4}\left[(W_3' c_3^2 + V_3' d_3^2)\Delta^2 f_n + (W_3' c_3 + V_3' d_3) f_n f_y + (W_2' \frac{b_{32}}{2} c_2 + V_3' \frac{\beta_{32}}{2} d_2) \Delta f_n f_{y'}\right] + 0(h^4)$$

(3.25b)

Comparing corresponding coefficients of the powers of h in equations (3.25a) and (3.25b) with the Taylor series in equations (3.10a) and (3.10b) respectively we get

$W_1 + W_2 + W_3 + V_1 + V_2 + V_3 = 1$

$W_2 c_2 + W_3 c_3 + V_2 d_2 + V_3 d_3 = \frac{1}{3}$

$c_3^2 W_3 + d_3^2 V_3 = \frac{1}{6}$

$c_3 W_3 + d_3 V_3 = \frac{1}{6}$

$c_2 \frac{b_{32}}{2} W_3 + d_2 \frac{\beta_{32}}{2} V_3 = \frac{1}{6}$

$W_1' + W_2' + W_3' + V_1' + V_2' + V_3' = 2$ (3.26)

$c_2 W_2' + c_3 W_3' + d_2 V_2' + d_3 V_3' = 1$

$c_3^2 W_3' + d_3^2 V_3' = \frac{2}{3}$

$c_3 W_3' + d_3 V_3' = \frac{2}{3}$

$c_2 \frac{b_{32}}{2} W_3' + d_2 \frac{\beta_{32}}{2} V_3' = \frac{2}{3}$

We now have ten equations with eighteen unknowns $W_1, W_2, W_3, W_1', W_2', W_3', V_1, V_2, V_3, V_1', V_2', V_3', c_2, c_3, d_2, d_3, b_{32}, \beta_{32}$ to be solved in order to determine the particular scheme in question. We can solve the equations in (3.26) by

i. Choosing the parameters

$$W_1 = W_2 = W_3 = 0, V_1 = \frac{2}{3}, V_2 = V_3 = \frac{1}{6}$$

$$W_1' = 1, W_2' = \frac{1}{3}, W_3' = \frac{2}{3}, 0, V_1' = V_2' = V_3' = 0$$

$$c_2 = c_3 = d_2 = d_3 = 1$$

Then equation (3.22a) and (3.22b) becomes

$$y_{n+1} = \frac{y_n + hy_n'}{1 + y_n(\frac{2}{3}H_1 + \frac{1}{6}H_2 + \frac{1}{6}H_3)}$$

and

$$y_{n+1}' = y_n' + \frac{1}{h}(k_1 + \frac{1}{3}k_2 + \frac{2}{3}k_3)$$

where

$K_1 = \frac{h^2}{2}f_n, H_1 = \frac{h^2}{2}g_n$

$$K_2 = \frac{h^2}{2}f_n + \frac{h^3}{2}\Delta f_n$$

$$H_2 = \frac{h^2}{2}g_n + \frac{h^3}{2}\Delta g_n$$

$K_3 = \frac{h^2}{2}f_n + \frac{h^3}{2}\Delta f_n + \frac{h^4}{4}(\Delta^2 f_n + f_n f_y + \frac{b_{32}}{2}\Delta f_n f_{y'})$

$H_3 = \frac{h^2}{2}g_n + \frac{h^3}{2}\Delta g_n + \frac{h^4}{4}(\Delta^2 g_n + g_n g_y + \frac{\beta_{32}}{2}\Delta g_n f_g).$

The parameters for this scheme are chosen such that they satisfy all the equations in (3.26).

3.4 ANALYSIS OF THE BASIC PROPERTIES

Like all computational methods, the use of this scheme for the numerical solution of initial value problems will generate errors at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during computer implementation.

The accuracy of the initial value problem depends on the magnitude of the errors. Thus, it is important that the numerical solution approximates the exact solution and that the numerical solution tends to the exact solution as the stepsize tends to zero. Because of this, the concepts of consistencyand convergence of this formula are important and they are considered in this section.

3.5 CONSISTENCY PROPERTY

A numerical method is said to be consistent with the differential equation if the numerical formula approximate the differential equation as the stepsize h approaches zero.

To show that this scheme is consistent we proceed as follows:-

Recall the equation

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{i=1}^{s} W_i K_i}{1 + y_n \sum_{i=1}^{s} V_i H_i} \tag{3.27}$$

Subtract y_n on both sides of equation (3.27) to get

$$y_{n+1} - y_n = \frac{y_n + hy'_n + \sum_{i=1}^{s} W_i K_i}{1 + y_n \sum_{i=1}^{s} V_i H_i} - y_n$$

$$y_{n+1} - y_n = \frac{y_n + hy'_n + \sum_{i=1}^{s} W_i K_i - y_n - y_n^2 \sum_{i=1}^{s} V_i H_i}{1 + y_n \sum_{i=1}^{s} V_i H_i}$$

$$y_{n+1} - y_n = \frac{hy'_n + \sum_{i=1}^{s} W_i K_i - y_n^2 \sum_{i=1}^{s} V_i H_i}{1 + y_n \sum_{i=1}^{s} V_i H_i}$$

but

$$K_i = \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{i} a_{ij} K_j, y_n' + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j)$$

$$H_i = \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z_n' + \sum_{j=1}^{i} \alpha_{ij} H_j, z_n' + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j)$$

andso we have

$$y_{n+1} - y_n$$
$$= \frac{hy_n' + \sum_{i=1}^{s} W_i \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{i} a_{ij} K_j, y_n' + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j) - y_n^2 \sum_{i=1}^{s} V_i \frac{h^2}{2} g(x_n}{1 + y_n \sum_{i=1}^{s} V_i \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z_n' + \sum_{j=1}^{i} \alpha_{ij} H_j, z_n'}$$

Dividing all through by h and taking limit as h tends to zero on both sides we have

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = y_n' \qquad (3.28)$$

weagain recall

$$y_{n+1}' = \frac{y_n' + \frac{1}{h}\sum_{i=1}^{s} W_i' K_i}{1 + \frac{1}{h} y_n' \sum_{i=1}^{s} V_i' H_i} \qquad (3.29)$$

subtracty_n' on both sides of (3.29) to get

$$y_{n+1}' - y_n' = \frac{y_n' + \frac{1}{h}\sum_{i=1}^{s} W_i' K_i'}{1 + \frac{1}{h} y_n' \sum_{i=1}^{s} V_i' H_i} - y_n'$$

simplifying

$$y'_{n+1} - y'_n = \frac{y'_n + \frac{1}{h}\sum_{i=1}^{s} W'_i K_i - y'_n - \frac{1}{h} y_n'^2 \sum_{i=1}^{s} V'_i H_i}{1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i}$$

furthersimplification

$$y'_{n+1} - y'_n = \frac{\frac{1}{h}\sum_{i=1}^{s} W'_i K_i - \frac{1}{h} y_n'^2 \sum_{i=1}^{s} V'_i H_i}{1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i}$$

but

$$K_i = \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{i} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j)$$

$$H_i = \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{i} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j)$$

hence

$$y'_{n+1} - y'_n$$
$$= \frac{\frac{1}{h}\sum_{i=1}^{s} W'_i \frac{h^2}{2} f(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{i} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j) - \frac{1}{h} y_n'^2 \sum_{i=1}^{s} V'_i \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z'_n}{1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i \frac{h^2}{2} g(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{i} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j)}$$

dividingall through by h and taking the limit as h tends to zero on both sides we get

$$\lim_{h \to 0} \frac{y'_{n+1} - y'_n}{h}$$
$$= \frac{\frac{1}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{i} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{i} b_{ij} K_j\right) - \frac{1}{2} y_n'^2 g(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=}^{i}}{y'_n \, g(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{i} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j)}$$

but

$$f_n = y_n'^2 g(x_n, z_n, z'_n)$$

hence

$$\lim_{h\to 0}\frac{y'_{n+1}+y'_n}{h}=f_n$$

Showing that the computational method is consistent.

3.6 CONVERGENCE

A numerical method is said to be convergent if the discretization error (e_{n+1}) which is the difference between the exact solution $y'(x_{n+1})$ and the numerical solution y'_{n+1} generated by the method tends to zero. That is, if

$$e_{n+1}=y'(x_{n+1})-y'_{n+1} \qquad (3.30)$$

tends to zero as n increases without bound ($n\to\infty$) .

Given that our computational scheme is

$$y'_{n+1}=\frac{y'_n+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i}{1+\frac{1}{h}y'_n\sum_{i=1}^{s}V_i'H_i} \qquad (3.31)$$

While the exact solution $y'(x_{n+1})$ is seen to satisfy the difference equation of the form

$$y'(x_{n+1})=\frac{y'(x_n)+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i}{1+\frac{1}{h}y'(x_n)\ \sum_{i=1}^{s}V_i'H_i}+T_{n+1} \qquad (3.32)$$

whereT_{n+1} is the truncation error,

Subtracting (3.31) from (3.32) we get

$$y'(x_{n+1})-y'_{n+1}=\frac{y'(x_n)+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i}{1+\frac{1}{h}y'(x_n)\ \sum_{i=1}^{s}V_i'H_i}-\frac{y'_n+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i}{1+\frac{1}{h}y'_n\sum_{i=1}^{s}V_i'H_i}+T_{n+1}$$

adoptingequation (3.30) we have

$$e_{n+1}=\frac{(y'(x_n)+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i)\left(1+\frac{1}{h}y'_n\sum_{i=1}^{s}V_i'H_i\right)-(y'_n+\frac{1}{h}\sum_{i=1}^{s}W_i'K_i)(1+\frac{1}{h}y'(x_n)\ \sum_{i=1}^{s}V_i'H_i)}{\left(1+\frac{1}{h}y'(x_n)\sum_{i=1}^{s}V_i'H_i\right)(1+\frac{1}{h}y'_n\sum_{i=1}^{s}V_i'H_i)}+T_{n+1}$$

simplifyingwe get

$$e_{n+1} = \frac{e_n + \frac{1}{h^2}(y'_n - y'(x_n))[\sum_{i=1}^{s} W'_i K_i \sum_{i=1}^{s} V'_i H_i]}{\left(1 + \frac{1}{h} y'(x_n) \sum_{i=1}^{s} V'_i H_i\right)\left(1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i\right)} + T_{n+1}$$

$$e_{n+1} = \frac{e_n + e_n \frac{1}{h^2}(\sum_{i=1}^{s} W'_i K_i \sum_{i=1}^{s} V'_i H_i)}{\left(1 + \frac{1}{h} y'(x_n) \sum_{i=1}^{s} V'_i H_i\right)\left(1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i\right)} + T_{n+1}$$

$$e_{n+1} = \frac{e_n\left[1 + \frac{1}{h^2}(\sum_{i=1}^{s} W'_i K_i \sum_{i=1}^{s} V'_i H_i)\right]}{\left(1 + \frac{1}{h} y'(x_n) \sum_{i=1}^{s} V'_i H_i\right)\left(1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i\right)} + T_{n+1} \qquad (3.33)$$

setting

$$A_n = \left[1 + \frac{1}{h^2}\left(\sum_{i=1}^{s} W'_i K_i \sum_{i=1}^{s} V'_i H_i\right)\right]$$

$$B_n = \left(1 + \frac{1}{h} y'(x_n) \sum_{i=1}^{s} V'_i H_i\right)$$

$$C_n = \left(1 + \frac{1}{h} y'_n \sum_{i=1}^{s} V'_i H_i\right)$$

$$T_{n+1} = T$$

thenthe equation (3.33) becomes

$e_{n+1} = \frac{A_n}{B_n C_n} e_n + T$.

Let

$B = maxB_n > 0, C = maxC_n > 0, A = maxA_n < 1$

then

$$e_{n+1} \leq \frac{A}{BC} e_n + T$$

set$\frac{A}{BC} = K < 1$

then

$$e_{n+1} \leq Ke_n + T$$

set$n = 0$

$$e_1 = Ke_0 + T$$

$$e_2 = Ke_1 + T$$

substitutefor e_1 into e_2 we get

$$e_2 = K^2 e_0 + KT + T$$

$$e_3 = Ke_2 + T$$

$$e_3 = K^3 e_0 + K^2 T + T$$

whichimplies that

$$e_{n+1} = Ke_0^{n+1} + \sum_{t=0}^{n+1} K^t T$$

Since $\frac{A}{BC} = K < 1$, we can see that as $n \to \infty$, then $e \to 0$. This proves that the scheme converges.

CHAPTER FOU5R

NUMERICAL EXAMPLES

4.0 INTRODUNCTION

In this chapter, we shall by way of illustration and computation test the efficiency of our methods by comparing its results with the exact solutions for some second order ordinary differential equations.

4.1 Example

Solve $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

The exact solution is given by

$y(x) = 9e^{-2x} - 7e^{-3x}$, $y'(x) = -18e^{-2x} + 21e^{-3x}$

4.2 Example

Solve $y'' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$,

The exact solution is given by

$y(x) = \cos x\sqrt{2}$, $y'(x) = -\sqrt{2}\sin x\sqrt{2}$

4.3 Example

Solve, $y''y' = 2$, $y(0) = 1$, $y'(0) = 2$ (a non-linear problem)

The exact solution is given by

$y(x) = \frac{4}{3}(x+1)^{\frac{3}{2}} - \frac{1}{3}$, $y'(x) = 2(x+1)^{\frac{1}{2}}$

4.4 DISCUSSION OF RESULTS

TABLE 4.1.1 SOLUTION TO EXAMPLE 4.1 when h=0.1

At h=0.1	**Exact Solutions**		**Approximate solutions**		**Error**	**Error**
h	$Y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	-2.182849233	0.820029047	2.18230877	0.814499999	$-5.40463X10^{-4}$	$5.529075x10^{-3}$
0.2	2.191198961	-0.540716472	2.192101098	-0.547699063	$-9.02137x\ 10^{-4}$	$6.982591x10^{-3}$
0.3	2.093317107	-1.340646595	2.09559265	-1.348042951	$-2.275543x10^{-3}$	$7.396356x10^{-3}$
0.4	1.935601194	-1.762842904	1.938709176	-1.77031471	$-3.107982x10^{-3}$	$7.471806x10^{-3}$
0.5	1.74900385	-1.936096578	1.752383838	-1.943413447	$-3.379988x10^{-3}$	$7.316869x10^{-3}$
0.6	1.55365569	-1.950219162	1.556867281	-1.957145375	$-3.211591x10^{-3}$	$6.926213x10^{-3}$
0.7	1.362177678	-1.867160358	1.364929653	-1.873469845	$-2.751975x10^{-3}$	$6.309495x10^{-3}$
0.8	1.182042989	-1.729060305	1.184181022	-1.734570642	$-2.138033x10^{-3}$	$5.510337x10^{-3}$
0.9	1.017251405	-1.56406422	1.018731998	-1.568659249	$-1.480593x10^{-3}$	$4.595029x10^{-3}$
1.0	0.86950807	-1.390506663	0.870370642	-1.394143752	$-8.62572x10^{-4}$	$3..3637089x10^{-3}$

TABLE 4.1.2 SOLUTION TO EXAMPLE 4.1 when h=0.01

At h=0.01	**Exact Solutions**		**Approximate solutions**		**Error**	**Error**
h	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	-2.182849233	0.820029047	2.182819699	0.820033988	2.9534×10^{-5}	-4.914×10^{-6}
0.2	2.191198961	-0.540716472	2.191171121	-0.540702322	2.7789×10^{-5}	-1.415×10^{-5}
0.3	2.093317107	-1.340646595	2.09329813	-1.340629017	1.8977×10^{-5}	-1.7578×10^{-5}
0.4	1.935601194	-1.762842904	1.93559026	-1.762826812	1.0934×10^{-5}	-1.6092×10^{-5}
0.5	1.74900385	-1.936096578	1.74899788	-1.936084062	5.97×10^{-6}	-1.2516×10^{-5}
0.6	1.55365569	-1.950219162	1.553651518	-1.950210085	4.172×10^{-6}	-9.077×10^{-6}
0.7	1.362177678	-1.867160358	1.362172828	-1.867153375	4.85×10^{-6}	-6.983×10^{-6}
0.8	1.182042989	-1.729060305	1.182035839	-1.729053697	7.15×10^{-6}	-6.608×10^{-6}
0.9	1.017251405	-1.56406422	1.017241114	-1.564056415	1.0291×10^{-5}	-7.805×10^{-6}
1.0	0.86950807	-1.390506663	0.869494415	-1.390496495	1.3655×10^{-5}	-1.0168×10^{-5}

TABLE 4.1.3 SOLUTION TO EXAMPLE 4.1 when h=0.001

at h=0.001	**Exact Solutions**		**Approximate solutions**		**Error**	**Error**
H	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	-2.182849233	0.820029047	2.182848907	0.820029179	$3.26x10^{-7}$	$-1.05x10^{-7}$
0.2	2.191198961	-0.540716472	2.19119864	-0.540716254	$-3.0x10^{-8}$	$-2.18x10^{-7}$
0.3	2.093317107	-1.340646595	2.093316869	-1.340646334	$2.38x10^{-7}$	$-2.61x10^{-7}$
0.4	1.935601194	-1.762842904	1.935601037	-1.762842657	$1.57x10^{-7}$	$-2.47x10^{-7}$
0.5	1.74900385	-1.936096578	1.749003745	-1.93609637	$1.05x10^{-7}$	$-2.08x10^{-7}$
0.6	1.55365569	-1.950219162	1.553655607	-1.950218994	$8.3x10^{-8}$	$-1.68x10^{-7}$
0.7	1.362177678	-1.867160358	1.362177592	-1.867160217	$8.6x10^{-8}$	$-1.41x10^{-7}$
0.8	1.182042989	-1.729060305	1.182042885	-1.729060176	$1.04x10^{-7}$	$-1.29x10^{-7}$
0.9	1.017251405	-1.56406422	1.017251273	-1.564064086	$1.32x10^{-7}$	$-1.34x10^{-7}$
1.0	0.86950807	-1.390506663	0.869507907	-1.39050651	$1.63x10^{-7}$	$-1.53x10^{-7}$

TABLE 4.2.1 SOLUTION TO EXAMPLE 4.2 when h=0.1

at h=0.1	Exact Solutions		Approximate solutions		Error	Error
h	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	0.99001665	-0.199334	0.9900826	-0.199333	$6.59x10^{-5}$	$1.00x10^{-6}$
0.2	0.960265956	-0.394688	0.9605914	-0.394695	$3.254x10^{-4}$	$7.00x10^{-6}$
0.3	0.911341926	-0.5821613	0.9115397	-0.582169	$1.978x10^{-4}$	$7.7x10^{-6}$
0.4	0.844221414	-0.7580108	0.84468	-0.75804	$4.586x10^{-4}$	$2.92x10^{-5}$
0.5	0.760244597	-0.9187254	0.7610204	-0.918832	$7.758x10^{-4}$	$1.066x10^{-4}$
0.6	0.661088212	-1.061096	0.6617606	-1.0613407	$6.724x10^{-4}$	$2.447x10^{-4}$
0.7	0.548732084	-1.1822801	0.5498161	-1.1826387	$1.084x10^{-3}$	$3.586x10^{-4}$
0.8	0.425419594	-1.279858	0.426953	-1.2804097	$1.5334x10^{-3}$	$5.517x10^{-4}$
0.9	0.293612883	-1.3518813	0.2958722	-1.3527117	$2.2593x10^{-3}$	$8.304x10^{-4}$
1.0	0.155943694	-1.396912	0.1587413	-1.398166	$2.7976x10^{-3}$	$1.254x10^{-4}$

TABLE 4.2.2 SOLUTION TO EXAMPLE 4.2 when h=0.01

at h=0.01	Exact Solutions		Approximate solutions		Error	Error
H	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y'	y
0.1	0.99001665	-0.199334	0.990017021	-0.19933402	$-3.66\text{x}10^{-7}$	$2.00\text{x}10^{-8}$
0.2	0.960265956	-0.394688	0.96026734	-0.394688129	$-1.384\text{x}10^{-6}$	$1.29\text{x}10^{-7}$
0.3	0.911341926	-0.5821613	0.911344944	-0.582161883	$-3.018\text{x}10^{-6}$	$5.83\text{x}10^{-7}$
0.4	0.844221414	-0.7580108	0.844226619	-0.758012177	$-5.205\text{x}10^{-6}$	$1.377\text{x}10^{-6}$
0.5	0.760244597	-0.9187254	0.760252455	-0.918727982	$-7.858\text{x}10^{-6}$	$2.582\text{x}10^{-6}$
0.6	0.661088212	-1.061096	0.661099086	-1.061100445	$-1.0874\text{x}10^{-5}$	$4.445\text{x}10^{-6}$
0.7	0.548732084	-1.1822801	0.548746216	-1.182286956	$-1.4132\text{x}10^{-5}$	$6.856\text{x}10^{-6}$
0.8	0.425419594	-1.279858	0.425437093	-1.279867908	$-1.7499\text{x}10^{-5}$	$9.908\text{x}10^{-6}$
0.9	0.293612883	-1.3518813	0.293633729	-1.351895003	$-2.0846\text{x}10^{-5}$	$1.3703\text{x}10^{-5}$
1.0	0.155943694	-1.396912	0.155967762	-1.39693016	$-2.4069\text{x}10^{-5}$	$1.816\text{x}10^{-5}$

TABLE 4.2.3 SOLUTION TO EXAMPLE 4.2 when h=0.001

At h=0.001	Exact Solutions		Approximate solutions		Error	Error
	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	0.99001665	-0.199334	0.99001659	-0.199334	$4.1\text{x}10^{-8}$	$0.0\text{x}10^{-0}$
0.2	0.960265956	-0.394688	0.960266	-0.394687961	$0.0\text{x}10^{-0}$	$-3.9\text{x}10^{-8}$
0.3	0.911341926	-0.5821613	0.911341955	-0.582161313	$-5.5\text{x}10^{-8}$	$1.3\text{x}10^{-8}$
0.4	0.844221414	-0.7580108	0.844221465	-0.758010836	$-6.5\text{x}10^{-8}$	$3.6\text{x}10^{-8}$
0.5	0.760244597	-0.9187254	0.760244674	-0.918725396	$-7.7\text{x}10^{-8}$	$-4.0\text{x}10^{-9}$
0.6	0.661088212	-1.061096	0.661088318	-1.061096059	$-1.06\text{x}10^{-7}$	$5.9\text{x}10^{-8}$
0.7	0.548732084	-1.1822801	0.5487732223	-1.182280153	$-1.39\text{x}10^{-7}$	$5.3\text{x}10^{-8}$
0.8	0.425419594	-1.279858	0.425419765	-1.279858036	$-1.71\text{x}10^{-7}$	$3.6\text{x}10^{-8}$
0.9	0.293612883	-1.3518813	0.293613086	-1.351881401	$-2.03\text{x}10^{-7}$	$1.01\text{x}10^{-8}$
1.0	0.155943694	-1.396912	0.155943928	-1.396912178	$-2.34\text{x}10^{-7}$	$1.78\text{x}10^{-8}$

TABLE 4.3.1 SOLUTION TO EXAMPLE 4.3 when h=0.1

at h=0.1	Exact Solutions		Approximate solutions		Error	Error
h	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	1.204919644	2.097617696	1.20499445	2.097625	$-7.4806\text{x}10^{-5}$	$-7.304\text{x}10^{-6}$
0.2	1.419378851	2.19089023	1.419991157	2.190882025	$-6.12306\text{x}10^{-4}$	$8.205\text{x}10^{-6}$
0.3	1.642970737	2.28035085	1.643817571	2.280320238	$-8.46834\text{x}10^{-4}$	$3.0612\text{x}10^{-5}$
0.4	1.875336452	2.366431913	1.876383986	2.366378131	$-1.047534\text{x}10^{-3}$	$5.3782\text{x}10^{-5}$
0.5	2.116156409	2.449489743	2.117377646	2.449414346	$-1.221237\text{x}10^{-3}$	$7.5397\text{x}10^{-5}$
0.6	2.365143603	2.529822128	2.366516619	2.529727433	$-1.373016\text{x}10^{-3}$	$9.4695\text{x}10^{-4}$
0.7	2.622038424	2.607680962	2.623542146	2.607569396	$-1.506726\text{x}10^{-3}$	$1.11566\text{x}10^{-4}$
0.8	2.886604554	2.683281573	2.888229896	2.683155416	$-1.625342\text{x}10^{-3}$	$1.26157\text{x}10^{-4}$
0.9	3.158625684	2.75680975	3.160356907	2.756671043	$-1.731223\text{x}10^{-3}$	$1.38707\text{x}10^{-4}$
1.0	3.437902833	2.828427127	3.439729077	2.828277653	$-1.826244\text{x}10^{-3}$	$1.49471746\text{x}10^{-4}$

TABLE 4.3.2 SOLUTION TO EXAMPLE 4.3 when h=0.01

At h=0.01	**Exact Solutions**		**Aproximate solutions**		**Error**	**Error**
h	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	yS'
0.1	1.204919644	2.097617696	1.20492269	2.097617594	$-3.046x10^{-6}$	$1.02x10^{-7}$
0.2	1.419378851	2.19089023	1.419384437	2.190889902	$-5.586x10^{-6}$	$3.28x10^{-7}$
0.3	1.642970737	2.28035085	1.642978471	2.280350272	$-7.734x10^{-6}$	$5.78x10^{-7}$
0.4	1.875336452	2.366431913	1.875346025	2.366431097	$-9.573x10^{-6}$	$8.16x10^{-7}$
0.5	2.116156409	2.449489743	2.116167572	2.449488713	$-1.1161x10^{-6}$	$1.03x10^{-6}$
0.6	2.365143603	2.529822128	2.365156151	2.529820911	$-1.2548x10^{-5}$	$1.217x10^{-6}$
0.7	2.622038424	2.607680962	2.622052188	2.607679585	$-1.3764x10^{-5}$	$1.377x10^{-6}$
0.8	2.886604554	2.683281573	2.886619393	2.683280058	$-1.4839x10^{-5}$	$1.515x10^{-6}$
0.9	3.158625684	2.75680975	3.158641476	2.756808118	$-1.5792x10^{-5}$	$1.632x10^{-6}$
1.0	3.437902833	2.828427127	3.437919475	2.828425393	$-1.6642x10^{-5}$	$1.734x10^{-6}$

TABLE 4.3.3 SOLUTION TO EXAMPLE 4.3 when h=0.001

at h=0.001	Exact Solutions		Aproximate Solutions		Error	Error
h	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y	y'
0.1	1.204919644	2.097617696	1.204919674	2.097617695	$-3.0x10^{-8}$	$1.0x10^{-9}$
0.2	1.419378851	2.19089023	1.419378906	2.190890227	$-5.5x10^{-8}$	$3.0x10^{-9}$
0.3	1.642970737	2.28035085	1.642970814	2.28035084	$-7.7x10^{-8}$	$1.0x10^{-8}$
0.4	1.875336452	2.366431913	1.875336548	2.366431905	$-9.6x10^{-8}$	$8.0x10^{-9}$
0.5	2.116156409	2.449489743	2.11615652	2.449489732	$-1.11x10^{-7}$	$1.1x10^{-8}$
0.6	2.365143603	2.529822128	2.365143728	2.529822116	$-1.25x10^{-7}$	$1.2x10^{-8}$
0.7	2.622038424	2.607680962	2.622038561	2.607680948	$-1.37x10^{-7}$	$1.4x10^{-8}$
0.8	2.886604554	2.683281573	2.886604701	2.683281558	$-1.47x10^{-7}$	$1.5x10^{-8}$
0.9	3.158625684	2.75680975	3.158625841	2.756809734	$-1.57x10^{-7}$	$1.6x10^{-8}$
1.0	3.437902833	2.828427127	3.437902998	2.828427107	$-1.65x10^{-7}$	$2.0x10^{-8}$

The first example 4.1 is an initial value second order ordinary differential equation. This equation suits the scheme properly as it is an equation of the form (x, y, y') . The results are better with smaller step lengths.

The example 4.2 is a special case of the second order ordinary differential equation. The results on table 4.2.1 are good approximations of the exact solution. We also see from table 4.2.2 and 4.2.3 that the results from this scheme become closer to the exact solution as the step size is further reduced.

The third example considered is a non-linear initial value second order ordinary differential equation. Explicit schemes are known to perform poorly on non-linear ordinary differential equations, but this scheme has done very well as seen in table 4.3.1, 4.3.2, 4.3.3.

In all three examples we see that the scheme has given good approximations and they are more accurate as the step size is reduced.

CHAPTER FIVE

5.0 INTRODUCTION

This project work developed an explicit rational Runge-Kutta scheme for the solution of initial value second order ordinary differential equations. The results obtained from the scheme when used to solve some numerical problems were found to give good approximations when compared with the exact solution.

5.1 SUMMARY

The analysis carried out from the results show that the new scheme (the explicit rational Runge-Kutta scheme) was found to perform well especially with smaller grid points.

5.2 CONCLUSION

We see that this scheme though it was developed to solve second order ordinary differential equations of the form $y'' = f(x, y, y')$ has been able to solve a second order ordinary differential equation of the form $y'' = f(x, y)$. The new scheme also shows significant move to accuracy whenever the step length is reduced. It has also shown to be capable of handling both linear and non-linear cases of initial value second order ordinary differential equations.

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