GAME VALUES FOR SOME PURSUIT-EVASION DIFFERENTIAL GAME PROBLEMS

BY

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DECLARATION

I hereby declare that this work is the product of my research efforts undertaken under the supervision of Dr. Abbas Ja'afaru Badakaya, and has not been presented in Nigeria and World at large for the award of a degree or any certificate. All sources have been duly acknowledged.

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CERTIFICATION

This is to certify that the research work for this dissertation and the sebsequent write-up were carried out by JAMILU ADAMU (SPS/17/MMT/00013) under the following supervision:

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Contents

CHAPTER FOUR 21

CHAPTER FIVE 51

ABSTRACT

In this dissertation, we consider three pursuit-evasion differential game problems of many pursuers and one evader in the Hilbert space l_2. In each of the problem the equations of motion of the pursuers are described by first order differential equations and that of the evader by second order differential equation. In first problem, control functions of the pursuers and evader are subject to integral and geometric constraints respectively. Control functions of both pursuers and evader are subject to geometric constraints in the second problem. In the third problem, the control functions of both pursuers and evader are subject to integral constraints. The stoppage time of the games considered is fixed and denoted by θ. During the game, the pursuers try to reduce the distance to the evader as much as possible and the evader tries to makes it as large as possible. The payoff functional is the distance between evader and the closest pursuer when the game is terminated. We obtain the value of the game and constructed optimal strategies of the players.

CHAPTER ONE

INTRODUCTION

1.1 PREAMBLE

Differential game is an important area of applied mathematics which evolved as a research area due to inter-field research activities in game theory, calculus of variation and optimal control theory. It was a product efforts to solve combat problems after world war II.

In context of game theory, Differential game is the mathematical theory that is concerned with problem of conflicts modeled as game problem in which the state of the players depend on time in a continuous way [43]. Differential game is an extension of sequential game theory categorized as a mathematical model designed to analyze conflict problem in the context of a dynamical system [9]. It is also a mathematical approach of investigating conflict problems which can be modeled as system of differential equations [61] .

Differential games also are related closely with optimal control problems. Optimal control problem involves single control $u(t)$ and a single criterion to be optimized. Differential game problem is an optimal control problem in which a second control function $v(t)$ is introduce and two criteria one for each player. Hence, each player attempts to control the state of the system so as to achieve the goal respectively, and the system responds to the inputs of both players.

There are numerous applications of differential game in various field of knowledge. These includes; Economics, Management science, Engineering design, Missiles guidance, Politics, Sports, Biology, Ecology, Military and other spheres to mention but few.

Differential game is applied in engineering design mostly in engineering system involving humans, where they use game theoretic tools to design protocols that will provide incentive for people to cooperate, video games and programmable robots. For instance scientist tend to use game theoretic tools to design optimal traffic flows of predicting or avoiding black-out in power networks or congestion in cyber-physical network control system. Computer game which uses a controlling device is constructed and designed such that for any user to play and achieve the target of the game, they must attempt to optimize their input which require a lot of strategy depending on the nature of the game. The player that responds to the users input is endowed with the ability of making several move such that it can move any direction in the game as instructed by the user. But every move of the player is programmed such that there's an optimal control to attain optimal result and this requires optimization control theory and dynamic of more than one system. Robot are build, programmed and controlled for optimal result in their executions and these require expertise in field like game which are mostly use in modern countries to carry out dangerous military assignment and jobs that are hazardous to people such as diffusing bombs, mining and exploration of shipwrecks.

The application of differential game in missiles guidance enable the guidance strategies to be derived for both the attacker and the target so that objectives of both parties are satisfy. Differential games have been applied to economics. Recent developments include adding stochasticity to differential games and the derivation of the stochastic differential game of capitalism .

Differential game was first studied by Rufus Isaacs in the early fifties (1951). The need of solving combat problem during this era was the motive behind this development. The earliest work published by Rufus Isaas include the Homicidal Chauffeur Game, the Game of Hamstrung squad car and Quasi-Discrete Game.

Homicidal Chauffeur game is a mathematical pursuit differential game which pits a hypothetical runner, who can only move slowly but is highly maneuverable again the driver of a motor vehicle, which is much faster but less maneuverable whose goal through out the game is to run him down. The question to be answered is ”under what circumstance and with what strategy can the driver of the car guarantee that he can always strike the pedestrain or the pedestrain guarantee that he can indefinitely elude being run down by the car?”

The ”lion and the man” problem is also among the earliest game studies. This is a pursuit-evasion problem posed as to determine a strategy for the pursuer (lion) to capture the evader (man) in a closed and bounded arena, both assumed to have equal maximum speed. The pursuer's (lion) goal is to catch the evader (man) for any control of the man, while the evader's (man) goal is to escape catch from the pursuer's (lion). What tactics should the pursuer (lion) employ to be sure of capturing the man? Is it possible for the evader (man) to escape catch? If the region is unbounded, can the evader (man) survive? This problem caught the attention of many researchers and after rigorous research, the solution of these problem have been applied to real life situation.

1.2 DIFFERENTIAL GAME PROBLEM

A typical differential game problem consist of system of equation of the form

$$\dot{x} = f(t,x,u,v), \quad x(0) = x_0, \quad t \in [0,\theta], \tag{1.2.1}$$

with payoff function

$$p(u,v) = H(x(t)) + \int_0^\theta G(x,u,v)dt, \tag{1.2.2}$$

where the state of the game is represented by the point $x \in \Omega \subseteq R^n$, (other spaces such as Hilbert space l_2 can host the state variable) ; control function of the players are measurable function $u(t) \in U$ and $v(t) \in V$; U and V are subsets of the Euclidean space, $H(x(t))$ is a single valued scalar function define on target set $C \in \Omega$ and is called the terminal cost. $H(x(t))$ and its derivative are allowed to be piecewise continuous. The function $G(x,u,v)$ is called the running cost and it has some mathematical properties with $f(x,u,v)$.

In relation to the output of the payoff function, differential game can be classified into two. These are differential game of kind and differential game of degree. Differential game of kind is the one in which the payoff function has a finite possible payoffs. Pursuit-evasion differential game is an example of differential game kind. Differential game of degree is the game in which the payoff function has a continuum possible payoffs. Example two-player zero sum and multi-objective differential game.

The following are possible questions related to this problem.

1. Which player wins the game? This involves providing the conditions which will guaranteed a given player to win the game.
2. What is the optimal strategies of the players? This is the best strategy adopted by the player to win the game.
3. What is the optimal trajectory? This is the path traced by the player when using its optimal strategy.

4. What is the value of the game. This is usually the value of the payoff when the players use their best strategy. This is defined as

$$V = \min_u \max_v P(t,u,v) \tag{1.2.3}$$

Alternatively, suppose that there exists control $\bar{u}(t)$ and $\bar{v}(t)$ such that

$$P(u,\bar{v}) \leq P(\bar{u},\bar{v}) \leq P(\bar{u},v), \tag{1.2.4}$$

for any control u and v of the pursuer and evader respectively. The value of the game defined as

$$V = P(\bar{u},\bar{v}), \tag{1.2.5}$$

where the pairs $(\bar{u},\bar{v})$ is called the saddle point of the game.

1.3 DEFINITION OF SOME TERMS

Definition 1.3.1 *Pursuit Differential Game Problem involve finding condition(s) that guaranteed pursuer to win the game.*

Definition 1.3.2 *Evasion Differential Game Problem involve finding condition(s) that guaranteed evader to win the game.*

Definition 1.3.3 *Pursuit-evasion Differential Game is game problem which require finding the value of the payoff functional when the players employed their optimal strategies.*

Definition 1.3.4 *State variable of a player is a set of variable used to described the position or state of that player at any given time in the space under consideration.*

Definition 1.3.5 *[80] A set E is lebesque measurable or simply measurable if for each set A we have*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{1.3.1}$$

Definition 1.3.6 *[80] Let X be a measurable subset of R. A function $f : X \longrightarrow R$ is called measurable or Lebesgue measurable if the set $\{x \in X : f(x) > \alpha\}$ is measurable for each $\alpha \in R$.*

Definition 1.3.7 *Control function of the player is a measurable function that determine the form or nature of the strategy applied. It is the one in which players use to make input in to the game in order to achieve his goal.*

Definition 1.3.8 *Payoff function: This is a functional which measure the performance of a player at any given time during the game.*

Definition 1.3.9 *Let* $f : [0,\infty] \longrightarrow \bar{R}$ *be Lebesgue measurable function. Then* f *is said to satisfy integral constraints if*

$$\int_0^\infty |f(t)|^2 dt \leq \lambda^2, \tag{1.3.2}$$

where λ *is a positive real number.*

Definition 1.3.10 *Let* $f : [0,\infty] \longrightarrow \bar{R}$ *be Lebesgue measurable function. Then* f *is said to satisfy geometric constraints if*

$$|f(t)| \leq \lambda, \quad t > 0 \tag{1.3.3}$$

where λ *is a positive real number.*

Definition 1.3.11 *[10] (Inner product space) Let* X *be a linear space. An inner product on* X *is a function,* $\langle\ ,\ \rangle : X \times X \longrightarrow K$ *defined on* $X \times X$ *with values in* K *(where* $K = R$ *or* C*) such that the following three conditions are satisfied: For* $x,y,z \in X, \alpha, \beta \in K$

IP1: $\langle x,x\rangle \geq 0$ *and* $\langle x,x\rangle = 0$ *if and only if* $x = 0$

IP2: $\langle x,y\rangle = \overline{\langle y,x\rangle}$ *, where the "bar" indicates complex conjugate; and*

IP3: $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

The pair, $(X\ , \langle\ ,\ \rangle)$ *is called an inner product space.*

Definition 1.3.12 *[10] An inner product space* X *is said to be complete, if every Cauchy sequence in* X *converges to a point of* X*.*

Definition 1.3.13 *[10] A complete inner product space is called a Hilbert space.*

1.4 OUTLINE OF THE DISSERTATION

This dissertation comprises of five chapters.

Chapter 1 consist of introduction of differential game problems, definition of some terms, differential game problem, statement of the problem, aim and objectives of the studies, methodology, scope and limitation of the studies.

Chapter 2 discussed literature review of some existing work that are closely related to the research topic of this dissertation.

Chapter 3 present some important tools and techniques that are used in the prove of the main result.

Next in chapter 4, we study pursuit-evasion differential game of many pursuers and one evader in the space l_2, the equation of motion of the pursuers and evader is described by first and second order differential equation respectively. We obtain the value of the game and constructed optimal strategies of the players.

Finally chapter 5 present summary, conclusion and recommendation for further research.

1.5 STATEMENT OF THE PROBLEM

Consider pursuit-evasion differential game described by the following equations:

$$\begin{cases} P_i : \dot{x}_i = u_i(t), \quad x_i(0) = x_{i0}, \\ E : \ddot{y} = \ v(t), \ \ \dot{y}(0) = y^1, \ y(0) = y^0, \end{cases} \tag{1.5.1}$$

where $x_i, x_{i0}, u_i, y, y^0, y^1, v \in l_2,\ u_i = (u_{i1}, u_{i2}, \dots)$ is a control function of the pursuer P_i $i \in I,\ \ I = \{1, 2, \dots, m\}$ and $v = (v_1, v_2, \dots)$ is that of the evader E.

1.5.1 Problems Description

1. The game in which motion of the players is described by (1.5.1). Control functions of the pursuers and evader are subject to integral and geometric constraints respectively is called problem I.
2. The game in which motion of the players is described by (1.5.1). Control functions of the pursuers and evader are subject to geometric constraints is called problem II.
3. The game in which motion of the players is described by (1.5.1). Control functions of the pursuers and evader are subject to integral constraints is called problem III.

In each of the problem:

i. The stoppage time of the games considered is fixed and denoted by θ.

ii. The payoff function measure the distance between evader and closest pursuer at any given time during the game.

iii. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it.

iv. The value of the game is the distance between the evader and the closest pursuers when the game is terminated.

RESEARCH QUESTION: What is the value of the game?

1.6 AIM AND OBJECTIVES

The aim of this research is to find the value of the game and construct admissible optimal strategies of the players in each of the problem stated above.
The objectives are as follows;

i. to solve pursuit differential game problem of one pursuer and one evader.

ii. to construct admissible optimal strategies of the players.

iii. to provide the formula for finding the value of the game.

iv. to give illustrative example to demonstrate our result.

1.7 METHODOLOGY

The problems we posed in this research are to be solve through:

- Using some ideas in the relevant papers such as [15], [18], [19], [28], [39], [44], [56], [65] and [66] which include solving an auxiliary differential game problem in half-space.
- Using some properties of spheres, balls and Half-space in Hilbert space.
- Using some existing results in the literature such as some well known mathematical inequalities, and knowledge of calculus.

1.8 SCOPE AND LIMITATIONS

This research is concerned with the study of pursuit-evasion differential game problems in the Hilbert space l_2. Motion of the pursuers and evader are described by first and second order differential equations respectively. The research is limited to the study of problems which consider finite number of pursuers and one evader.

CHAPTER TWO

REVIEW OF RELATED LITERATURES

2.1 INTRODUCTION

This chapter contains review of some works that are closely related to the research problems studied in this Dissertation.

2.2 LITERATURE REVIEW

Many books have been devoted to differential games, such as books by Isaacs [14], Kuhn and Szego [47], Friedman [13], Petrosyan [59], Subbotin and Chentsov [74], Pontryagin [60], Krasovskii and Subbotin [46], Rikhsiev [64], Lewin [52], and other fundamentals in the theory of differential games.

There are numerous papers devoted to the study of differential game problems by many researchers. The papers [1]- [8], [12], [13], [15] - [46], [48] - [51], [54], [56] - [58], [62], [63], [65] - [73], [75] -[79] and [82] and some references therein are examples.

The problems studied in the works cited above are either pursuit problem, evasion problem or pursuit-evasion problem. The works [1], [3], [17], [20], [21], [24], [31], [35], [40], [41], [44], [56], [58], [59], [62], [67], [69], [75], [78], etc. are concerned with pursuit problem. In the other hand the works [4], [32], [38], [41], [49], [54], [68], etc. are concerned with evasion problem.

The works [7], [15], [18], [19],[28], [33] [39], [51], [65], [66], [73], etc., which are concerned with pursuit-evasion differential game problems are of special interest in this dissertation. Among these works some studied pursuit-evasion differential game problem in which motion of the players are described by first order differential equation(for example see, [19], [28], [65] and [66]). The pursuit-evasion differential game problem studied in [15], [18], [38], [70] and [79] involves motion of players described by higher order differential equation.

In [40], Ivanov and Ledyeav studied simple motion differential game with many players and geometric constraints on the control functions of the players. By using lyapunov function method for an auxiliary problem, they obtained sufficient conditions to find the pursuit time in R^n.

Levchenkov, A.Yu., and Pashkov, G.A., in [51] investigated differential game of optimal approach of two identical inertial pursuers to a non-inertial evader on a fixed time interval. Control parameters were subjected to geometric constraints. They constructed the value functions of the game and used necessary and sufficient conditions which a functions must satisfy to be the valued function.

Ibragimov, G.I in [36] studied pursuit-evasion differential game of one pursuer and one evader with integral constraints was investigated on a closed convex subset of R^n. Formula for optimal pursuit time was obtained and optimal strategies of the players was constructed.

Vagin, A.A., and Petrov, N.N., in [79] studied a pursuit differential game problem with finite number pursuers and evader in the space R^n. Motions of the players are described by n^{th} order differential equations. Control functions of the players are subject to geometric constraints. sufficient condition for completion of pursuit was obtained.

In [37], Ibragimov, G.I., investigate two problem of pursuit in linear discrete games. Both problems were considered under geometric constraints on control function of the pursuers. The evader, in the first game problem was subject to geometric constraints and in the second game problem, it satisfies integral constraints. Necessary and sufficient condition were obtained for completion of pursuit from initial position of players.

In the work of Ibragimov, G.I [19] a pursuit-evasion differential game of many players with geometric constraints being imposed on the control parameters of players was studied. Game is described by an infinite system of differential equation of first order in Hilbert space l_2. Duration of the game is fixed. Payoff is the infimum of the distance between the evader and the pursuers when the game is terminated. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. A condition to find the value of the differential game is obtained. Optimal strategies of players are also constructed.

Attamurat Kuchkarov, in [2] consider pursuit and evasion differential game of a group of m pursuers and one evader on manifolds with Euclidean metric. They found a condition under which pursuit can be completed, and if the condition is not satisfied, then evasion is possible. They constructed a strategies for the pursuers in pursuit game which ensure completion of the game for a finite time and give a formula for this time, which they also construct a strategy for the evader as well.

Ibragimov, G.I., and Salimi, M., in [18] studied a pursuit-evasion differential game of many players with integral constraints being imposed on the control functions of the players. Game is described by an infinite system of differential equation of second order in Hilbert space l_2. Duration of the game is fixed. Payoff functional is the infimum of the distance between the evader and the pursuers when the game is terminated. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. Under certain condition, the value of the game has been found and the Optimal strategies of players has been constructed.

In [15], Ibragimov G.I., and Hussin, N.A., studied a pursuit-evasion differential game of many players with geometric constraints imposed on the control parameters of players. Game is described by an infinite system of differential equations of second order in Hilbert space. Duration of the game is fixed. Payoff is the infimum of the distance between the evader and the pursuer when the game is terminated. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. A condition to find the value of the differential game is obtained. Optimal strategies of players were also constructed.

In [25], Ibragimov, G.I., et.al., studied an evasion problem from several pursuers and one evader in a simple motion differential game with integral constraints. They assume that the total resource of the pursuers does not exceed that of the evader. They present an explicit strategy for the evader which guarantees evasion.

In [41], Ja'afaru, A.B., and Ibragimov, G.I., studied pursuit and evasion differential game problem described by an infinite number of first-order differential equation with function coefficients in Hilbert space l_2. Problems involving integral, geometric, and mix constraints to the control functions of the players are considered. They obtained sufficient condition for completion of pursuit and for which evasion is possible. Strategy of the pursuer and control function of the evader are constructed.

Salimi, M., in [67] investigates an evasion problem with a finite number of dynamical object trying to catch a single evader in Hilbert space l_2. Geometric constraints are imposed on the control functions of the pursuers and the evader. He solved the game by applying a suitable strategy for the evader that ensure its evasion.

Ibragimov, G.I., and Kuchkarov A.Sh., in [28] investigate pursuit-evasion differential game of countably many pursuers and one evader. Game is described by an infinite system of differential equation of first order in Hilbert space l_2. Control of the pursuers and evader are subjected to integral restrictions. The duration of the game is fixed and the payoff functional is the infimum of the distance between the evader and the pursuers when the game is terminated. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. Optimal strategies of the players are constructed, and value of the game was found.

Ibragimov, G.I., et.al., in [38] studied an inertial pursuit-evasion differential game with a finite or countable number of pursuers and evader in Hilbert space l_2. Control of the pursuers and evader are subjected to integral restrictions. The period of the game is fixed. They formulated the value of the game and identified explicitly optimal strategies of the players. They assumed that there is no relation between the control resource of any pursuer and that of the evader.

In [7], Badakaya, A.J., studied differential game problem involving countable number of pursuers and one evader in the space l_2. Players motion obey ordinary differential equation with integral constraints subjected on the control functions of the players. Duration of the game is fixed. Payoff is the infimum of the distance between the evader and the pursuer when the game is terminated. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. A condition to find the value of the differential game is obtained. Optimal strategies of players were also constructed.

Nargiza, S., et.al., in [56] studied pursuit differential game problem for the so-called "boy and crocodile" in the space R^n. Boy's motion is described by first order differential equations and that of the crocodile by second order differential equation. Control functions of the pursuer and evader are subject to integral and geometric constraints respectively. They obtained Sufficient conditions of completion of pursuit.

Salimi, M., et.al., in [65] investigate a pursuit-evasion differential game in which countably many dynamical object pursue a single one. All the players perform simple motions. The duration of the game is fixed. The controls of a group of pursuers are subject to integral constraints , and the controls of the other pursuers and the evader are subjected to geometric constraints. The payoff of the game is the distance between the evader and the closest pursuer when the game is terminated. They construct optimal strategies for players and find the value of the game.

Salimi, M., and Ferrara, M., in [66] studied differential game in which a finite or countable number of pursuers pursue a single evader. The control function of the players satisfy the integral constraints. The period of the game is determined. The farness between the evader and the closest pursuer when the game is finished is the payoff function of the game. He introduce the value of the game and identify optimal strategies of the pursuers. He assume that there is no relation between the energy resources of the pursuer and evader.

Badakaya, A.J., in [44] studied a pursuit differential game problem with finite number pursuers and one evader in the space l_2. Pursuers' motions are described by first order differential equations and that of the evader by second order differential equation. Control functions of the pursuers and evader are subject to integral and geometric constraints respectively. Duration of the game is denoted by the positive number θ. Theorems are stated and proved each of which provides a condition for completion of pursuit. Consequently, strategies of the pursuers that ensure completion of pursuit are constructed.

In this piece of research, we studied a number of pursuit-evasion differential game problems in a Hilbert space l_2, in which motions of the pursuers and evader are described by first and second order differential equations respectively. In problem I, control functions of the pursuers and evader are subjected to integral and geometric constrains respectively. Control function of both the pursuers and evader are subjected to geometric constraints in Problem II and integral constraints in Problem III.

CHAPTER THREE

MATERIALS AND METHODOLOGY

3.1 INTRODUCTION

In this chapter we present some important tools and techniques that are used in the proof of our main result. Some known results that are useful to our work are also stated.

3.1.1 Some Basic Properties of Inner Product Space

In this section we present some basic properties of inner product space.

Remark 3.1.1 *[10] From the definition of inner product, we have for all* $x,y,z \in X and \alpha,\beta \in K$

$$\langle x,\alpha y+\beta z\rangle = \alpha\langle x,y\rangle+\beta\langle x,z\rangle$$

Proposition 3.1.1 *[10] (Cauchy Schwartz Inequality) Let* X *be an inner product space, then for all* $x,y \in X$*, we have*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$$

Lemma 3.1.1 *[10] Let* X *be an inner product space. A map* $||\cdot|| : X \to R$*, define by* $||x|| = \sqrt{\langle x,x\rangle}$ *is a norm on* X*. That is,* $(X,||\cdot||)$ *is a norm linear space.*

Remark 3.1.2 *[10] From the above lemma, Cauchy schwartz inequality can be written as*

$$|\langle x,y\rangle| \leq ||x||||y||,\ for\ all\ x,y \in X$$

Proposition 3.1.2 *[10] (The Parallelogram Law) Let* X *be an inner product space. Then*

$$||x+y||^2+||x-y||^2 = 2\left(||x||^2+||y||^2\right),\ for\ all\ x,y \in X.$$

Definition 3.1.1 *[10] Let* E *be a real normed linear space, the following identity hold,*

$$||x+y||^2 = ||x||^2+2\langle x,y\rangle+||y||^2,\ for\ all\ x,y \in E.$$

Definition 3.1.2 *[10] Let $(X, ||\cdot||)$ be a normed space.*

i. The set $\{x \in X : ||x - x_0|| < r\}$, denoted by $S(x_0, r)$, is called the open sphere (or open ball) of radius r with center x_0

ii. The set $\{x \in X : ||x - x_0|| \leq r\}$, denoted by $S(x_0, r)$, is called the closed sphere (or closed ball) of radius r with center x_0

Definition 3.1.3 *[10] Let X be a real normed linear space. For $\alpha \in R$, define*

$$\begin{aligned} H_{f,\alpha} : &= \{x \in X : f(x) = \alpha,\ f \in X^*\} \\ &= \{x \in X : \langle x, f\rangle = \alpha\} \end{aligned}$$

where $\langle x, f\rangle = f(x)$. Then, $H_{f,\alpha}$ is called a hyperplane of X.

Proposition 3.1.3 *[55] (Triangle Inequality): Let $a, b \in K$, (where $K = R$ or C). Then*

$$|a + b| \leq |a| + |b|$$

Proposition 3.1.4 *. [55] (Holder's inequality For Integral): Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let Ω be a bounded open set in R^n and f, g measurable function on Ω such that*

$$\int_\Omega |f|^p dt < \infty \ \textit{and} \ \int_\Omega |g|^q dt < \infty.$$

Then,

$$\int_\Omega |fg| dt \leq \left(\int_\Omega |f|^p ds\right)^{\frac{1}{p}} \left(\int_\Omega |g|^p dt\right)^{\frac{1}{q}} \tag{3.1.1}$$

The special case $p = q = 2$ is often referred as the Cauchy-Schwarz inequality

Proposition 3.1.5 *[55] Let $f : [a, b] \longrightarrow R$ be integrable, then $|f|$ is integrable and we have*

$$\left|\int_a^b f(x) dx\right| \leq \int_a^b |f(x)| dx$$

3.1.2 Some Properties of Balls, Spheres and Half-space in Hilbert Space

Consider the spheres $S(x_0, R)$ and $S(y_0, r)$ and balls $H(x_0, R)$ and $H(y_0, r)$ in the Hilbert space, where $x_0 \neq y_0$, R and r are positive numbers.

Proposition 3.1.6 *[19] A point $\bar{y} \in S(y_0, r)$ belong to the sphere $S(x_0, R)$ if and only if it lies on the hyperplane.*

$$2\langle y_0 - x_0, \bar{y} \rangle = R^2 - r^2 + ||y_0||^2 - ||x_0||^2 \tag{3.1.2}$$

Proof . let $\bar{y}$ be a common point of the spheres; then $||\bar{y} - x_0||^2 = R^2$ and $||\bar{y} - y_0||^2 = r^2$ by subtracting the second relation from the first relation, we have

$$2\langle y_0 - x_0, \bar{y} \rangle = R^2 - r^2 + ||y_0||^2 - ||x_0||^2 .$$

Conversely, let $\bar{y} \in S(y_0, r)$ lie on the hyperplane(3.1.2). Let us show that $\bar{y} \in S(x_0, R)$. Indeed, by (3.1.2), we have

$$\begin{aligned}
||\bar{y} - x_0||^2 &= ||\bar{y} - y_0 + y_0 - x_0||^2 \\
&= ||\bar{y} - y_0||^2 + 2\langle \bar{y} - y_0, y_0 - x_0 \rangle + ||y_0 - x_0||^2 \\
&= r^2 + 2\langle \bar{y}, y_0 - x_0 \rangle - 2\langle y_0, y_0 - x_0 \rangle + ||y_0 - x_0||^2 \\
&= r^2 + R^2 - r^2 + ||y_0||^2 - ||x_0||^2 - 2\,||y_0||^2 + 2\langle y_0, x_0 \rangle + ||y_0||^2 - 2\langle y_0, x_0 \rangle + ||x_0||^2 \\
&= R^2.
\end{aligned}$$

■

Proposition 3.1.7 *[19] A point $\bar{y} \in S(y_0, r)$ does not belong to the ball $H(x_0, R)$ if and only if*

$$2\langle y_0 - x_0, \bar{y} \rangle > R^2 - r^2 + ||y_0||^2 - ||x_0||^2 \tag{3.1.3}$$

Proof . Suppose that $\bar{y} \notin H(x_0, R)$; then $||\bar{y} - x_0|| > R$. We set $||\bar{y} - x_0|| = R_1$. Obviously, $R_1 > R$, and the point $\bar{y}$ lies on the sphere $S(x_0, R_1)$. By proposition (3.1.6), we obtain

$$2\langle y_0 - x_0, \bar{y} \rangle = R_1^2 - r^2 + ||y_0||^2 - ||x_0||^2 > R^2 - r^2 + ||y_0||^2 - ||x_0||^2$$

. Conversely, Let a point $\bar{y} \in S(y_0, r)$ satisfy inequality (3.1.3). Then

$$\begin{aligned}
||\bar{y} - x_0||^2 &= ||\bar{y} - y_0 + y_0 - x_0||^2 \\
&= ||\bar{y} - y_0||^2 + 2\langle \bar{y} - y_0, y_0 - x_0 \rangle + ||y_0 - x_0||^2 \\
&= r^2 + 2\langle \bar{y}, y_0 - x_0 \rangle - 2\langle y_0, y_0 - x_0 \rangle + ||y_0 - x_0||^2 \\
&> r^2 + R^2 - r^2 + ||y_0||^2 - ||x_0||^2 - 2\,||y_0||^2 + 2\langle y_0, x_0 \rangle + ||y_0||^2 - 2\langle y_0, x_0 \rangle + ||x_0||^2 \\
&= R^2.
\end{aligned}$$

Therefore, $\bar{y} \notin H(x_0, R)$. ■

Corollary 3.1.1 *[19] A point $\bar{y} \in S(y_0, r)$ belong to the ball $H(x_0, R)$ if and only if lies in the half-space.* $X = \left\{ z \in l_2 : 2\langle y_0 - x_0, z\rangle \leq R^2 - r^2 + ||y_0||^2 - ||x_0||^2 \right\}$

Remark 3.1.3 *[19] In our notation, this corollary acquires the form*

$$S(y_0, r) \cap X \subset H(x_0, R), \quad S(y_0, r) \cap H(x_0, R) \subset X$$

.

Proposition 3.1.8 *[19] One has* $H(y_0, r) \cap X \subset H(x_0, R)$.

Proof
Indeed, if $z \in H(y_0, r) \subset X$, then $||z - y_0|| \leq r$ and $2\langle y_0 - x_0, z\rangle \leq R^2 - r^2 + ||y_0||^2 - ||x_0||^2$. Then,

$$\begin{aligned}
||z - x_0||^2 &= ||z - y_0 + y_0 - x_0||^2 \\
&= ||z - y_0||^2 + 2\langle z - y_0, y_0 - x_0\rangle + ||y_0 - x_0||^2 \\
&\leq r^2 + 2\langle z, y_0 - x_0\rangle - 2\langle y_0, y_0 - x_0\rangle + ||y_0 - x_0||^2 \\
&\leq r^2 + R^2 - r^2 + ||y_0||^2 - ||x_0||^2 - 2\,||y_0||^2 + 2\langle y_0, x_0\rangle + ||y_0||^2 - 2\langle y_0, x_0\rangle + ||x_0||^2 \\
&\leq R^2.
\end{aligned}$$

■

Now suppose that we have finitely or countably many balls $H(y_0, r)$ and $H(x_{i0}, R_i)$, $x_{i0} \neq y_0$, $i \in I$, where I is the finite or countable indexing set and r, R_i, $i \in I$ are positive numbers.

Proposition 3.1.9 *[19] If there exists a nonzero vector p_0 such that $\langle y_0 - x_{i0}, p_0\rangle \geq 0$, for all $i \in I$, and*

$$H\left(y_0, r\right) \subset \bigcup_{i \in I} H\left(x_{i0}, R_i\right), \tag{3.1.4}$$

then

$$H(y_0, r) \subset \bigcup_{i \in I_0} X_i, \tag{3.1.5}$$

where

$$I_0 = \{i \in I : S(y_0, r) \cap H(x_{i0}, R_i) \neq \emptyset\},$$

$$X_i = \begin{cases} \left\{ z \in l_2 : 2\langle y_0 - x_{i0}, z\rangle \leq R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2 \right\}, & x_{i0} \neq y_0, \\ \{z \in l_2 : \langle z - y_0, p_0\rangle \leq R_i\}, & x_{i0} = y_0. \end{cases} \tag{3.1.6}$$

Proof . Indeed, if $x_{i0} \neq y_0,\ i \in I_0$, then by corollary (3.1.1) the inclusion

$$S(y_0,r) \cap H(x_{i0},R_i) \subset X_i,\ \ i \in I_0, \tag{3.1.7}$$

is valid. We show that the inclusion (3.1.7) is also true for the case $x_{i0} = y_0,\ i \in I_0$. If this is the case, then either $r > R_i$ or $r \leq R_i$. For $r > R_i$, the intersection $S(y_0,r) \cap H(x_{i0},R_i)$ is empty and therefore by definition of I_0 we get $i \notin I_o$. We drop this case since we deal only with $i \in I_0$. In the latter case, i.e. if $r \leq R_i$, we have

$$S(y_0,r) \cap H(x_{i0},R_i) = S(x_{i0},r) \cap H(x_{i0},R_i) = H(x_{i0},R_i) \subset X_i,\ i \in I_0.$$

Thus, for all $i \in I_0$ the inclusion (3.1.7) is true and therefore from it we get

$$\bigcup_{i \in I_0} (S(y_0,r) \cap H(x_{i0},R_i)) \subset \bigcup_{i \in I_0} X_i.$$

Hence,

$$S(y_0,r) \cap \left(\bigcup_{i \in I_0} H(x_{i0},R_i) \right) \subset \bigcup_{i \in I_0} X_i. \tag{3.1.8}$$

On the other hand, from (3.1.4) we obtain

$$H(y_0,r) \cap S(y_0,r) \subset \left(\bigcup_{i \in I_0} H(x_{i0},R_i) \right) \cap S(y_0,r).$$

Consequently,

$$S(y_0,r) \subset \bigcup_{i \in I} (H(x_{i0},R_i) \cap S(y_0,r)) = \bigcup_{i \in I_0} (H(x_{i0},R_i) \cap S(y_0,r)) \subset \bigcup_{i \in I_0} H(x_{i0},R_i), \tag{3.1.9}$$

since $H(x_{i0},R_i) \cap S(y_0,r) = \emptyset,\ i \in I \backslash I_0$. Then combining (3.1.8) and (3.1.9) yields

$$S(y_0,r) \subset \bigcup_{i \in I_0} X_i \tag{3.1.10}$$

We proceed to show the inclusion (3.1.5). Suppose for contradiction. There exist a point $\bar{y} \in H(y_0,r)$ such that $\bar{y} \notin\subset \bigcup_{i \in I_0} X_i$. This implies that $\bar{y} \notin X_i$, for all $i \in I_0$. Then it follows from the definition of X_i (3.1.6), we have

$$\begin{array}{ll} 2\langle y_0 - x_{i0}, \bar{y} \rangle > R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2, & x_{i0} \neq y_0, \\ \langle \bar{y} - y_0, p_0 \rangle > R_i, & x_{i0} = y_0. \end{array} \tag{3.1.11}$$

Therefore taking account of the condition $\langle y_0 - x_{i0}, p_0\rangle \geq 0, i \in I_0$, for the points of the half-line $y(t) = \bar{y} + p_0 t,\ t \geq 0$, we have,

$$\begin{aligned} 2\langle y_0 - x_{i0}, y(t)\rangle =& 2\langle y_0 - x_{i0}, \bar{y}\rangle + 2\langle y_0 - x_{i0}, p_0\rangle \\ &\geq 2\langle y_0 - x_{i0}, \bar{y}\rangle \\ &> R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2, \quad i \in I_0, \end{aligned}$$

for $x_{i0} \neq y_0$ and

$$\langle y(t) - y_0, p_0\rangle = \langle \bar{y} - y_0, p_0\rangle + t \geq \langle \bar{y} - y_0, p_0\rangle > R_i, i \in I_0,$$

for $x_{i0} = y_0$ which implies that the points of the half-line $y(t)$ do not lie in the half-space $X_i, i \in I_0$ and $t \geq 0$, and hence $y(t) \notin\subset \bigcup_{i\in I_0} X_i$ for all $t \geq 0$.

On the other hand, by (3.1.10), a point of the sphere $S(y_0, r)$ lying on this half-line belongs to the set $\bigcup_{i\in I_0} X_i$. which is a contradiction. Hence the proof. ■

Proposition 3.1.10 *[19] Let* $\inf_{i\in I} R_i = R_0 > 0$, *If there exists a nonzero vector* p_0 *such that* $\langle y_0 - x_{i0}, p_0\rangle \geq 0$ *for all* $i \in I$ *and for any* $\varepsilon > 0$ *the set* $\bigcup_{i\in I} H(x_{i0}, R_i - \varepsilon)$ *does not contain the ball* $H(y_0, r)$*, then there exist a point* $\bar{y} \in S(y_0, r)$ *such that* $||\bar{y} - x_{i0}|| \geq R_i$ *for all* $i \in I$.

Proof Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of real numbers tending to zero. Obviously, $\varepsilon_k \leq R_0$. Then, by the assumption of the propositions, the set $\bigcup_{i\in I} H(x_{i0}, R_i - \varepsilon_k), k = 1, 2, \ldots$, does not contain the ball $H(y_0, r)$. Therefore, for each index k, there exist a point $y_k \in H(y_0, r)$ that does not belong to the balls $H(x_{i0}, R_i - \varepsilon_k), i \in I$. The point y_k satisfies the inequality

$$2\langle y_0 - x_{i0}, y_k\rangle \geq (R_i - \varepsilon_k)^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2, \qquad x_{i0} \neq y_0 \tag{3.1.12}$$

Otherwise by using the condition $y_k \in H(y_0, r)$, from proposition (3.1.8), we would obtain the inclusion $y_k \in H(x_{i0}, R_i - \varepsilon_k)$ which contradicts the definition of y_k.

Since the ball $H(y_0, r)$ is weakly compact, it follows that the sequence $y_k, k = 1, 2, \ldots$, contains a convergent subsequence. For simplicity, we suppose that the sequence itself is convergent and $\bar{y}$ is its weak limit. One can readily see that $y_k \in H(y_o, r)$. In equation (3.1.12), we pass the limit as $k \to \infty$. Then

$$2\langle y_0 - x_{i0}, \bar{y}\rangle \geq R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2, \ i \in I. \tag{3.1.13}$$

One can assume that $\bar{y} \in S(y_0, r)$. Indeed, should the inequality $||\bar{y} - y_0|| < r$ be valid, the inclusion $\bar{y} + p_0 t_0$ would satisfy the inequality

$$2\langle y_0 - x_{i0}, \bar{y} + p_0 t_0 \rangle \geq 2\langle y_0 - x_{i0}, \bar{y} \rangle \geq R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2, \;\; i \in I \qquad (3.1.14)$$

Thus we assume that inequalities (3.1.13) are valid for a point $\bar{y}$ of the sphere $S(y_0, r)$. Then by proposition (3.1.6) and (3.1.7), $||\bar{y} - x_{i0}|| \geq R_i, \;\; x_{i0} \neq y_0, \;\; i \in I.$

Now consider the ball $H(x_{k0}, R_k)$ such that $x_{k0} \neq y_0$. By the assumptions of the proposition, the ball $H(y_0, r)$ does not lies in the ball $H(y_0, R_k - \varepsilon)$ for any $\varepsilon > 0$. Consequently, $r > R_k - \varepsilon$. Hence we find that $r \geq R_k$. Then the point $\bar{y} \in S(y_0, r)$ satisfies the inequality

$$||\bar{y} - x_{k0}|| = ||\bar{y} - y_0|| = r \geq R_k$$

The proof of the proposition is complete. ■

CHAPTER FOUR

RESULT

4.1 INTRODUCTION

In this chapter, we present the main results of the dissertation which consist of six propositions, three lemmas and three theorems.

4.2 Formulation of the Problem

Consider the space $l_2 = \left\{ \rho = (\rho_1, \rho_2, \dots) : \sum\limits_{k=1}^{\infty} \rho_k^2 < \infty \right\}$, with inner product $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to \mathbb{R}$ and norm $|| \cdot || : l_2 \to [0, +\infty)$, defined as follows:

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k, \ ||\rho|| = \left(\sum_{k=1}^{\infty} \rho_k^2 \right)^{1/2},$$

where $x, y, \rho \in l_2$, respectively.
Instead of differential game described by (1.5.1) we can consider an equivalent differential game with the same payoff function described by the following system:

$$\begin{cases} P_i : \dot{x}_i = u_i(t), & x_i(0) = x_{i0}, \\ E : \dot{y} = \ (\theta - t) v(t), & y(0) = y_0 = y^1 \theta + y^0. \end{cases} \tag{4.2.1}$$

where $x_i, x_{i0}, u_i, y, y^0, y^1, v \in l_2, \ u_i = (u_{i1}, u_{i2}, \dots)$ is a control function of the i^{th} pursuer $P_i, \ i \in I = \{1, 2, \dots, m\}$ and $v = (v_1, v_2, \dots)$ is that of the evader E.

If the pursuer P_i and evader E use admissible controls $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots)$ and $v(t) = (v_1(t), v_2(t), \dots)$ respectively, then by (1.5.1) their corresponding motions is given by

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \dots), \quad y(t) = (y_1(t), y_2(t), \dots). \tag{4.2.2}$$

where

$$x_{ik}(t) = x_{i0k} + \int_0^t u_{ik}(s)ds, \tag{4.2.3}$$

and

$$y_k(t) = y_{0k} + \int_0^t (\theta - s)v_k(s)ds. \tag{4.2.4}$$

Definition 4.2.1 *A function $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots)$ with measurable coordinates such that*

$$\int_0^\theta ||u_i(t)||^2 dt \le \rho_i^2, \tag{4.2.5}$$

or

$$||u_i(t)|| \le \rho_i, \ \ 0 \le t \le \theta, \tag{4.2.6}$$

where ρ_i is the positive number, is called admissible control of the pursuer.

Definition 4.2.2 *A function $v(t) = (v_1(t), v_2(t), \dots)$ with measurable coordinates such that*

$$\int_0^\theta ||v(t)||^2 dt \le \sigma^2, \tag{4.2.7}$$

or

$$||v(t)|| \le \sigma, \ \ 0 \le t \le \theta, \tag{4.2.8}$$

where σ is the positive number, is called admissible control of the evader.

Definition 4.2.3 *A function $U_i(t, x_i, y, v(t))$, $U_i : [0, \infty) \times l_2 \times l_2 \times l_2 \to l_2$, such that the system*

$$\begin{cases} \dot{x}_i = U_i(t, x, y, v(t)), \quad x_i(0) = x_{i0}, \\ \ddot{y} = \ v(t), \ \ \dot{y}(0) = y^1, \ \ y(0) = y^0, \end{cases} \tag{4.2.9}$$

has a unique solution $(x_i(\cdot),\ y(\cdot))$ with continuous components $x_i(\cdot),\ y(\cdot)$ in l_2, for an arbitrary admissible control $v = v(t)$, $0 \le t \le \theta$, of the evader E is called strategy of the pursuer P_i. A strategy U_i is said to be admissible if each control formed by this strategy is admissible.

Definition 4.2.4 *Strategies $\bar{U}_i$ of the pursuers P_i are said to be optimal if*

$$\inf_{U_1, \dots, U_n, \dots} \Gamma_1(U_1, \dots, U_n, \dots) = \Gamma_1(\bar{U}_1, \bar{U}_2, \dots, \bar{U}_m, \dots), \tag{4.2.10}$$

where

$$\Gamma_1(U_1, U_2, \dots, U_m, \dots,) = \sup_{v(\cdot)} \inf_{i \in I} ||x_i(\theta) - y(\theta)||, \tag{4.2.11}$$

and U_i are admissible strategies of the pursuers P_i and $v(\cdot)$ is an admissible control of the evader E.

Definition 4.2.5 *A function* $V(t, x_i, \ldots, x_m, \ldots, y)$*,* $V : [0, \infty) \times l_2 \times, \ldots, \times l_2 \times, \ldots, \times l_2 \to l_2$*, such that the system*

$$\begin{cases} \dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \\ \ddot{y} = \ V(t, x_1, \ldots, x_m \ldots, y), \quad \dot{y}(0) = y^1, \quad y(0) = y^0, \end{cases} \tag{4.2.12}$$

has a unique solution $(x_1(\cdot), \ldots, x_m(\cdot), \ldots, y(\cdot))$ *with continuous components* $x_1(\cdot), \ldots, x_m(\cdot), \ldots, y(\cdot)$ *in* l_2*, for an arbitrary admissible control* $u_i = u_i(t)$*,* $0 \le t \le \theta$*, of the pursuers* P_i*, is called strategy of the evader* E*. A strategy* V *is said to be admissible if each control formed by this strategy is admissible.*

Definition 4.2.6 *Strategy* $\bar{V}$ *of the evader* E *are said to be optimal if*

$$\sup_V \Gamma_2(V) = \Gamma_2(\bar{V}) \tag{4.2.13}$$

where

$$\Gamma_2(V) = \inf_{u_1(\cdot), \ldots, u_m(\cdot), \ldots} \inf_{i \in I} ||x_i(\theta) - y(\theta)||, \tag{4.2.14}$$

and $u_i(\cdot)$ *are admissible control of the pursuers* P_i *and* V *is an admissible strategy of the evader* E*.*

The following are the problems considered in this research work:

PROBLEM I: This problem finds the value of the differential game problem described by (4.2.1), with the control $u_i(t)$ satisfying (4.2.5) and $v(t)$ satisfying (4.2.8).

PROBLEM II: This problem finds the value of the differential game problem described by (4.2.1), with the control $u_i(t)$ satisfying (4.2.6) and $v(t)$ satisfying (4.2.8).

PROBLEM III: This problem finds the value of the differential game problem described by (4.2.1), with the control $u_i(t)$ satisfying (4.2.5) and $v(t)$ satisfying (4.2.7).

4.3 Solution of the Problems

In this section we present solutions to our problems.

4.3.1 Problem I

4.3.2 Attainability Domain I

Proposition 4.3.1 *The attainability domain of the pursuers P_i from the initial state x_{i0} at time $t_0 = 0$ is the ball $H_{p_i}(x_{i0}, \rho_i\sqrt{\theta})$.*

Proof To prove this, we show that

i. $||x_i(\theta) - x_{i0}|| \leq \rho_i\sqrt{\theta}$.

ii. For any point $\bar{x} \in H_{p_i}(x_{i0}, \rho_i\sqrt{\theta})$, there exist a pursuer's control $u_i(t)$, $0 \leq t \leq \theta$ such that $x_i(\theta) = \bar{x}$.

Indeed,

$$\begin{aligned}
||x_i(\theta) - x_{i0}|| &= \left|\left| x_{i0} + \int_0^\theta u_i(s)ds - x_{i0} \right|\right| \\
&= \left|\left| \int_0^\theta u_i(s)ds \right|\right| \\
&\leq \int_0^\theta ||u_i(s)||ds \\
&\leq \left(\int_0^\theta 1^2 ds \right)^{\frac{1}{2}} \left(\int_0^\theta ||u_i(s)||^2 ds \right)^{\frac{1}{2}} \\
&\leq \sqrt{\theta}.\rho_i \\
&= \rho_i\sqrt{\theta}.
\end{aligned}$$

If the pursuers P_i uses the control,

$$u_i(t) = \frac{\bar{x} - x_{i0}}{\theta},$$

then we have,

$$\begin{aligned}
x_i(\theta) &= x_{i0} + \int_0^\theta u_i(s)ds \\
&= x_{i0} + \int_0^\theta \frac{\bar{x} - x_{i0}}{\theta} ds \\
&= x_{i0} + \frac{(\bar{x} - x_{i0})}{\theta} \int_0^\theta ds \\
&= x_{i0} + \frac{(\bar{x} - x_{i0})}{\theta}.\theta \\
&= x_{i0} + \bar{x} - x_{i0} \\
&= \bar{x}.
\end{aligned}$$

■

Proposition 4.3.2 *The attainability domain of the evader E from the initial state y_0 at time $t_0 = 0$ is the ball $H_E(y_0, \sigma\frac{\theta^2}{2})$.*

Proof To prove this, we show that

i. $||y(\theta) - y_0|| \leq \sigma\frac{\theta^2}{2}$

ii. For any point $\bar{y} \in H_E(y_0, \sigma\frac{\theta^2}{2})$, there exist a evader's control $v(t)$, $0 \leq t \leq \theta$ such that $y(\theta) = \bar{y}$.

Indeed,

$$\begin{aligned}
||y(\theta) - y_0|| &= \left|\left| y_0 + \int_0^\theta (\theta - s)v(s)ds - y_0 \right|\right| \\
&= \left|\left| \int_0^\theta (\theta - s)v(s)ds \right|\right| \\
&\leq \int_0^\theta |\theta - s|\, ||v(s)||ds \\
&\leq \sigma \int_0^\theta |\theta - s|\, ds \\
&= \sigma\frac{\theta^2}{2}.
\end{aligned}$$

If the evader uses the control,

$$v(t) = \frac{2}{\theta^2}(\bar{y} - y_0),$$

then we have,

$$\begin{aligned}
y(\theta) &= y_0 + \int_0^\theta (\theta - s)v(s)ds \\
&= y_0 + \int_0^\theta (\theta - s)\frac{2}{\theta^2}(\bar{y} - y_0)ds \\
&= y_0 + \frac{2}{\theta^2}(\bar{y} - y_0)\int_0^\theta (\theta - s)ds \\
&= y_0 + \frac{2}{\theta^2}(\bar{y} - y_0).\frac{\theta^2}{2} \\
&= y_0 + \bar{y} - y_0 \\
&= \bar{y}.
\end{aligned}$$

■

4.3.3 Auxiliary Differential Game I

In this section, we consider differential game problem described by the equations (4.2.1), (4.2.5) and (4.2.8) but with one pursuer and evader.

We define the set

$$\Omega = \left\{ z \in l_2 : 2\langle y_0 - x_0, z\rangle \leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) + ||y_0||^2 - ||x_0||^2 \right\}.$$

The goal of the pursuer P is to ensure the equality $x(\theta) = y(\theta)$ and that of evader E is the opposite. The problem is to construct a strategy for the pursuer that ensure the equality $x(\theta) = y(\theta)$ for any admissible control of the evader.

Lemma 4.3.1 *If $y(\theta) \in \Omega$, then, there exist a strategy of the pursuer P such that $x(\theta) = y(\theta)$ in the game.*

Proof Let the pursuer uses the strategy defined by

$$U(t) = \frac{y_0 - x_0}{\theta} + (\theta - t)v(t),\ 0 \leq t \leq \theta. \tag{4.3.1}$$

To show that this strategy is admissible, we use the fact that $y(\theta) \in \Omega$. This means that,

$$2\langle y_0 - x_0, y(\theta)\rangle \leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) + ||y_0||^2 - ||x_0||^2.$$

In accordance with the last inequality and using the state equation of the evader (4.2.4), we have the following:

$$\begin{aligned}
2\left\langle y_0 - x_0, \int_0^\theta (\theta - s)v(s)ds \right\rangle &= 2\langle y_0 - x_0, y(\theta) - y_0\rangle = 2\langle y_0 - x_0, y(\theta)\rangle - 2\langle y_0 - x_0, y_0\rangle \\
&= 2\langle y_0 - x_0, y(\theta)\rangle - 2||y_0||^2 + 2\langle x_0, y_0\rangle \\
&\leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) + ||y_0||^2 - ||x_0||^2 - 2||y_0||^2 + 2\langle x_0, y_0\rangle \\
&\leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) - ||y_0||^2 - ||x_0||^2 + 2\langle x_0, y_0\rangle \\
&\leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) - \left(||y_0||^2 + ||x_0||^2 - 2\langle x_0, y_0\rangle\right) \\
&\leq \theta\left(\rho^2 - \sigma^2\frac{\theta^3}{3}\right) - ||y_0 - x_0||^2.
\end{aligned}$$

Therefore,

$$2\left\langle y_0-x_0,\int_0^\theta(\theta-s)v(s)ds\right\rangle\leq\theta\left(\rho^2-\sigma^2\frac{\theta^3}{3}\right)-||y_0-x_0||^2. \qquad (4.3.2)$$

We can now show that the strategy (4.3.1) is admissible, using the inequality (4.3.2), as follows:

$$\begin{aligned}\int_0^\theta||u(t)||^2dt&=\int_0^\theta\left|\left|\frac{y_0-x_0}{\theta}+(\theta-s)v(s)\right|\right|^2ds\\&=\int_0^\theta\left(\left|\left|\frac{y_0-x_0}{\theta}\right|\right|^2+2\left\langle\frac{y_0-x_0}{\theta},(\theta-s)v(s)\right\rangle+||(\theta-s)v(s)||^2\right)ds\\&=\int_0^\theta\frac{||y_0-x_0||^2}{\theta^2}ds+2\int_0^\theta\left\langle\frac{y_0-x_0}{\theta},(\theta-s)v(s)\right\rangle ds+\int_0^\theta(\theta-s)^2||v(s)||^2ds\\&\leq\frac{||y_0-x_0||^2}{\theta}+\frac{2}{\theta}\left\langle y_0-x_0,\int_0^\theta(\theta-s)v(s)ds\right\rangle+\sigma^2\int_0^\theta(\theta-s)^2ds\\&\leq\frac{||y_0-x_0||^2}{\theta}+\frac{1}{\theta}\left(\theta\left(\rho^2-\sigma^2\frac{\theta^3}{3}\right)-||y_0-x_0||^2\right)+\sigma^2\frac{\theta^3}{3}\\&\leq\rho^2.\end{aligned}$$

Lastly, if the pursuer P uses the strategy (4.3.1), then we have $x(\theta)=y(\theta)$. Indeed,

$$\begin{aligned}x(\theta)&=x_0+\int_0^\theta\left(\frac{y_0-x_0}{\theta}+(\theta-s)v(s)\right)ds\\&=x_0+\int_0^\theta\left(\frac{y_0-x_0}{\theta}\right)ds+\int_0^\theta(\theta-s)v(s)ds\\&=x_0+\left(\frac{y_0-x_0}{\theta}\right)\int_0^\theta ds+\int_0^\theta(\theta-s)v(s)ds\\&=x_0+\left(\frac{y_0-x_0}{\theta}\right)\theta+\int_0^\theta(\theta-s)v(s)ds\\&=x_0+y_0-x_0+\int_0^\theta(\theta-s)v(s)ds\\&=y_0+\int_0^\theta(\theta-s)v(s)ds=y(\theta)\end{aligned}$$

■

Theorem 4.3.1 *If there exists a nonzero vector* p_0 *such that* $\langle y_0-x_{i0},p_0\rangle\geq 0$ *,* $i\in I$*, the number*

$$\phi=\inf\left\{l\geq 0:H_E\left(y_0,\sigma\frac{\theta^2}{2}\right)\subset\bigcup_{i\in I}H_{p_i}\left(x_{i0},\rho_i\sqrt{\theta}+l\right)\right\}$$

is the value of the game.

Proof To prove this theorem, we first introduce dummy pursuers whose state variables are z_i, $i \in I$ and motions described by the equations

$$\dot{z}_i = w_i(t), \;\; z_i(0) = x_{i0}, \tag{4.3.3}$$

where the control function $w_i(t)$ is such that

$$\left(\int_0^\theta ||w_i(s)||^2 ds\right)^{\frac{1}{2}} \leq \bar{\rho}_i = \rho_i + \frac{\phi}{\sqrt{\theta}}. \tag{4.3.4}$$

The attainability domain of the dummy pursuer z_i from the initial state x_{i0} is the ball

$$H_{D_i}(x_{i0}, \bar{\rho}_i\sqrt{\theta}) = H_{D_i}(x_{i0}, \rho_i\sqrt{\theta} + \phi).$$

Let strategies of the dummy pursuers $z_i, \; i \in I$ be defined by

$$w_i(t) = \begin{cases} \frac{y_0 - x_{i0}}{\theta} + (\theta - t)v(t), & 0 \leq t \leq \vartheta \\ 0, & \vartheta < t \leq \theta, \end{cases} \tag{4.3.5}$$

where ϑ is the time such that

$$\int_0^\vartheta \left|\left|\frac{y_0 - x_{i0}}{\theta} + (\theta - t)v(t)\right|\right|^2 dt = \bar{\rho}_i^2. \tag{4.3.6}$$

Now, we define the strategy of the real pursuers by

$$u_i(t) = \frac{\rho_i}{\bar{\rho}_i} w_i(t), \; 0 \leq t \leq \theta. \tag{4.3.7}$$

In accordance with the payoff of the game, the number ϕ is the value of the game if the following inequality holds

$$\sup_{v(\cdot)} \inf_{i \in I} ||y(\theta) - x_i(\theta)|| \leq \phi \leq \inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots} \inf_{i \in I} ||y(\theta) - x_i(\theta)||. \tag{4.3.8}$$

In view of this, we prove the inequality (4.3.8). Firstly, we show that left hand side of the inequality (4.3.8) holds. By definition of ϕ, we have

$$H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i=I}^{\infty} H_{P_i}\left(x_{i0}, \rho_i\sqrt{\theta} + \phi\right).$$

Using proposition (3.1.9), we have

$$H_E\left(y_0,\sigma\frac{\theta^2}{2}\right)\subset\bigcup_{i\in\bar{1}}X_i,$$

where

$$\bar{I}=\left\{i\in I:S\left(y_0,\sigma\frac{\theta^2}{2}\right)\cap H_{p_i}(x_{i0},\rho_i\sqrt{\theta}+\phi)\neq\emptyset\right\};$$

$$X_i=\begin{cases}\left\{z\in l_2:2\langle y_0-x_{i0},z\rangle\le\left(\rho_i\sqrt{\theta}+\phi\right)^2-\left(\sigma\frac{\theta^2}{2}\right)^2+||y_0||^2-||x_{i0}||^2\right\}, & x_{i0}\neq y_0,\\ \left\{z\in l_2:2\langle z-y_0,p_0\rangle\le\rho_i\sqrt{\theta}+\phi\right\}, & x_{i0}=y_0.\end{cases}$$

Consequently, the point $y(\theta)\in H_E\left(y_0,\sigma\frac{\theta^2}{2}\right)$ belong to some half space X_s, $s\in\bar{I}$.
If $x_{s0}\neq y_0$, then according to lemma (4.3.1) with the strategy of the dummy pursuer (4.3.1) we have

$$\int_0^\theta||w_i(s)||^2ds\le\bar{\rho}_i^2,$$

and $z_s(\theta)=y(\theta)$. In the other hand, if $x_{s0}=y_0$, $\bar{\rho}_i=\rho_i+\frac{\phi}{\sqrt{\theta}}\ge\sigma(\frac{\theta^3}{3})^{\frac{1}{2}}$ with the dummy pursuer's strategy defined by $w_s(\theta)=(\theta-t)v(t)$, we can have $z_s(\theta)=y(\theta)$. This shows that for each of the two cases we established that $z_s(\theta)=y(\theta)$. With this and if the real pursuers use the strategies (4.3.7), we show that

$$||y(\theta)-x_s(\theta)||\le\phi.$$

Indeed,

$$\begin{aligned}||y(\theta)-x_s(\theta)||=||z_s(\theta)-x_s(\theta)||&=\left|\left|x_{i0}+\int_0^\theta w_s(t)dt-x_{i0}-\int_0^\theta u_s(t)dt\right|\right|\\&=\left|\left|\int_0^\theta w_s(t)dt-\int_0^\theta u_s(t)dt\right|\right|=\left|\left|\int_0^\theta w_s(t)dt-\int_0^\theta\frac{\rho_s}{\bar{\rho}_s}w_s(t)dt\right|\right|\\&\le\int_0^\theta\left|\left|\left(1-\frac{\rho_s}{\bar{\rho}_s}\right)w_s(t)\right|\right|dt\le\left(\frac{\bar{\rho}_s-\rho_s}{\bar{\rho}_s}\right)\int_0^\theta||w_s(t)||dt\\&\le\left(\frac{\bar{\rho}_s-\rho_s}{\bar{\rho}_s}\right)\left[\left(\int_0^\theta1^2dt\right)^{\frac{1}{2}}\left(\int_0^\theta||w_s(t)||^2dt\right)^{\frac{1}{2}}\right]\\&\le\left(\frac{\bar{\rho}_s-\rho_s}{\bar{\rho}_s}\right)\bar{\rho}_s\sqrt{\theta}\\&\le(\bar{\rho}_s-\rho_s)\sqrt{\theta}=(\rho_s+\frac{\phi}{\sqrt{\theta}}-\rho_s)\sqrt{\theta}=\phi.\end{aligned}$$

This shows that the left hand inequality in (4.3.8) holds. Which means that the value ϕ is guaranteed for the pursuers.

Secondly, we prove the right hand inequality in (4.3.8). That is, we prove that the value φ is guaranteed for the evader. If $\phi = 0$, the inequality follows for any admissible control of the evader. Suppose that $\phi > 0$ and let the evader uses the strategy

$$v(t) = \frac{2}{\theta^2}(\bar{y} - y_0), 0 \leq t \leq \theta. \tag{4.3.9}$$

By definition of ϕ, for any $0 < \varepsilon < \phi$. The set $\bigcup_{i=1}^{\infty} H_p\left(x_{i0}, \rho_i\sqrt{\theta} + \phi - \varepsilon\right)$ does not contain the ball $H_E\left(y_0, \sigma\frac{\theta^2}{2}\right)$. Then by proposition (3.1.10), there exist a point $\bar{y} \in S\left(y_0, \sigma\frac{\theta^2}{2}\right)$, such that

$$||\bar{y} - x_{i0}|| \geq \rho_i\sqrt{\theta} + \phi. \tag{4.3.10}$$

Moreover from proposition (4.3.1), we have

$$||x_i(\theta) - x_{i0}|| \leq \rho_i\sqrt{\theta}. \tag{4.3.11}$$

In accordance with the inequalities (4.3.10) and (4.3.11), we have for all $i \in I$

$$\begin{aligned} ||\bar{y} - x_i(\theta)|| &\geq ||\bar{y} - x_{i0}|| - ||x_i(\theta) - x_{i0}|| \\ &\geq \rho_i\sqrt{\theta} + \phi - \rho_i\sqrt{\theta} = \phi. \end{aligned}$$

The control (4.3.9) brings the evader to the point $\bar{y}$ at the time θ. This claim can be proved as follows

$$\begin{aligned} y(\theta) &= y_0 + \int_0^\theta (\theta - s)v(s)ds \\ &= y_0 + \int_0^\theta (\theta - s).\frac{2}{\theta^2}(\bar{y} - y_0)ds \\ &= y_0 + \frac{2}{\theta^2}(\bar{y} - y_0)\int_0^\theta (\theta - s)ds \\ &= y_0 + \frac{2}{\theta^2}(\bar{y} - y_0).\frac{\theta^2}{2} \\ &= y_0 + \bar{y} - y_0 \\ &= \bar{y}. \end{aligned}$$

Therefore, the value ϕ is guaranteed for the evader.This proves the right hand inequality of (4.3.8). Hence the proof of the theorem is complete. ■

Example 4.3.2 *Let* $\rho_i = 2,\ \sigma = 1,\ \theta = 4$ *in the game problem 1. We consider the following initial positions* $x_{i0} = (0,\ 0,\ \dots,\ 6,\ 0,\ \dots),\ y_0 = (0,\ 0,\ \dots)$ *of the players, where the number 6 is the* i^{th} *coordinate of the point* x_i*. Note that* $\rho_i\sqrt{\theta} = 4,\ \sigma\frac{\theta^2}{2} = 8$*. Let us obtain the value of the game. To this end, it is sufficient to show that*

1. *For any* $\varepsilon > 0$ *the inclusion*

$$H_E(0,\ 8) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10+\varepsilon),$$

holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 8)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10)$*.*

Indeed, let $z = (z_1,\ z_2,\ \dots)$ *be arbitrary point of the ball* $H_E(0,\ 8)\ :\sum_{i=1}^{\infty} z_i^2 \le 64$*. Then either the vector z has a nonnegative coordinate or all the coordinate of the vector z are negative.*

In the former case, z has a nonnegative coordinates z_k*. Then*

$$\begin{aligned}||z-x_{k0}|| &= \left(z_1^2+\dots+z_{k-1}^2+(6-z_k)^2+z_{k+1}^2+\dots\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^{\infty} z_i^2+36-12z_k\right)^{\frac{1}{2}}\\ &\le (64+36-12z_k)^{\frac{1}{2}}\\ &\le (100-12z_k)^{\frac{1}{2}} \le 10 < 10+\varepsilon.\end{aligned}$$

Hence, $z_k \in H_{P_i}(x_{k0},\ 10+\varepsilon)$*.*

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z-x_{k0}|| \le (100-12z_k)^{\frac{1}{2}} \le 10 < 10+\varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0\ 8)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10),$$

since for any number i

$$||z-x_{i0}|| = (100-12z_i)^{\frac{1}{2}} > 10.$$

Therefore by theorem (4.3.1), the number

$$\begin{aligned}\phi &= \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta}+l\right)\right\} \\ &= \inf\left\{ l \geq 0 : H_E\left(0, 8\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, 4+l\right)\right\} \\ &= 6.\end{aligned}$$

Example 4.3.3 *Let* $\rho_i = 2,\ \sigma = 2,\ \theta = 4$ *in the game problem 1. We consider the following initial positions* $x_{i0} = (0,\ 0,\ \ldots,\ 12,\ 0,\ \ldots),\ y_0 = (0,\ 0,\ \ldots)$ *of the players, where the number 12 is the* i^{th} *coordinate of the point* x_i*. Note that* $\rho_i\sqrt{\theta} = 4,\ \sigma\frac{\theta^2}{2} = 16$*. Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *For any* $\varepsilon > 0$ *the inclusion*

   $$H_E(0,\ 16) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20+\varepsilon),$$

   holds where 0 *is the origin.*

2. *the ball* $H_E(0,\ 16)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20)$.

Indeed, let $z = (z_1,\ z_2,\ \ldots)$ *be arbitrary point of the ball* $H_E(0,\ 16)\ :\sum_{i=1}^{\infty} z_i^2 \leq 256$*. Then either the vector* z *has a nonnegative coordinate or all the coordinate of the vector* z *are negative.*

In the former case, z *has a nonnegative coordinates* z_k*. Then*

$$\begin{aligned}||z - x_{k0}|| &= \left(z_1^2 + \cdots + z_{k-1}^2 + (12-z_k)^2 + z_{k+1}^2 + \ldots\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{\infty} z_i^2 + 144 - 24z_k\right)^{\frac{1}{2}} \\ &\leq (256 + 144 - 24z_k)^{\frac{1}{2}} \\ &\leq (400 - 24z_k)^{\frac{1}{2}} \leq 20 < 20 + \varepsilon.\end{aligned}$$

Hence, $z_k \in H_{P_l}(x_{k0},\ 20+\varepsilon)$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z - x_{k0}|| \leq (400 - 24z_k)^{\frac{1}{2}} \leq 20 < 20 + \varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0\ 16)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_I}(x_{i0},\ 20),$$

since for any number i

$$||z - x_{i0}|| = (400 - 24z_i)^{\frac{1}{2}} > 20.$$

Therefore by theorem (4.3.1), the number

$$\begin{aligned}
\phi =& \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta} + l\right)\right\} \\
&= \inf\left\{ l \geq 0 : H_E\left(0, 16\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, 4 + l\right)\right\} \\
&= 16.
\end{aligned}$$

Example 4.3.4 *Let* $\rho_i = 2,\ \sigma = 8,\ \theta = 1$ *in the game problem 1. We consider the following initial positions* $x_{i0} = (0,\ 0,\ \ldots,\ 3,\ 0,\ \ldots),\ y_0 = (0,\ 0,\ \ldots)$ *of the players, where the number 3 is the* i^{th} *coordinate of the point* x_i*. Note that* $\rho_i\sqrt{\theta} = 2,\ \sigma\frac{\theta^2}{2} = 4$*. Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *For any* $\varepsilon > 0$ *the inclusion*

$$H_E(0,\ 4) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 5 + \varepsilon),$$

   holds where 0 *is the origin.*

2. *the ball* $H_E(0,\ 4)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 5)$*.*

Indeed, let $z = (z_1,\ z_2,\ \ldots)$ *be arbitrary point of the ball* $H_E(0,\ 4)\ : \sum_{i=1}^{\infty} z_i^2 \leq 16$*. Then either the vector z has a nonnegative coordinate or all the coordinate of the vector z are negative.*

In the former case, z has a nonnegative coordinates z_k*. Then*

$$\begin{aligned}
||z - x_{k0}|| &= \left(z_1^2 + \cdots + z_{k-1}^2 + (3 - z_k)^2 + z_{k+1}^2 + \ldots\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{\infty} z_i^2 + 9 - 6z_k\right)^{\frac{1}{2}} \\
&\leq (16 + 9 - 6z_k)^{\frac{1}{2}} \\
&\leq (25 - 6z_k)^{\frac{1}{2}} \leq 5 < 5 + \varepsilon.
\end{aligned}$$

Hence, $z_k \in H_{P_i}(x_{k0},\ 5+\varepsilon)$*. In the letter case, since* $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z - x_{k0}|| \leq (25 - 6z_k)^{\frac{1}{2}} \leq 5 < 5 + \varepsilon,$$

for an index k. On the other hand, any point $z \in S(0\ 8)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 5),$$

since for any number i

$$||z - x_{i0}|| = (25 - 6z_i)^{\frac{1}{2}} > 5.$$

Therefore by theorem (4.3.1), the number

$$\begin{aligned}
\phi &= \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta} + l\right) \right\} \\
&= \inf\left\{ l \geq 0 : H_E\left(0, 4\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, 2 + l\right) \right\} \\
&= 3.
\end{aligned}$$

4.3.4 Problem II

4.3.5 Attainability Domain II

Proposition 4.3.3 *The attainability domain of the pursuers* P_i *from the initial state* x_{i0} *at time* $t_0 = 0$ *is the ball* $H_{P_i}(x_{i0}, \rho_i\theta)$*.*

Proof To prove this, we show that

i. $||x_i(\theta) - x_{i0}|| \leq \rho_i\theta$

ii. For any point $\bar{x} \in H_{P_i}(x_{i0}, \rho_i\theta)$, there exist a pursuer's control $u_i(t)$, $0 \leq t \leq \theta$ such that $x_i(\theta) = \bar{x}$, indeed,

$$\begin{aligned}
||x_i(\theta)-x_{i0}|| &= \left|\left|x_{i0}+\int_0^{\theta} u_i(s)ds - x_{i0}\right|\right| \\
&= \left|\left|\int_0^{\theta} u_i(s)ds\right|\right| \\
&\leq \int_0^{\theta} ||u_i(s)||ds \\
&\leq \rho_i\theta,
\end{aligned}$$

we defined the pursuer's control as follows,

$$u_i(t)=\frac{\bar{x}-x_{i0}}{\theta}.$$

Using equation (4.2.2), we have

$$\begin{aligned}
x_i(\theta) &= x_{i0}+\int_0^{\theta} u_i(s)ds \\
&= x_{i0}+\int_0^{\theta} \frac{\bar{x}-x_{i0}}{\theta}ds \\
&= x_{i0}+\frac{(\bar{x}-x_{i0})}{\theta}\int_0^{\theta} ds \\
&= x_{i0}+\bar{x}-x_{i0}=\bar{x}.
\end{aligned}$$

■

Proposition 4.3.4 *The attainability domain of the evader E from the initial state* y_0 *at time* $t_0=0$ *is the ball* $H_E(y_0,\sigma\frac{\theta^2}{2})$.

Proof . (see proposition 4.3.2) ■

4.3.6 Auxiliary Differential Game II

In this section we consider differential game problem described by the following equations (4.2.1), (4.2.6) and (4.2.8) but with one pursuer and evader. The purpose of the pursuer P is to ensure the equality $x(\theta)=y(\theta)$ and that of evader E is the opposite. The problem is to construct a strategy for the pursuer that ensure the equality $x(\theta)=y(\theta)$ for any admissible control of the evader.

Lemma 4.3.2 *If* $\rho>\sigma\theta$ *and* $||y_0-x_0||\leq\theta(\rho-\sigma\theta)$, *then pursuit can be completed in the game.*

Proof Let the pursuer uses the strategy (4.3.1). Then the admissibility of this strategy can be shown as follows:

$$\begin{aligned}||u(t)|| &= \left|\left|\frac{y_0-x_0}{\theta}+(\theta-t)v(t)\right|\right| \leq \left|\left|\frac{y_0-x_0}{\theta}\right|\right| + ||(\theta-t)v(t)|| \\ &\leq \frac{||y_0-x_0||}{\theta}+(\theta-t)\,||v(t)|| \leq \frac{||y_0-x_0||}{\theta}+\sigma(\theta-t) \\ &\leq \frac{\theta(\rho-\sigma\theta)}{\theta}+\sigma\theta-\sigma t \\ &= \rho-\sigma t \\ &< \rho.\end{aligned}$$

If the pursuer P uses the strategy (4.3.1). From the prove of the Lemma (4.3.1), we have seen that this strategy ensures completion of pursuit. This complete the proof of the lemma (4.3.2). ■

Theorem 4.3.1 *If there exists a nonzero vector p_0 such that $\langle y_0-x_{i0},p_0\rangle \geq 0$, for all $i \in I$, the number*

$$\varphi = \inf\left\{ l \geq 0 : H_E\left(y_0,\sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in I} H_{P_i}(x_{i0},\rho_i\theta+l)\right\} \tag{4.3.12}$$

is the value of the game.

Proof To prove this theorem, we first introduce dummy pursuers whose state variables are z_i, $i \in I$, and motions described by the equations (4.3.3) where the control function $w_i(t)$ is such that

$$||w_i(s)|| \leq \bar{\rho}_i = \rho_i + \frac{\varphi}{\theta}.$$

The attainability domain of the dummy pursuer z_i from the initial state x_{i0} is the ball

$$H_{D_i}(x_{i0},\bar{\rho}_i\theta) = H_{D_i}(x_{i0},\rho_i\theta+\varphi).$$

Let the dummy pursuers z_i uses the strategies defined by (4.3.5).

In accordance with the payoff of the game, the number φ is the value of the game if the following inequality holds

$$\sup_{v(\cdot)}\inf_{i\in I}||y(\theta)-x_i(\theta)|| \leq \varphi \leq \inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots}\inf_{i\in I}||y(\theta)-x_i(\theta)||\,. \tag{4.3.13}$$

In view of this, we prove the inequality (4.3.13). Firstly, we show that left hand side of the inequality (4.3.13) holds. Indeed, by definition of φ, we have

$$H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i=1}^{\infty} H_{P_i}\left(x_{i0}, \rho_i\theta + \varphi\right).$$

Using proposition (3.1.9), we have

$$H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in\bar{I}} X_i,$$

where

$$\bar{I} = \left\{i \in I : S\left(y_0, \sigma\frac{\theta^2}{2}\right) \cap H_{p_i}(x_{i0}, \rho_i\theta + \varphi) \neq \emptyset\right\};$$

$$X_i = \begin{cases} \left\{z \in l_2 : 2\langle y_0 - x_{i0}, z\rangle \leq (\rho_i\theta + \varphi)^2 - \left(\sigma\frac{\theta^2}{2}\right)^2 + ||y_0||^2 - ||x_{i0}||^2\right\}, & x_{i0} \neq y_0, \\ \{z \in l_2 : 2\langle z - y_0, p_0\rangle \leq \rho_i\theta + \varphi\}, & x_{i0} = y_0. \end{cases}$$

Consequently, the point $y(\theta) \in H_E\left(y_0, \sigma\frac{\theta^2}{2}\right)$ belong to some half space X_s, $s \in \bar{I}$
If $x_{s0} \neq y_0$ by lemma (4.3.2) for strategy (4.3.5) of the dummy pursuers z_i, we obtain $||w_i(s)|| \leq \bar{\rho}$ and $z_s(\theta) = y(\theta)$. In the other hand, if $x_{s0} = y_0$, $\bar{\rho} \geq \theta\sigma$ and with dummy pursuer's strategy define by $w_s(t) = (\theta - t)v(t)$ we can have $z_s(\theta) = y(\theta)$. This shows that in both cases we can have $z_s = y(\theta)$. With this and if the real pursuers uses the strategy (4.3.7), we show that

$$||y(\theta) - x_s(\theta)|| \leq \varphi.$$

Indeed,

$$\begin{aligned}
||y(\theta)-x_s(\theta)|| &= ||z_s(\theta)-x_s(\theta)|| \\
&= \left|\left|x_{i0}+\int_0^\theta w_s(t)dt - x_{i0} - \int_0^\theta u_s(t)dt\right|\right| \\
&= \left|\left|\int_0^\theta w_s(t)dt - \int_0^\theta \frac{\rho_s}{\bar{\rho}_s} w_s(t)dt\right|\right| \\
&\leq \int_0^\theta \left|\left|\left(1-\frac{\rho_s}{\bar{\rho}_s}\right) w_s(t)\right|\right| dt \\
&\leq \left(\frac{\bar{\rho}_s-\rho_s}{\bar{\rho}_s}\right)\int_0^\theta ||w_s(t)||dt \\
&\leq \left(\frac{\bar{\rho}_s-\rho_s}{\bar{\rho}_s}\right)\bar{\rho}_s\theta \\
&\leq (\bar{\rho}_s-\rho_s)\theta \\
&= (\rho_s+\frac{\varphi}{\theta}-\rho_s)\theta \\
&= \varphi.
\end{aligned}$$

This shows that the left hand inequality in (4.3.13) holds. Which means that the value φ is guaranteed for the pursuers.

Secondly, we prove the right hand inequality in (4.3.13). That is, we prove that the value φ is guaranteed for the evader. If $\varphi = 0$, the inequality follows for any admissible control of the evader. Suppose that $\varphi > 0$ and let the evader uses the strategy

$$v(t) = \frac{2}{\theta^2}(\bar{y}-y_0),\ 0 \leq t \leq \theta. \tag{4.3.14}$$

By the definition of φ, and for any $0 < \varepsilon < \varphi$, the ball $H_E\left(y_0, \sigma\frac{\theta^2}{2}\right)$ is not contained in the set

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0}, \rho_i\theta + \varphi - \varepsilon).$$

Then by proposition (3.1.10), there exist a point $\bar{y} \in S\left(y_0, \sigma\frac{\theta^2}{2}\right)$, such that

$$||\bar{y}-x_{i0}|| \geq \rho_i\theta + \varphi. \tag{4.3.15}$$

Moreover, from proposition (4.3.3), we have

$$||x_i(\theta)-x_{i0}|| \leq \rho_i\theta. \tag{4.3.16}$$

In accordance with inequalities (4.3.15) and (4.3.16), we have for all $i \in I$

$$||\bar{y} - x_i(\theta)|| \geq ||\bar{y} - x_{i0}|| - ||x_{i0} - x_i(\theta)|| \geq \rho_i\theta + \varphi - \rho_i\theta = \varphi.$$

The control (4.3.14) brings the evader to the point $\bar{y}$ at time θ. This claim can be proved as follows:

$$\begin{aligned} y(\theta) &= y_0 + \int_0^\theta (\theta - s)v(s)ds \\ &= y_0 + \int_0^\theta (\theta - s)\frac{2}{\theta^2}(\bar{y} - y_0)ds \\ &= y_0 + \frac{2}{\theta^2}(\bar{y} - y_0)\frac{\theta^2}{2} \\ &= y_0 + \bar{y} - y_0 = \bar{y}. \end{aligned}$$

Therefore, the value φ is guaranteed for the evader. This proves the right hand inequality of (4.3.13). Hence, the proof of the theorem is complete. ■

Example 4.3.2 *Let* $\rho_i = 1,\ \sigma = 4,\ \theta = 2$ *in the game problem 2. We consider the following initial positions* $x_{i0} = (0,\ 0,\ \dots,\ 6,\ 0,\ \dots)$, $y_0 = (0,\ 0,\ \dots)$ *of the players, where the number 6 is the* i^{th} *coordinate of the point* x_i*. Note that* $\rho_i\theta = 2,\ \sigma\frac{\theta^2}{2} = 8.$ *Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *For any* $\varepsilon > 0$ *the inclusion*

$$H_E(0,\ 8) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10 + \varepsilon),$$

   holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 8)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10)$.

Indeed, let $z = (z_1,\ z_2,\ \dots)$ *be arbitrary point of the ball* $H_E(0,\ 8)\ : \sum_{i=1}^{\infty} z_i^2 \leq 64$*. Then either the vector* z *has a nonnegative coordinate or all the coordinate of the vector* z *are negative. In the former case,* z *has a nonnegative coordinates* z_k*. Then*

$$\begin{aligned} ||z - x_{k0}|| &= \left(z_1^2 + \dots + z_{k-1}^2 + (6 - z_k)^2 + z_{k+1}^2 + \dots\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{\infty} z_i^2 + 36 - 12z_k\right)^{\frac{1}{2}} \\ &\leq (64 + 36 - 12z_k)^{\frac{1}{2}} \\ &\leq (100 - 12z_k)^{\frac{1}{2}} \leq 10 < 10 + \varepsilon. \end{aligned}$$

Hence, $z_k \in H_{P_i}(x_{k0},\ 10+\varepsilon)$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z - x_{k0}|| \leq (100 - 12z_k)^{\frac{1}{2}} \leq 10 < 10 + \varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0\ 8)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10),$$

since for any number i

$$||z - x_{i0}|| = (100 - 12z_i)^{\frac{1}{2}} > 10.$$

Therefore by the theorem (4.3.1) the number

$$\begin{aligned} \varphi &= \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, \rho_i\theta + l\right) \right\} \\ &= \inf\left\{ l \geq 0 : H_E\left(0, 8\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, 2 + l\right) \right\} \\ &= 8. \end{aligned}$$

Example 4.3.3 *Let* $\rho_i = 3,\ \sigma = 2,\ \theta = 4$ *in the game problem 2. We consider the following initial positions* $x_{i0} = (0,\ 0,\ \ldots,\ 12,\ 0,\ \ldots),\ y_0 = (0,\ 0,\ \ldots)$ *of the players, where the number 12 is the* i^{th} *coordinate of the point* x_i. *Note that* $\rho_i\theta = 12,\ \sigma\frac{\theta^2}{2} = 16$. *Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *For any* $\varepsilon > 0$ *the inclusion*

   $$H_E(0,\ 12) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20+\varepsilon),$$

   holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 16)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20)$.

Indeed, let $z = (z_1,\ z_2,\ \ldots)$ *be arbitrary point of the ball* $H_E(0,\ 16)\ : \sum_{i=1}^{\infty} z_i^2 \leq 256$. *Then either the vector z has a nonnegative coordinate or all the coordinate of the vector z are* *negative. In the former case, z has a nonnegative coordinates z_k. Then*

$$\begin{aligned}||z-x_{k0}|| &= \left(z_1^2+\cdots+z_{k-1}^2+(12-z_k)^2+z_{k+1}^2+\dots\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^{\infty} z_i^2+256-144z_k\right)^{\frac{1}{2}}\\ &\leq (256+144-24z_k)^{\frac{1}{2}}\\ &\leq (400-24z_k)^{\frac{1}{2}} \leq 20 < 20+\varepsilon.\end{aligned}$$

Hence, $z_k \in H_{P_i}(x_{k0},\ 20+\varepsilon)$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ is convergent, then $z_k \longrightarrow 0$ as $k \longrightarrow \infty$ and therefore

$$||z-x_{k0}|| \leq (400-24z_k)^{\frac{1}{2}} \leq 20 < 20+\varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0\ 16)$ with negative coordinates does not belong to the set

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20),$$

since for any number i

$$||z-x_{i0}|| = (400-24z_i)^{\frac{1}{2}} > 20.$$

Therefore by the theorem (4.3.1) the number

$$\begin{aligned}\varphi &= \inf\left\{l \geq 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in I} H_{p_i}(x_{i0}, \rho_i\theta+l)\right\}\\ &= \inf\left\{l \geq 0 : H_E(0,8) \subset \bigcup_{i\in I} H_{p_i}(x_{i0}, 12+l)\right\}\\ &= 8.\end{aligned}$$

Example 4.3.4 *Let $\rho_i = 1,\ \sigma = 1,\ \theta = 4$ in the game problem 2. We consider the following initial positions $x_{i0} = (0,\ 0,\ \dots,\ 6,\ 0,\ \dots),\ y_0 = (0,\ 0,\ \dots)$ of the players, where the number 6 is the i^{th} coordinate of the point x_i. Note that $\rho_i\theta = 4,\ \sigma\frac{\theta^2}{2} = 8$. Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *For any $\varepsilon > 0$ the inclusion*

$$H_E(0,\ 8) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10+\varepsilon),$$

holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 8)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10)$.

Indeed, let $z=(z_1,\ z_2,\ \dots)$ *be arbitrary point of the ball* $H_E(0,\ 8)\ :\sum_{i=1}^{\infty} z_i^2 \le 64$. *Then either the vector z has a nonnegative coordinate or all the coordinate of the vector z are negative. In the former case, z has a nonnegative coordinates* z_k. *Then*

$$\begin{aligned}||z-x_{k0}|| &= \left(z_1^2+\cdots+z_{k-1}^2+(6-z_k)^2+z_{k+1}^2+\dots\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^{\infty} z_i^2+36-12z_k\right)^{\frac{1}{2}}\\ &\le (64+36-12z_k)^{\frac{1}{2}}\\ &\le (100-12z_k)^{\frac{1}{2}} \le 10 < 10+\varepsilon.\end{aligned}$$

Hence, $z_k \in H_{P_i}(x_{k0},\ 10+\varepsilon)$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z-x_{k0}|| \le (100-12z_k)^{\frac{1}{2}} \le 10 < 10+\varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0\ 8)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10),$$

since for any number i

$$||z-x_{i0}|| = (100-12z_i)^{\frac{1}{2}} > 10.$$

Therefore by the theorem (4.3.1) the number

$$\begin{aligned}\varphi &= \inf\left\{ l \ge 0 : H_E\left(y_0, \sigma\frac{\theta^2}{2}\right) \subset \bigcup_{i\in I} H_{p_i}(x_{i0}, \rho_i\theta+l)\right\}\\ &= \inf\left\{ l \ge 0 : H_E(0,8) \subset \bigcup_{i\in I} H_{p_i}(x_{i0}, 4+l)\right\}\\ &= 6.\end{aligned}$$

4.3.7 Problem III

4.3.8 Attainability Domain III

Proposition 4.3.5 *The attainability domain of the pursuers P_i from the initial state x_{i0} at time $t_0 = 0$ is the ball $H_{p_i}(x_{i0}, \rho_i\sqrt{\theta})$.*

Proof . (see proposition 4.3.1) ■

Proposition 4.3.6 *The attainability domain of the evader E from the initial state y_0 at time $t_0 = 0$ is the ball $H_E\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{\frac{1}{2}}\right)$.*

Proof To prove this proposition, we show that;

i. $||y(\theta) - y_0|| \leq \sigma\left(\frac{\theta^3}{3}\right)^{\frac{1}{2}}$.

ii. For any point $\bar{y} \in H_E\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{\frac{1}{2}}\right)$, there exist a evader's control $v(t),\ 0 \leq t \leq \theta$ such that $y(\theta) = \bar{y}$.

Indeed,

$$\begin{aligned}
||y(\theta) - y_0|| &= \left|\left|y_0 + \int_0^\theta (\theta - s)v(s)ds - y_0\right|\right| \\
&= \left|\left|\int_0^\theta (\theta - s)v(s)ds\right|\right| \\
&\leq \int_0^\theta (\theta - s)||v(s)||ds \\
&\leq \left(\int_0^\theta (\theta - s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^\theta ||v(s)||^2 ds\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\theta^3}{3}\right)^{\frac{1}{2}} \sigma = \sigma\left(\frac{\theta^3}{3}\right)^{\frac{1}{2}}
\end{aligned}$$

If the evader E uses the control

$$v(t) = \frac{3}{\theta^3}(\theta - s)(\bar{y} - y_0),\ \ 0 \leq t \leq \theta$$

, then we have,

$$\begin{aligned}
y(\theta) &= y_0 + \int_0^\theta (\theta - s)v(s)ds \\
&= y_0 + \int_0^\theta \frac{3}{\theta^3}(\theta - s)^2(\bar{y} - y_0)ds \\
&= y_0 + \frac{3}{\theta^3}(\bar{y} - y_0)\int_0^\theta (\theta - s)^2 ds \\
&= y_0 + \frac{3}{\theta^3}(\bar{y} - y_0)\frac{\theta^3}{3} \\
&= y_0 + \bar{y} - y_0 = \bar{y}.
\end{aligned}$$

■

4.3.9 Auxiliary Differential Game III

In this section we consider differential game problem described by the following equations (4.2.1), (4.2.5) and (4.2.7) but with one pursuer and evader.
We define the set

$$\Omega_3 = \left\{ z \in l_2 : 2\langle y_0 - x_0, z\rangle \leq \theta(\rho^2 - 2\sigma^2\theta^2) + ||y_0||^2 - ||x_0||^2 \right\}.$$

The goal of the pursuer P is to ensure the equality $x(\theta) = y(\theta)$ and that of evader E is the opposite. The problem is to construct a strategy for the pursuer that ensure the equality $x(\theta) = y(\theta)$ for any admissible control of the evader.

Lemma 4.3.3 *If $y(\theta) \in \Omega_3$, then, there exist a strategy of the pursuer P such that $x(\theta) = y(\theta)$ in the game.*

Proof Let the pursuer P uses the strategy (4.3.1). From the prove of Lemma (4.3.1), we have seen that this strategy ensures completion of pursuit. This means that for the proof of this lemma it remains to show the admissibility of this strategy. To show that the strategy (4.3.1) is admissible, we use the fact that $y(\theta) \in \Omega_3$. This means that,

$$2\langle y_0 - x_0, y(\theta)\rangle \leq \theta(\rho^2 - 2\sigma^2\theta^2) + ||y_0||^2 - ||x_0||^2. \tag{4.3.17}$$

In accordance with the inequality (4.3.17) and using the state equation of the evader (4.2.4), we have

$$\begin{aligned}
2\left\langle y_0-x_0,\int_0^\theta(\theta-t)v(t)dt\right\rangle &= 2\langle y_0-x_0,y(\theta)-y_0\rangle = 2\langle y_0-x_0,y(\theta)\rangle - 2\langle y_0-x_0,y_0\rangle\\
&= 2\langle y_0-x_0,y(\theta)\rangle - 2||y_0||^2+2\langle x_0,y_0\rangle\\
&\le \theta(\rho^2-2\sigma^2\theta^2)+||y_0||^2-||x_0||^2-2||y_0||^2+2\langle x_0,y_0\rangle\\
&\le \theta(\rho^2-2\sigma^2\theta^2)-||y_0||^2-||x_0||^2+2\langle x_0,y_0\rangle\\
&\le \theta(\rho^2-2\sigma^2\theta^2)-\left(||y_0||^2+||x_0||^2-2\langle x_0,y_0\rangle\right)\\
&\le \theta(\rho^2-2\sigma^2\theta^2)-||y_0-x_0||^2.
\end{aligned}$$

Therefore,

$$2\left\langle y_0-x_0,\int_0^\theta(\theta-t)v(t)dt\right\rangle \le \theta(\rho^2-2\sigma^2\theta^2)-||y_0-x_0||^2. \tag{4.3.18}$$

Using inequality (4.3.18), we now show that the strategy (4.3.1) is admissible

$$\begin{aligned}
\int_0^\theta ||u(t)||^2dt &= \int_0^\theta\left|\left|\frac{y_0-x_0}{\theta}+(\theta-t)v(t)\right|\right|^2dt\\
&= \int_0^\theta\left(\left|\left|\frac{y_0-x_0}{\theta}\right|\right|^2+2\left\langle\frac{y_0-x_0}{\theta},(\theta-t)v(t)\right\rangle+||(\theta-t)v(t)||^2\right)dt\\
&= \int_0^\theta\frac{||y_0-x_0||^2}{\theta^2}dt+2\int_0^\theta\left\langle\frac{y_0-x_0}{\theta},(\theta-t)v(t)\right\rangle dt+\int_0^\theta(\theta-t)^2||v(t)||^2dt\\
&\le \frac{||y_0-x_0||^2}{\theta}+\frac{2}{\theta}\left\langle y_0-x_0,\int_0^\theta(\theta-t)v(t)dt\right\rangle+2\sigma^2\theta^2\\
&\le \frac{||y_0-x_0||^2}{\theta}+\frac{1}{\theta}\left(\theta\left(\rho^2-2\sigma^2\theta^2\right)-||y_0-x_0||^2\right)+2\sigma^2\theta^2=\rho^2.
\end{aligned}$$

Thus, the strategy (4.3.1) of the pursuer p is admissible. This complete the proof of the lemma. ■

Theorem 4.3.1 *If there exists a nonzero vector p_0 such that $\langle y_0-x_{i0},p_0\rangle\ge 0$, $i\in I$, the number*

$$\phi_3=\inf\left\{l\ge 0: H_E\left(y_0,\sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right)\subset\bigcup_{i\in I}H_{p_i}\left(x_{i0},\rho_i\sqrt{\theta}+l\right)\right\}$$

is the value of the game.

Proof To prove this theorem, we first introduce dummy pursuers whose state variables are z_i, $i \in I$ and motions described by the equations (4.3.3) where the control function $w_i(t)$ is such that

$$\left(\int_0^\theta ||w_i(s)||^2 ds\right)^{\frac{1}{2}} \le \bar{\rho}_i = \rho_i + \frac{\phi_3}{\sqrt{\theta}}.$$

The attainability domain of the dummy pursuer z_i from the initial state x_{i0} is the ball

$$H_{D_i}(x_{i0}, \bar{\rho}_i\sqrt{\theta}) = H_{D_i}(x_{i0}, \rho_i\sqrt{\theta} + \phi_3).$$

Let the dummy pursuers $z_i, i \in I$ uses the strategies (4.3.5).

In accordance with the payoff of the game, the number ϕ_3 is the value of the game if the following inequality holds

$$\sup_{v(\cdot)} \inf_{i \in I} ||y(\theta) - x_i(\theta)|| \le \phi_3 \le \inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots} \inf_{i \in I} ||y(\theta) - x_i(\theta)||. \tag{4.3.19}$$

In view of this, we prove the inequality (4.3.19). Firstly, we show that left hand side of the inequality (4.3.19) holds. By definition of ϕ_3, we have

$$H_E\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right) \subset \bigcup_{i=1}^{\infty} H_{P_i}\left(x_{i0}, \rho_i\sqrt{\theta} + \phi_3\right).$$

Using proposition (3.1.9), we have

$$H_E\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right) \subset \bigcup_{i \in \bar{I}} X_i,$$

where

$$\bar{I} = \left\{ i \in I : S\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right) \cap H_{P_i}(x_{i0}, \rho_i\sqrt{\theta} + \phi_3) \neq \emptyset \right\};$$

$$X_i = \begin{cases} \left\{ z \in l_2 : 2\langle y_0 - x_{i0}, z\rangle \le \left(\rho_i\sqrt{\theta} + \phi_3\right)^2 - \sigma^2\dfrac{\theta^3}{3} + ||y_0||^2 - ||x_{i0}||^2 \right\}, & x_{i0} \neq y_0, \\ \left\{ z \in l_2 : 2\langle z - y_0, p_0\rangle \le \rho_i\sqrt{\theta} + \phi_3 \right\}, & x_{i0} = y_0. \end{cases}$$

Consequently, the point $y(\theta) \in H_E\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right)$ belong to some half space X_s, $s \in \bar{I}$ If $x_{s0} \neq y_0$ by lemma (4.3.3) for strategy (4.3.5) of the dummy pursuers z_i, we obtain $\int_0^\theta ||w_i(s)||^2 ds \le \bar{\rho}_i^2$ and $z_s(\theta) = y(\theta)$. In the other hand, if $x_{s0} = y_0$, $\bar{\rho}_i = \rho_i + \frac{\phi}{\sqrt{\theta}} \ge$

$\sigma(\frac{\theta^3}{3})^{\frac{1}{2}}$, with dummy pursuer's strategies defined by $w_s(\theta) = (\theta - t)v(t)$, we can have $z_s(\theta) = y(\theta)$. This shows that for each cases we established that $z_s(\theta) = y(\theta)$. With this if the real pursuers uses the strategies (4.3.7), we show that

$$||y(\theta) - x_s(\theta)|| \leq \phi.$$

Indeed,

$$\begin{aligned}
||y(\theta) - x_s(\theta)|| = ||z_s(\theta) - x_s(\theta)|| &= \left|\left|x_{i0} + \int_0^\theta w_s(t)dt - x_{i0} - \int_0^\theta u_s(t)dt\right|\right| \\
&= \left|\left|\int_0^\theta w_s(t)dt - \int_0^\theta u_s(t)dt\right|\right| = \left|\left|\int_0^\theta w_s(t)dt - \int_0^\theta \frac{\rho_s}{\bar{\rho}_s} w_s(t)dt\right|\right| \\
&\leq \int_0^\theta \left|\left|\left(1 - \frac{\rho_s}{\bar{\rho}_s}\right) w_s(t)\right|\right| dt \leq \left(\frac{\bar{\rho}_s - \rho_s}{\bar{\rho}_s}\right) \int_0^\theta ||w_s(t)||dt \\
&\leq \left(\frac{\bar{\rho}_s - \rho_s}{\bar{\rho}_s}\right) \left[\left(\int_0^\theta 1^2 dt\right)^{\frac{1}{2}} \left(\int_0^\theta ||w_s(t)||^2 dt\right)^{\frac{1}{2}}\right] \\
&\leq \left(\frac{\bar{\rho}_s - \rho_s}{\bar{\rho}_s}\right) \bar{\rho}_s \sqrt{\theta} \leq (\bar{\rho}_s - \rho_s)\sqrt{\theta} = (\rho_s + \frac{\phi_3}{\sqrt{\theta}} - \rho_s)\sqrt{\theta} = \phi_3.
\end{aligned}$$

This shows that the left hand inequality in (4.3.19) holds. Which means that the value ϕ_3 is guaranteed for the pursuers.
Secondly, we prove the right hand inequality in (4.3.19). That is, we prove that the value ϕ_3 is guaranteed for the evader. If $\phi_3 = 0$, the inequality follows for any admissible control of the evader. Suppose that $\phi_3 > 0$ and let the evader uses the strategy

$$v(t) = \sigma \frac{3}{\theta^3}(\theta - t)e, \ \ 0 \leq t \leq \theta, \ \ e = \frac{\bar{y} - y_0}{||\bar{y} - y_0||}. \tag{4.3.20}$$

By definition of ϕ_3, for any $0 < \varepsilon < \phi_3$. The set $\bigcup_{i=1}^{\infty} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta} + \phi_3 - \varepsilon\right)$ does not contain the ball $H_E\left(y_0, \sigma(\theta^3/3)^{1/2}\right)$.
Then by proposition (3.1.10), there exist a point $\bar{y} \in S\left(y_0, \sigma(\theta^3/3)^{1/2}\right)$ i.e. $||\bar{y} - y_0|| = \sigma(\theta^3/3)^{1/2}$ such that

$$||\bar{y} - x_{i0}|| \geq \rho_i\sqrt{\theta} + \phi_3. \tag{4.3.21}$$

Moreover, from proposition (4.3.5), we have

$$||x_i(\theta) - x_{i0}|| \leq \rho_i\sqrt{\theta}. \tag{4.3.22}$$

From inequalities (4.3.21) and (4.3.22), we have

$$\begin{aligned}||\bar{y}-x_i(\theta)|| &\geq ||\bar{y}-x_{i0}|| - ||x_i(\theta)-x_{i0}|| \\ &\geq \rho_i\sqrt{\theta}+\phi_3-\rho_i\sqrt{\theta}=\phi_3.\end{aligned}$$

Therefore, $||\bar{y}-x_i(\theta)|| \geq \phi_3$.
The control (4.3.20) brings the evader to the point $\bar{y}$ at time θ.
Indeed,

$$\begin{aligned}y(\theta) &= y_0+\int_0^\theta (\theta-s)v(s)ds \\ &= y_0+\int_0^\theta (\theta-s)^2\sigma\left(\frac{3}{\theta^3}\right)^{1/2} eds \\ &= y_0+\sigma\left(\frac{\theta^3}{3}\right)^{1/2} e=\bar{y}.\end{aligned}$$

Therefore, the value ϕ_3 is guaranteed for the evader. This proves the right hand inequality of (4.3.19). Hence, this complete of the proof of the theorem. ■

Now we give an illustrative example.

Example 4.3.2 *Let* $\rho_i=\sqrt{3},\ \sigma=2,\ \theta=3$ *in the game problem 3, let the initial positions* $x_{i0}=(0,\ 0,\ \dots,\ 8,\ 0,\ \dots)$, $y_0=(0,\ 0,\ \dots)$ *of the players, where the number 8 is the* i^{th} *coordinate of the point* x_i. *Therefore,* $\rho_i\sqrt{\theta}=3,\ \sigma(\theta^3/3)^{1/2}=6$. *Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *for any* $\varepsilon>0$ *the inclusion*

$$H_E(0,\ 6)\subset\bigcup_{i=1}^{\infty}H_{P_i}(x_{i0},\ 10+\varepsilon),$$

   holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 6)$ *does not contain in the set* $\bigcup_{i=1}^{\infty}H_{P_i}(x_{i0},\ 10)$.

Indeed, let $z=(z_1,\ z_2,\ \dots)$ *be arbitrary point of the ball* $H_E(0,\ 6)\ :\sum_{i=1}^{\infty}z_i^2\leq 36$. *Then either the vector* z *has a nonnegative coordinate or all the coordinate of the vector* z *are* *negative. In the former case, z has a nonnegative coordinates z_k. Then*

$$\begin{aligned} ||z-x_{k0}|| &= \left(z_1^2+\dots+z_{k-1}^2+(8-z_k)^2+z_{k+1}^2+\dots\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{\infty} z_i^2+64-16z_k\right)^{\frac{1}{2}} \leq (36+64-16z_k)^{\frac{1}{2}} \\ &\leq (100-16z_k)^{\frac{1}{2}} \leq 10 < 10+\varepsilon. \end{aligned}$$

Hence, $z_k \in H_{P_k}(x_{k0},\ 10+\varepsilon)$, for some $k \in I$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ is convergent, then $z_k \longrightarrow 0$ as $k \longrightarrow \infty$ and therefore

$$||z-x_{k0}|| \leq (100-16z_k)^{\frac{1}{2}} \leq 10 < 10+\varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0,\ 6)$ with negative coordinates does not belong to the set

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 10),$$

since for any number i

$$||z-x_{i0}|| = (100-16z_i)^{\frac{1}{2}} > 10.$$

Therefore by the theorem (4.3.1), the number

$$\begin{aligned} \phi_3 &= \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma(\theta^3/3)^{1/2}\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta}+l\right)\right\} \\ &= \inf\left\{ l \geq 0 : H_E\left(0,6\right) \subset \bigcup_{i\in I} H_{p_i}\left(x_{i0}, 3+l\right)\right\} \\ &= 7. \end{aligned}$$

Example 4.3.3 *Let $\rho_i = 4\sqrt{3},\ \sigma = 4,\ \theta = 3$ in the game problem 3, let the initial positions $x_{i0} = (0,\ 0,\ \dots,\ 16,\ 0,\ \dots),\ y_0 = (0,\ 0,\ \dots)$ of the players, where the number 16 is the i^{th} coordinate of the point x_i. Therefore, $\rho_i\sqrt{\theta} = 12,\ \sigma(\theta^3/3)^{1/2} = 12$. Let us obtain the value of the game. To this end, it is sufficient to show that;*

1. *for any $\varepsilon > 0$ the inclusion*

$$H_E(0,\ 12) \subset \bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20+\varepsilon),$$

holds where 0 *is the origin.*

2. *The ball* $H_E(0,\ 12)$ *does not contain in the set* $\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20)$.

Indeed, let $z = (z_1,\ z_2,\ \dots)$ *be arbitrary point of the ball* $H_E(0,\ 12)\ : \sum_{i=1}^{\infty} z_i^2 \leq 144$. *Then either the vector z has a nonnegative coordinate or all the coordinate of the vector z are negative. In the former case, z has a nonnegative coordinates* z_k. *Then*

$$\begin{aligned} ||z - x_{k0}|| &= \left(z_1^2 + \dots + z_{k-1}^2 + (16 - z_k)^2 + z_{k+1}^2 + \dots\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{\infty} z_i^2 + 256 - 32z_k\right)^{\frac{1}{2}} \leq (144 + 256 - 32z_k)^{\frac{1}{2}} \\ &\leq (400 - 32z_k)^{\frac{1}{2}} \leq 20 < 20 + \varepsilon. \end{aligned}$$

Hence, $z_k \in H_{P_k}(x_{k0},\ 20 + \varepsilon)$, *for some* $k \in I$.

In the letter case, since $\sum_{i=1}^{\infty} z_i^2$ *is convergent, then* $z_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *and therefore*

$$||z - x_{k0}|| \leq (400 - 24z_k)^{\frac{1}{2}} \leq 20 < 20 + \varepsilon,$$

for an index k.

On the other hand, any point $z \in S(0,\ 12)$ *with negative coordinates does not belong to the set*

$$\bigcup_{i=1}^{\infty} H_{P_i}(x_{i0},\ 20),$$

since for any number i

$$||z - x_{i0}|| = (400 - 32z_i)^{\frac{1}{2}} > 20.$$

Therefore by the theorem (4.3.1), the number

$$\begin{aligned} \phi_3 &= \inf\left\{ l \geq 0 : H_E\left(y_0, \sigma(\theta^3/3)^{1/2}\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, \rho_i\sqrt{\theta} + l\right) \right\} \\ &= \inf\left\{ l \geq 0 : H_E\left(0, 12\right) \subset \bigcup_{i \in I} H_{p_i}\left(x_{i0}, 12 + l\right) \right\} \\ &= 8. \end{aligned}$$

CHAPTER FIVE

SUMMARY, RECOMMENDATION AND CONCLUSION

5.1 INTRODUCTION

The summary and conclusion of the entire research is presented in this chapter. Suggestions and recommendation for further research is also given.

5.2 Summary and Conclusion

In this research, we studied a number of pursuit-evasion differential game problems of fixed duration with many pursuers and one evader in the Hilbert space l_2. Equation of motion for each of the pursuer is described by first order differential equation and that of evader by second order differential equation. Three different problems has been presented and studied. In the first problem Control functions of the pursuers and evader are subject to integral and geometric constraints respectively. Control function of both pursuers and evader are subject to geometric constraints in the second problem, lastly in the third problem control functions of the players are subject to integral constraints. For each of the problem, we obtained the value of the game under certain, moreover, optimal strategies of the players were found. These were presented as theorems and their proves. The proofs of the main results relied on the solution of an auxiliary differential game problems in half-space and on some properties of spheres.

5.3 Recommendation

We observe that there are numerous literature on pursuit-evasion differential game problems in which the players perform the same motion with the same constraints on the control functions. Pursuit-evasion differential game problem in which players have different behaviors in their motion with different constraints on the control functions was studied. Also pursuit-evasion differential game problem in which players have different behaviors in their motion with the same types of constraints on the control functions of the players was studied. Evasion problem with respect to the differential games studied in this dissertation can be investigated.

For further research, pursuit-evasion differential game problems in the Hilbert space l_2 described by (1.5.1), with control functions of the pursuers and evader subjected to geometric and integral constraints respectively can be investigated. Also the problem is still open further investigation if the motion of the pursuers is described by second order differential equations and that of evader by first order differential equation. As there are four different possible combination of geometric and integral constraints on the control function of the players, there would be four different problems to be studied.

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